Available online at http://scik.org
Adv. Fixed Point Theory, 2023, 13:27
https://doi.org/10.28919/afpt/8215
ISSN: 1927-6303

SOME CONVERGENCE RESULTS FOR A SEQUENCE OF GÓRNICKI TYPE CONTRACTION MAPPINGS

REKHA PANICKER*, RAHUL SHUKLA, DEEPA VIJAYASENAN

Department of Mathematical Sciences and Computing, Walter Sisulu University, Mthatha 5117, South Africa

Abstract. The stability of fixed points for a sequence of mappings $\{T_n\}$ satisfying the conditions introduced by Górnicki is studied in a metric space (X,d). In particular, these mappings are only defined on a subset X_n of the metric space X. In this paper we study the convergence of $\{T_n\}$ and the convergence of their fixed points $\{x_n\}$. We also illustrate our results by applying them to an initial value problem for an ordinary differential equation.

Keywords: metric space; Gornicki contraction mapping; (G)-convergence; (H)-convergence.

2020 AMS Subject Classification: Primary 47H09; Secondary 47H10, 52A23, 46B20.

1. INTRODUCTION

Stability is a term that refers to the limiting characteristics of a system. The stability of fixed points is the study of the relationship between the convergence of a sequence of mappings on a metric space or a Banach space and the sequence of it's fixed points. If the fixed point sets of a sequence of mappings converge to the set of fixed points of their limit mappings, the sequence is said to be stable. We recommend the references [1, 5, 6, 10, 15, 21, 27, 28] for some interesting results on stability of fixed point sets of families of mappings in various settings. In this connection, see also the paper by S. Reich and I. Shafrir [25]. The problem can be traced

*Corresponding author

E-mail address: rpanicker@wsu.ac.za

Received September 13, 2023

back to a classical result of Bonsall [5] on contraction mappings. Later, Nadler Jr. [21] and others addressed the problem of replacing the completeness of the space X with the existence of fixed points (which is guaranteed by the completeness of X) and relaxing the contraction constant. Pointwise and uniform convergence play a significant role in the study of fixed point stability. The preceding concepts, however, do not hold true when the domain of definition of all mappings in the study is neither the same space nor a unique nonempty subset of it. Barbet and Nachi [4] overcame this problem by introducing two new ideas of convergence called (G)-convergence and (H)-convergence, which they used to derive stability results in metric spaces. These results generalize the corresponding results of Bonsall [5], Fraser and Nadler Jr. [10] and Nadler Jr. [21]. The Barbet-Nachi work is unique in the sense that it redefines pointwise and uniform convergence for operators defined on the subsets of a space rather than the entire metric space. (G)-convergence generalized pointwise convergence, while (H)-convergence extended uniform convergence. Afterwards, many authors generalized these results in different settings for various type of mappings, see for example [2, 16, 17, 18, 19, 22].

However, Górnicki's work [11, Theorem 2.6] stands as a captivating extension of the renowned Banach contraction principle. Within this manuscript, we leverage the broader Górnicki generalized contraction concept in conjunction with the novel convergence concepts introduced by Barbet and Nachi [4]. This combined framework enables us to explore the stability of fixed points across varying domains. The results obtained in this study provide insights into the stability of a wider range of mappings. We also explore its application to initial value problems.

2. Preliminaries

We present some definitions, notions and facts from the literature in this section. Throughout this paper, we use the notation $\mathbb{N}$ to denote the set of real numbers and $\overline{\mathbb{N}}$ to denote $\mathbb{N} \cup \{\infty\}$. Let (X,d) be a metric space and $T : X \to X$. Then T is called a contraction mapping if there exists a constant $k \in (0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y) \tag{2.1}$$

for all $x,y \in X$.

In 1968, Kannan [12] established the following theorem.

Theorem 1. [12] *If T is a map of the complete metric space (X,d) into itself and if there exists K such that $0 \leq K < \frac{1}{2}$ satisfying*

$$d(Tx,Ty) \leq K\{d(x,Tx)+d(y,Ty)\} \tag{2.2}$$

for all $x,y \in X$, then T has a unique fixed point u in X and $T^n x \to u$ for each $x \in X$.

Kannan's theorem is important because Subrahmanyam [29] proved that Kannan's theorem characterizes the metric completeness. That is, a metric space (X,d) is complete if and only if every mapping satisfying (2.2) on X with constant $K < \frac{1}{2}$ has a fixed point. Contractions do not have this property; Connell [9] gave an example of a metric space X such that X is not complete and every contraction on X has a fixed point. We recall, asymptotic regularity is a fundamentally important concept in metric fixed point theory. It was formally introduced by Browder and Petryshyn [7].

Definition 1. A mapping T of a metric space (X,d) into itself is said to be asymptotically regular if $\lim\limits_{n\to\infty}(T^n x, T^{n+1}x) = 0$ for all $x \in X$.

Obviously, if a mapping $T : X \to X$ is a contraction or satisfy 2.2 with $K < \frac{1}{2}$, then T is asymptotically regular.

3. GÓRNICKI GENERALISED CONTRACTION MAPPING

We recall the following definitions and results from Górnicki [11] which play an important role in this paper.

Definition 2. Let (X,d) be a complete metric space. A mapping $T : X \to X$ is said to be Górnicki generalised contraction if there exist $0 \leq M < 1$ and $0 \leq K < \infty$ such that

$$d(Tx,Ty) \leq Md(x,y)+K\{d(x,Tx)+d(y,Ty)\} \tag{3.1}$$

for all $x,y \in X$.

He proved that if X is a complete metric space and $T : X \to X$ is a continuous asymptotically regular Górnicki generalised contraction mapping, then T has a unique fixed point in X.
It is pointed out in [11] that every contraction mapping is Górnicki generalised contraction. The

above class of mappings contains many important classes of mappings. A number of contractions listed in [24] are particular cases of the following mapping:

Definition 3. [8]. Let (X,d) be a metric space and let $T : X \to X$ be a mapping such that for all $x, y \in X$,

$$d(Tx,Ty) \leq k \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}, \tag{3.2}$$

where $k \in [0,1)$ is fixed.

Now, we show that a mapping satisfying (3.2) also satisfies (3.1).

Proposition 1. *Let* (X,d) *be a metric space and let* $T : X \to X$ *satisfies* (3.2)*. Then* T *also satisfies* (3.1) *but the converse need not be true.*

Proof. We consider the following cases:

Case (i): $d(Tx,Ty) \leq kd(x,y)$.

Case (ii): $d(Tx,Ty) \leq kd(x,Tx)$.

Case (iii): $d(Tx,Ty) \leq kd(y,Ty)$.

Case (iv): $d(Tx,Ty) \leq kd(y,Tx)$. Then

$$d(Tx,Ty) \leq kd(y,x) + kd(x,Tx).$$

Case (v): $d(Tx,Ty) \leq kd(x,Ty)$. Then

$$d(Tx,Ty) \leq kd(y,x) + kd(y,Ty).$$

Thus T satisfies (3.1) with $M = k$, $K = k$. The example below illustrates that the converse of the preceding proposition may not hold.

Example 1. [23] Let $X = \{(0,0),(1,0),(0,1),(2,0),(0,2),(2,3),(3,2)\}$ be equipped with the metric d defined as follows:

$$d((x^{(1)},x^{(2)}),(y^{(1)},y^{(2)})) = |x^{(1)} - y^{(1)}| + |x^{(2)} - y^{(2)}|.$$

Define $T : X \to X$ by

$$T(0,0) = (0,0),\ T(1,0) = (0,0),\ T(0,1) = (0,0),\ T(2,0) = (1,0),\ T(0,2) = (0,1),$$

$$T(2,3) = (2,0),\ T(3,2) = (0,2).$$

It can be easily verified that T satisfies (3.1) for any $M \geq 0.5$ and $K \geq 6$. However, for $x = (2,3)$, $y = (3,2)$ and any $k \in [0,1)$, we have

$$\begin{aligned} d(Tx,Ty) = 4 &> k\max\{2,3,3,3,3\} \\ &= k\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}. \end{aligned}$$

Hence T does not satisfy (3.2).

□

4. Barbet-Nachi Type Convergence

Following the conditions developed by Barbet and Nachi [4], we define the following concepts of convergence for Górnicki-type mappings (3.1) in metric spaces, which is a generalization of pointwise and uniform convergence.

Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of non empty subsets of the metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be a family of Górnicki-type mappings.

Property I: The family of mappings $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ satisfying Property I if for every convergent sequence (x_n) in $\prod\limits_{n\in\mathbb{N}} X_n$, we have $\lim\limits_{n\to\infty} d(x_n, T_n x_n) = 0$.

Consider two nonempty sets, X and Y and a mapping $T : X \to Y$. The graph of T, which is represented as $Gr(T)$ is defined as the set $Gr(T) = \{(x,Tx) | x \in X\}$.

The mapping T_∞ is said to be a (G)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G) if the following condition holds:

(G)**:** $Gr(T_\infty) \subset \liminf Gr(T_n)$: for every $x \in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ such that:

$$\lim_{n\to\infty} d(x_n,x) = 0 \text{ and } \lim_{n\to\infty} d(T_n x_n, T_\infty x) = 0,$$

[4] Further, T_∞ is a (H)-limit of the sequence of Górnicki generalized contraction mappings $\{T_n\}_{n\in\mathbb{N}}$ or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H) if the following condition holds:

(H)**:** For all sequences $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$, there exists a sequence $\{y_n\}$ in X_∞ such that:

$$\lim_{n\to\infty} d(x_n, y_n) = 0 \text{ and } \lim_{n\to\infty} d(T_n x_n, T_\infty y_n) = 0.$$

5. Main Results

Following Lemma 1.1 of Latif, Nazir and Abbas [14], we proceed to demonstrate the proof of the sequential form of limits as indicated in (G) in a metric space.

Lemma 1. *Consider a metric space X and a family of mappings $\{T_n : X_n \to X\}$ where $\{X_n\}_{n\in\overline{\mathbb{N}}}$ is a family of nonempty subsets of X. Let $T_\infty : X_\infty \to X$ is a mapping that serves as the G-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$, meaning that $Gr(T_\infty) \subset \liminf Gr(T_n)$. For every $x \in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(T_n x_n, T_\infty x) = 0$.*

Proof. Note that $Gr(T_n) = \{(x_n, T_n x_n) : x_n \in X_n\}_{n\in\mathbb{N}}$ and

$$Gr(T_\infty) = \{(x, T_\infty x) : x \in X_\infty\}. \tag{5.1}$$

$$\liminf Gr(T_n) = \liminf\{(x_n, T_n x_n) : x_n \in X_n\}_{n\in\mathbb{N}}$$

which implies that

$$\liminf Gr(T_n) = \lim_{n\to\infty}[\inf\{(x_k, T_k x_k) : x_k \in X_k, k \geq n\}].$$

Moreover, we have

$$\begin{aligned} \liminf Gr(T_n) &= \sup[\inf\{(x_k, T_k x_k), x_k \in X_k\}] \\ &= \bigcup_{n=1}^{\infty}\left[\bigcap_{n=k}^{\infty}(x_k, T_k x_k), x_k \in X_k\right] \end{aligned} \tag{5.2}$$

From (5.1) and (5.2), for all $x \in X_\infty$,
$(x, T_\infty x) \in \bigcup_{n=1}^{\infty}\left[\bigcap_{n=k}^{\infty}(x_k, T_k x_k), x_k \in X_k\right]$ which implies that for all $x \in X_\infty$, there exists $n \in \mathbb{N}$, $k \geq n$ such that $(x, T_\infty x) = (x_k, T_k x_k), x_k \in X_k$.
Thus, for all $x \in X_\infty, x_n \in \prod_{n\in\mathbb{N}} X_n$, there exists $k \geq n$ such that $x_k = x$ and $T_k x_k = T_\infty x$. Hence for every $x \in X_\infty$, we have a sequence $\{x_n\} \in \prod_{n\in\mathbb{N}} X_n$ such that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(T_n x_n, T_\infty x) = 0$. □

Proposition 2. *Let (X,d) be a metric space and $\{X_n\}_{n\in\overline{\mathbb{N}}}$, a family of nonempty subsets of X. Consider a family of Górnicki type mappings $T_n : X_n \to X$. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$, then T_∞ is unique.*

Proof. Suppose T_∞ and T_∞^* are (G)-limit mappings of the sequence $\{T_n\}$. This means that for any $x \in X_\infty$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that $T_n x_n$ converges to $T_\infty x$ and $T_n y_n$ converges to $T_\infty^* x$ as $n \to \infty$.

$$\begin{aligned} d(T_\infty x, T_\infty^* x) &\leq d(T_\infty x, T_n x_n) + d(T_n x_n, T_n y_n) + d(T_n y_n, T_\infty^* x) \\ &\leq d(T_\infty x, T_n x_n) + Md(x_n, y_n) + K[d(x_n, T_n x_n) + d(y_n, T_n y_n)] + d(T_n y_n, T_\infty^* x) \end{aligned}$$

which tends to 0 as $n \to \infty$. Hence $T_\infty x = T_\infty^* x$. □

Theorem 2. *Let (X,d) be a metric space and $\{X_n\}_{n\in\overline{\mathbb{N}}}$, a family of nonempty subsets of X. $T_n : X_n \to X$, a family of Górnicki type mappings satisfying property (G) for all $n \in \mathbb{N}$. If for all $n \in \overline{\mathbb{N}}$, $x_n \in X_n$ is a fixed point of T_n, then the sequence $\{x_n\}$ converges to x_∞.*

Proof. Let $x_n \in X_n$ be a fixed point of T_n for each $n \in \overline{\mathbb{N}}$. Since T_n satisfies property (G), there exists a sequence $\{y_n\}$ where $y_n \in X_n$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} d(y_n, x_\infty) = 0$ and $\lim_{n\to\infty} d(T_n y_n, T_\infty x_\infty) = 0$ for $x \in X_\infty$. Thus we get

$$\begin{aligned} d(x_n, x_\infty) &\leq d(T_n x_n, T_n y_n) + d(T_n y_n, T_\infty x_\infty) \\ &\leq Md(x_n, y_n) + K[d(x_n, T_n x_n) + d(y_n, T_n y_n)] + d(T_n y_n, T_\infty x_\infty) \\ &\leq M[d(x_n, x_\infty) + d(y_n, x_\infty)] + K[d(x_n, T_n x_n) + d(y_n, x_\infty) + d(T_n y_n, T_\infty x_\infty)] \\ &\quad + d(T_n y_n, T_\infty x_\infty) \\ (1-M)d(x_n, x_\infty) &\leq Md(y_n, x_\infty) + K[d(y_n, x_\infty) + d(T_n y_n, T_\infty x_\infty)] + d(T_n y_n, T_\infty x_\infty) \end{aligned}$$

which implies,

$$d(x_n, x_\infty) \leq \frac{1}{1-M}\{Md(y_n, x_\infty) + K[d(y_n, x_\infty) + d(T_n y_n, T_\infty x_\infty)] + d(T_n y_n, T_\infty x_\infty)\}.$$

Hence we conclude that $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.

□

When all the subsets $X_n = X$, we obtain the following corollary as a consequence.

Corollary 1. *Let (X,d) be a metric space and $\{T_n : X \to X\}_{n\in\overline{\mathbb{N}}}$ a family of Górnicki type mappings satisfying the property (G). If for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. By taking $X_n = X$ for $n \in \overline{\mathbb{N}}$ in Theorem 2, we find the desired conclusion easily. □

The following proposition demonstrates that if $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of Górnicki type contractions, then its (G)-limit map T_∞ also belongs to the class of Górnicki type mappings.

Proposition 3. *Let (X,d) be a metric space and $\{X_n\}_{n\in\overline{\mathbb{N}}}$, a family of nonempty subsets of X. $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying property (G) and T_n is a Górnicki type contraction mapping. Then the (G)-limit map T_∞ of the sequence $\{T_n\}_{n\in\overline{\mathbb{N}}}$ is also a Górnicki type mapping.*

Proof. Let x and y be two points in X_∞. By property (G), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ converging respectively to x and y such that the sequences $\{T_nx_n\}$ and $\{T_ny_n\}$ converging respectively to $T_\infty x$ and $T_\infty y$. By applying the triangle inequality and Górnicki condition, we deduce that

$$\begin{aligned} d(T_\infty x, T_\infty y) &\leq d(T_\infty x, T_nx_n) + d(T_nx_n, T_ny_n) + d(T_ny_n, T_\infty y) \\ &\leq d(T_\infty x, T_nx_n) + Md(x_n, y_n) + K[d(x_n, T_nx_n) + d(y_n, T_ny_n)] + d(T_ny_n, T_\infty y). \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(T_\infty x, T_\infty y) &\leq \lim_{n\to\infty}\{M[d(x_n,x) + d(x,y) + d(y_n,y)] + K[d(x_n,x) + d(x,T_\infty x) + \\ &\quad d(T_\infty x, T_nx_n) + d(y_n,y) + d(y,T_\infty y) + d(T_ny_n, T_\infty y)]\} \\ &= Md(x,y) + K[d(x,T_\infty x) + d(y,T_\infty y)]. \end{aligned}$$

□

Theorem 3. *Let (X,d) be a metric space and $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X. Suppose that $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ is a sequence of Górnicki contractions satisfying property (G).*

If $x_n \in X_n$ is a fixed point of T_n for each $n \in \mathbb{N}$ and the sequence $\{x_n\}$ admits a subsequence converging to $x_\infty \in X_\infty$, then x_∞ is a fixed point of T_∞.

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $\lim\limits_{k\to\infty} x_{n_k} = x_\infty \in X_\infty$. By the property (G), there exists a sequence $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ such that $d(y_n, x_\infty) = 0$ and $d(T_n yn, T_\infty x_\infty) = 0$. For any $k \in \mathbb{N}$, we have

$$d\left(x_\infty, T_\infty x_\infty\right) \leq d(x_\infty, x_{n_k}) + d\left(T_{n_k} x_{n_k}, T_{n_k} y_{n_k}\right) + d\left(T_{n_k} y_{n_k}, T_\infty x_\infty\right). \tag{5.3}$$

Since T_{n_k} is a Górnicki type contraction,

$d(T_{n_k} x_{n_k}, T_{n_k} y_{n_k}) \leq M d(x_{n_k}, y_{n_k}) + K[d(x_{n_k}, T_{n_k} x_{n_k}) + d(y_{n_k}, T_{n_k} y_{n_k})]$, which tends to 0 as $k \to \infty$.

Then we have

$$\lim_{k\to\infty} d\left(x_\infty, T_\infty x_\infty\right) = 0,$$

which proves that x_∞ is a fixed point of T_∞.

□

6. Stability Results for (H)-Convergence

Next, we present another stability result by employing the (H)-convergence.

Theorem 4. *Let (X,d) be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X. Let $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the property (H) such that T_∞ is a Górnicki type contraction mapping. If for each $n \in \overline{\mathbb{N}}$, $x_n \in X_n$ is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. By property (H), there exists a sequence $\{y_n\}$ in X_∞ such that $d(x_n, y_n) \to 0$ and $d\left(T_n x_n, T_\infty y_n\right) \to 0$.

$$\begin{aligned}
d(x_n, x_\infty) &\leq d(T_n x_n, T_\infty y_n) + d(T_\infty y_n, T_\infty x_\infty) \\
&\leq d(T_n x_n, T_\infty y_n) + M d(y_n, x_\infty) + K[d(y_n, T_\infty y_n) + d(x_\infty, T_\infty x_\infty)] \\
&\leq d(T_n x_n, T_\infty y_n) + M[d(x_n, y_n) + d(x_n, x_\infty)] + K[d(y_n, T_n x_n) + d(T_n x_n, T_\infty y_n)].
\end{aligned}$$

Then

$(1-M)d(x_n,x_\infty) \leq Md(x_n,y_n)+K[d(y_n,T_nx_n)+d(T_nx_n,T_\infty y_n)]$.

Letting $n\to\infty$, we get $(1-M)d(x_n,x_\infty)\leq 0$ which implies that the sequence $\{x_n\}$ is (H)-convergent to x_∞.

□

Corollary 2. *Let (X,d) be a metric space and $\{T_n : X\to X\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the property (H) converging to the Górnicki contraction mapping $T_\infty : X\to X$. If for each $n\in\overline{\mathbb{N}}$, x_n represents the fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to T_∞.*

Proof. By taking $X_n = X$ for all $n\in\overline{\mathbb{N}}$ in Theorem 4, we obtain the desired result immediately.

□

Example 2. Let $X=[0,1]\cup[\frac{3}{2},\frac{5}{3}]$. Consider the family $T_n : X\to X$ defined by

$$\begin{aligned} T_n(x) &= \frac{nx}{1+nx}\ln(1+\frac{x}{n});\ 0\leq x\leq 1\\ &= 1+\frac{x}{n};\ \frac{3}{2}\leq x\leq\frac{5}{3} \end{aligned}$$

and $T : X\to X$ defined by

$$\begin{aligned} Tx &= 0;\ 0\leq x\leq 1\\ &= 1;\ \frac{3}{2}\leq x\leq\frac{5}{3}. \end{aligned}$$

Then

(1) T is a Górnicki contraction mapping with unique fixed point $x^*=0$.
(2) T is a (G)-limit of $\{T_n\}$.
(3) But $\{T_n\}$ does not converge pointwise to T. The problem arises when $x=0$. We can illustrate this by taking the $\{x_n\}=\{\frac{1}{\sqrt{n}}\}$ to verify that property (G) is satisfied at $x=0$.

7. Application

We consider the following application of our results. The following theorem is motivated due to S. B. Nadler, Jr. [21].

Theorem 5. *Let C be an open subset of $\mathbb{R}^2$, $(a,b) \in C$, $M \in \mathbb{R}$ and $0 < M < 1$. Suppose that:*

(a) (T_n) is a sequence of real valued continuous functions defined on C such that $|T_n(x,y)| \leq M$ for all $(x,y) \in C$ with a (G)-limit T_∞, a continuous function on C.

(b) the set $A = \{(x,y) : |x-a| \leq p \text{ and } |y-b| \leq M|y-a|\}$ is a subset of C with $p > 0$.

(c) let $M_n \in \mathbb{R}$ and $M_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$, (M_n) be a bounded sequence. For all (x,y), $(x,z) \in C$, we have $|T_n(x,y) - T_n(x,z)| \leq M_n|y-z| + K_n[|y-Ty| + |z-Tz|]$ where $0 \leq M_n.p < 1$ and $0 \leq K_n.p < \infty$ for all $n \in \mathbb{N} \cup \{0\}$.

Then the sequence $\{y_n\}$ converges uniformly on $I = [a-p, a+p]$ to y_0, where for each $n \in \mathbb{N} \cup \{0\}$, y_n is the unique solution on I of the initial value problem

$$y'(x) = T_n(x, y(x));\ y(a) = b.$$

Proof. Let X be the set of all real valued continuous functions on I with graph lying in A. Then X with the supremum metric d is a compact metric space. For each $g \in X$, define

$$[F_n(g)](x) = b + \int_a^x T_n(t, g(t))d(t)\ \forall\, x \in I.$$

For each $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned}
& |[F_n(g)](x) - [F_n(h)](x)| \\
= \quad & \left| \int_a^x T_n(t, g(t))d(t) - \int_a^x T_n(t, h(t))dt \right| \\
\leq \quad & \{M_n \sup \lim_{t\in[a,x]} |g(t) - h(t)| + K_n \sup \lim_{t\in[a,x]} [|g(t) - T(g(t)) + h(t) - T(h(t))|]\} \int_a^x dt \\
= \quad & \{M_n \sup \lim_{t\in[a,x]} |g(t) - h(t)| + K_n \sup \lim_{t\in[a,x]} [|g(t) - T(g(t)) + h(t) - T(h(t))|]\} |x-a| \\
\leq \quad & M_n.p|g-h| + K_n.p[|g - T(g)| + |h - T(h)|] \\
= \quad & Md(g,h) + K[d(g,Tg) + d(h,Th)]
\end{aligned}$$

where $M = M_n.p$; $0 \leq M < 1$ and $K = K_n.p$; $0 \leq K < \infty$. That is, F_n is a Górnicki type contraction mapping from X to X for all $n \in \mathbb{N} \cup \{0\}$. By Proposition 3, F is also a Górnicki type contraction mapping on X. For each $x \in I$, $g \in X$, $n \in \mathbb{N} \cup \{0\}$,

$$[F_n(g)](x) - [F(g)](x) = \int_a^x [T_n(t, g(t)) - T_\infty(t, g(t))]dt.$$

Since T_∞ is the (G)-limit of T_n, the sequence of integrands converges to zero and is uniformly bounded by $2M$. The Lebesgue bounded convergence theorem guarantees that the sequence of integrals on the R.H.S goes to 0 as $n \to \infty$. Therefore $F(g)$ is the (G)-limit of $F_n(g)$ on I. Now by [3, Proposition 5], (G)-limit is equivalent to pointwise limit. It is easy to see that $F_n(g)$ is uniformly continuous on I for each $n \in \mathbb{N} \cup \{0\}$ and hence the sequence $\{F_n(g)\}$ is equicontinuous on the compact set I. Therefore the sequence $\{F_n(g)\}$ converges uniformly to $F(g)$ on I. Hence the sequence $\{F_n\}$ converges pointwise to F on X. Since X is a compact metric space, the sequence $\{y_n\}$ has a convergent subsequence $\{y_{n_j}\}$ converging to y. By Theorem 3, y is a fixed point of F. From Theorem 2, the sequence $\{y_n\}$, where y_n is the unique fixed point of F_n for each $n \in \mathbb{N} \cup \{0\}$, converges to the fixed point y of F. The result follows since these fixed points are the unique solutions of the initial value problem. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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