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# A GENERALIZATION OF CARISTI'S THEOREM ON TWO MENGER SPACES

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**Abstract.** In this paper, we establish a fixed point theorem in a product of two probabilistic metric spaces by considering a system of some Caristi-type contractions. An application is given to support the main result.

**Keywords:** common fixed point; Menger space; distribution function; complete metric space.

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## 1. Introduction and Preliminaries

In 1942, thanks to a brilliant idea to see the distance between two points $x$ and $y$ as a random variable characterized by its distribution function $F_{x,y}(t) = P(d(x,y) < t)$, Menger introduced the notion of a probabilistic metric space [10]. But, the main study of the properties of Menger space is due to Schweizer and Sklar and their collaborators (see [15, 16, 17]). Sehgal and Bharucha-Reid [18] were the first to extend the fixed point theory to this space. Since then, many authors investigated the fixed point theory in this setting (see for example [1, 4, 5, 6, 8, 9, 12, 14, 18, 19]). Many fixed point theorems presented in probabilistic metric spaces were inspired by

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their corresponding results on metric spaces. One of the most attractive extensions is the random version of Caristi's fixed point theorem established by Zhang et al. [21] in 1991. This result was extended by Chaira et al. [3] to study the existence of a common fixed point of two mappings $S$ and $T$ satisfying a system of Caristi-type contractions. Recently, like what was done within the framework of standard metric spaces, fixed point theorems in probabilistic metric spaces are used to prove some random optimization principle and to resolve some differential and integral equations. For more details, see for example [2, 7, 13, 21] and the references therein.

In this work, we propose a generalization of the above results by considering two Menger spaces $(X,\mathscr{F},\tau)$ and $(Y,\mathscr{G},\tau)$ and two mappings $T : X \longrightarrow Y$ and $S : Y \longrightarrow X$ satisfying the system

$$\begin{cases} F_{p,STp}(t) \geq H(t - a[\psi(p) - \phi(Tp)]) \\ G_{q,TSq}(t) \geq H(t - b[\phi(q) - \psi(Sq)]) \end{cases}$$

where $\psi$ and $\phi$ are two lower semi-continuous functions. To demonstrate our main theorem, we first give conditions under which the product of two complete Menger spaces is a complete Menger space. This theorem is used, in Application section, to prove the existence of a solution of an integral equation system.

In order to fix the framework needed to state our main results, we recall the following notions.

**Definition 1.1.** A function $F : \mathbb{R} \longrightarrow \mathbb{R}_+$ is called a distribution function, if it satisfies the following conditions:

(1) $F$ is nondecreasing,
(2) $F$ is left continuous,
(3) $\inf F = 0$ and $\sup F = 1$.

We denote by $\mathscr{D}$ the set of all distribution functions.

**Example 1.2 (Heaviside function).**

$$H(t) = \begin{cases} 0 & if\ t \leq 0 \\ 1 & if\ t > 0. \end{cases}$$

**Example 1.3.** *Let $\mu$ be a finite measure on $(\mathbb{R},\mathscr{B}_R)$. We call the distribution function of $\mu$ the function $F_\mu$ defined on $\mathbb{R}$ by*

$$F_\mu(x)=\mu(]-\infty,x[)$$

**Definition 1.4.** [6] A triangular norm (briefly a t-norm) is a binary operation $\tau$ on the unit interval $[0,1]$ that is associative, commutative, nondecreasing in each of its variables and such that $\tau(x,1)=x$ for every $x\in[0,1]$.

**Example 1.5.** *The following are the two basic t-norms:*

- *Minimm:* $\tau_M(x,y)=\min\{x,y\}$.
- *Lukasiewicz t-norm:* $\tau_L(x,y)=\max\{x+y-1,0\}$.

**Definition 1.6.** [15] A probabilistic metric space (briefly a PM-space) is an ordered pair $(X,\mathscr{F})$, where $X$ is a non-empty set and $\mathscr{F}$ is a mapping from $X\times X$ in to $\mathscr{D}$ defined by $\mathscr{F}(p,q)=F_{p,q}$ where the distribution functions $F_{p,q}$ satisfy the following conditions:

(1) $\forall t>0$, $F_{p,q}(t)=1$ if and only if $p=q$;
(2) $F_{p,q}(0)=0$;
(3) $F_{p,q}=F_{q,p}$;
(4) If $F_{p,q}(t_1)=1$ and $F_{q,r}(t_2)=1$, then $F_{p,r}(t_1+t_2)=1$.

If furthermore $\tau$ is a t-norm verifying the following triangle inequality:

$$F_{p,r}(t_1+t_2)\geq\tau\left(F_{p,q}(t_1),F_{q,r}(t_2)\right),\ \forall p,q,r\in X,\ t_1,t_2\geq 0,$$

the triplet $(X,\mathscr{F},\tau)$ is called Menger space.

**Definition 1.7.** [12] The $(\varepsilon,\lambda)-$topology on $(X,\mathscr{F})$ is the topology introduced on $X$ by the family of the neighbourhoods $\left\{U_p(\varepsilon,\lambda)\right\}_{p\in X,\varepsilon>0,\lambda>0}$, where

$$U_p(\varepsilon,\lambda)=\left\{q\in X\,:\,F_{p,q}(\varepsilon)>1-\lambda\right\}.$$

It is assumed in the remainder of this paragraph that $\sup_{t<1}\tau(t,t)=1$.

The following result is due to Schweizer and al. [17].

**Theorem 1.8.** *If $(X,\mathscr{F},\tau)$ is a Menger space such that $\sup_{t<1}\tau(t,t)=1$, then $(X,\mathscr{F},\tau)$ is a Hausdorff space in the topology induced by the family $\left\{U_p(\varepsilon,\lambda)\right\}_{p\in X,\varepsilon>0,\lambda>0}$ of neighborhoods.*

**Definition 1.9.** [21] Let $(X,\mathscr{F},\tau)$ be a Menger space. A sequence $(p_n)_{n\in\mathbb{N}}$ in $X$ is said to be

(i) $\tau$-convergent to $p\in X$, if

$$\forall\varepsilon,\lambda>0\quad\exists N\in\mathbb{N}\quad\forall n\in\mathbb{N},\quad n\geq N\implies F_{p_n,p}(\varepsilon)>1-\lambda$$

(ii) $\tau$-Cauchy sequence, if for any

$$\forall\varepsilon,\lambda>0\quad\exists N\in\mathbb{N}\quad\forall n,m\in\mathbb{N},\quad n,m\geq N\implies F_{p_n,p_m}(\varepsilon)>1-\lambda$$

**Definition 1.10.** [21] A Menger space $(X,\mathscr{F},\tau)$ is said to be $\tau$-complete, if each $\tau$-Cauchy sequence in $X$ is $\tau$-convergent to some point in $X$.

The following result was established by Schweizer and Sklar in [16].

**Theorem 1.11.** *Let $(X,\mathscr{F},\tau)$ be a Menger space such that $\sup_{t<1}\tau(t,t)=1$. Then, a sequence $(p_n)_{n\in\mathbb{N}}$ in $X$ is $\tau$-convergent to $p\in X$ if and only if for each $t\in\mathbb{R}$,*

$$\lim_{n\longrightarrow\infty}F_{p_n,p}(t)=H(t).$$

The following theorem establishes a connection between metric spaces and Menger spaces.

**Theorem 1.12.** [18] *Let $(X,d)$ be a metric space. Let $\mathscr{F}:X\times X\longrightarrow\mathscr{D}$ be the mapping defined by:*

$$F_{p,q}(t)=H(t-d(p,q))\qquad(p,q\in X,\,t\in\mathbb{R}).$$

*Then $(X,\mathscr{F},\tau_M)$ is a Menger space. It is complete if the metric $d$ is complete.*

We say that a t-norm $\tau$ satisfies the condition $(\mathscr{P})$ if, for all $y\in[0,1]$,

$$\lim_{x\to 1^-}\tau(x,y)=y.$$

**Remark 1.13.** It is easy to see that if a t-norm $\tau$ satisfies the condition $(\mathscr{P})$, then

$$\sup_{t<1}\tau(t,t)=1.$$

**Theorem 1.14.** [21] *Let $(X, \mathscr{F}, \tau)$ be a Menger space with $\tau$ satisfying the condition $(\mathscr{P})$. Let $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ be two sequences in $X$ such that $(p_n)_{n\in\mathbb{N}}$ is $\tau$-convergent to $p \in X$ and $(q_n)_{n\in\mathbb{N}}$ is $\tau$-convergent to $q \in X$. Then*

*(1) for any given $t \in \mathbb{R}$, we have:*

$$\liminf_{n\longrightarrow\infty} F_{p_n,q_n}(t) \geq F_{p,q}(t).$$

*(2) If $t \in \mathbb{R}$ is any continuous point of $F_{p,q}$, then*

$$\lim_{n\longrightarrow\infty} F_{p_n,q_n}(t) = F_{p,q}(t).$$

**Theorem 1.15.** [21] *If $(X, \mathscr{F}, \tau)$ is a Menger space with $\tau$ satisfying the condition $(\mathscr{P})$, then it is metrizable. In addition, if $(X, \mathscr{F}, \tau)$ is sequentially complete, then it must be net-complete.*

## 2. Main Results

The proof of our main result is based on the following lemma:

**Lemma 2.1.** *Let $(X,\mathscr{F},\tau)$ and $(X,\mathscr{G},\tau)$ be two complete Menger spaces with $\tau$ satisfying the condition $(\mathscr{P})$. Then $(X\times Y,\mathscr{F}^\infty,\tau)$ is a complete Menger space, where*

$$\mathscr{F}^\infty((p_1,q_1),(p,q_2))(t) = F^\infty_{(p_1,q_1),(p,q_2)}(t) := \min\left\{F_{p_1,p}(t); G_{q_1,q_2}(t)\right\},$$

*for all $(p_1,q_1)$, $(p,q_2) \in X\times Y$ and $t>0$.*

Proof. It is easy to see that $(X\times Y,\mathscr{F}^\infty)$ is a probabilistic metric space. Let's prove that it is a complete Menger space.

Let $(p_1,q_1)$, $(p,q_2)$, $(p_3,q_3) \in X\times Y$ and $t_1,t_2>0$.

$$\begin{aligned} F^\infty_{(p_1,q_1),(p_3,q_3)}(t_1+t_2) &= \min\left\{F_{p_1,p_3}(t_1+t_2); G_{q_1,q_3}(t_1+t_2)\right\} \\ &\geq \min\left\{\tau\left(F_{p_1,p}(t_1), F_{p,p_3}(t_2)\right); \tau\left(G_{q_1,q_2}(t_1), G_{q_2,q_3}(t_2)\right)\right\}. \end{aligned}$$

Since $\tau$ is nondecreasing in each of its variables and

$$\begin{cases} F^\infty_{(p_1,q_1),(p,q_2)}(t_1) = \min\left\{F_{p_1,p}(t_1); G_{q_1,q_2}(t_1)\right\} \\ F^\infty_{(p,q_2),(p_3,q_3)}(t_2) = \min\left\{F_{p,p_3}(t_2); G_{q_2,q_3}(t_2)\right\}, \end{cases}$$

then $F^{\infty}_{(p_1,q_1),(p_3,q_3)}(t_1+t_2) \geq \tau\left(F^{\infty}_{(p_1,q_1),(p,q_2)}(t_1), F^{\infty}_{(p,q_2),(p_3,q_3)}(t_2)\right)$.

Let $((p_n,q_n))_n$ be a Cauchy sequence in $X \times Y$. For each $\varepsilon > 0$ and $\lambda > 0$, there exists $N \in \mathbb{N}$ such that, for all $n,m \geq N$, we have $F^{\infty}_{(p_n,q_n),(p_m,q_m)}(\varepsilon) \geq 1-\lambda$. Then $F_{p_n,p_m}(\varepsilon) \geq 1-\lambda$ and $G_{q_n,q_m}(\varepsilon) \geq 1-\lambda$. Which implies that $(p_n)_n$ and $(q_n)_n$ are two Cauchy sequences in $X$ and $Y$ respectively. Since $X$ and $Y$ are complete, there exist $p \in X$ and $q \in Y$ such that $p_n \underset{n\to\infty}{\longrightarrow} p$ for $\mathscr{F}$ and $q_n \underset{n\to\infty}{\longrightarrow} q$ for $\mathscr{G}$. Thus, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $F_{p_n,p}(\varepsilon) \geq 1-\lambda$ and $G_{q_n,q}(\varepsilon) \geq 1-\lambda$. Hence, $F^{\infty}_{(p_n,q_n),(p,q)}(\varepsilon) \geq 1-\lambda$. This proves that $X \times Y$ is complete. □

**Theorem 2.2.** *Let $(X,\mathscr{F},\tau)$ and $(Y,\mathscr{G},\tau)$ be two complete Menger spaces with $\tau$ satisfying the condition $(\mathscr{P})$. Let $T : X \longrightarrow Y$ and $S : Y \longrightarrow X$ be two mappings such that, for all $(p,q) \in X \times Y$ and $t > 0$,*

$$\begin{cases} F_{p,STp}(t) \geq H\left(t - a[\psi(p) - \phi(Tp)]\right) \\ G_{q,TSq}(t) \geq H\left(t - b[\phi(q) - \psi(Sq)]\right) \end{cases} \tag{2.1}$$

*where $\psi : X \longrightarrow \mathbb{R}_+$ , $\phi : Y \longrightarrow \mathbb{R}_+$ are two lower semi-continuous functions.*

*Then the mapping $\mathscr{L} : X \times Y \longrightarrow X \times Y$, defined by $\mathscr{L}(p,q) = (Sq,Tp)$, admits a fixed point $(p^*,q^*)$ in $X \times Y$.*

Proof. We distinguish three cases:

**Case.1** If there exists $p^* \in X$ such that $\psi(p^*) < \phi(Tp^*)$, then by the system (2.1), for all $t > 0$, we have $F_{p^*,STp^*}(t) = 1$. Thus, $p^* = STp^*$. Setting $q^* = Tp^*$, we have $Sq^* = p^*$ and then $(p^*,q^*)$ is a fixed point of $\mathscr{L}$.

**Case.2** If there exists $q^* \in Y$ such that $\phi(q^*) < \psi(Sq^*)$, then the same argument used in Case.1 gives the desired result.

**Case.3** Suppose that, for all $(p,q) \in X \times Y$, we have $\phi(Tp) \leq \psi(p)$ and $\psi(Sq) \leq \phi(q)$.

We put $\nu = min(a,b)$

**Claim.** The binary relation " $\preceq$ " on $X \times Y$ defined by:

$$(p,q) \preceq (p',q') \iff \begin{cases} F_{p,p'}(t) \geq H\left(t - \nu\left(\psi(p) - \psi\left(p'\right)\right)\right) \\ G_{q,q'}(t) \geq H\left(t - \nu\left(\phi(q) - \phi\left(q'\right)\right)\right) \end{cases} \tag{2.2}$$

is a partial order. The reflexivity and the antisymmetry of "$\preceq$ "are obvious. For the transitivity, observe that

$$\text{if } (p,q) \preceq \left(p',q'\right), \text{ then } \phi\left(q'\right) \leq \phi(q) \text{ and } \psi\left(p'\right) \leq \psi(p). \tag{2.3}$$

Indeed,

- if $\phi(q') > \phi(q)$, by the definition of " $\preceq$ ", we have $G_{q,q'}(t) = 1$, for all $t > 0$. This implies that $q' = q$, which is a contradiction;
- if $\psi(p') > \psi(p)$, then $F_{p,p'}(t) = 1$, for all $t > 0$. This implies that $p' = p$, which is a contradiction.

Let $(p_1,q_1),(p_2,q_2),(p_3,q_3) \in X \times Y$ such that $(p_1,q_1) \preceq (p_2,q_2)$ and $(p,q_2) \preceq (p_3,q_3)$. Applying (2.3), we get

$$\phi(q_3) \leq \phi(q_2) \leq \phi(q_1) \text{ and } \psi(p_3) \leq \psi(p_2) \leq \psi(p_1).$$

Let $t > 0$.

- If $t \leq \nu(\phi(q_1) - \phi(q_3))$, then $G_{q_1,q_3}(t) \geq 0 = H(t - \nu(\phi(q_1) - \phi(q_3)))$.
- If $t > \nu(\phi(q_1) - \phi(q_3))$, taking $t_1, t_2 > 0$ such that $t = t_1 + t_2$ and $t_1 > \nu(\phi(q_1) - \phi(q_2))$ and $t_2 > \nu(\phi(q_2) - \phi(q_3))$, we have

$$\begin{aligned} G_{q_1,q_3}(t) &\geq \tau\left(G_{q_1,q_2}(t_1), G_{q_2,q_3}(t_2)\right) \\ &\geq \tau(H(t_1 - \nu(\phi(q_1) - \phi(q_2))), H(t_2 - \nu(\phi(q_2) - \phi(q_3)))) \\ &= \tau(1,1) = 1 = H(t - \nu(\phi(q_1) - \phi(q_3))). \end{aligned}$$

By the same argument, we obtain

$$F_{p_1,p_3}(t) \geq H(t - \nu(\psi(p_1) - \psi(p_3))).$$

It follows from (2.2) that $(p_1,q_1) \preceq (p_3,q_3)$. Consequently, "$\preceq$ "is a partial order on $X \times Y$.

**Claim.** The partially ordered set $(X \times Y, \preceq)$ has a maximal element.

Let $\{(p_\alpha, q_\alpha)\}_{\alpha \in I}$ be a chain of $X \times Y$ indexed by a totally ordered set $I$. The nets $(\psi(p_\alpha))_{\alpha \in I}$ and $(\phi(q_\alpha))_{\alpha \in I}$ are decreasing and thus convergent respectively to some reals

$l_1$ and $l_2$. For all $\varepsilon > 0$ and $\lambda > 0$ such that $\varepsilon > \lambda$, there exists $\alpha_0 \in I$ such that for all $\alpha \in I$,

$$\alpha_0 \leq \alpha \Longrightarrow l_1 \leq \psi(p_\alpha) < l_1 + \frac{\lambda}{\nu}.$$

Let $\alpha, \beta \in I$ such that $\alpha_0 \leqslant \alpha \leqslant \beta$. Then $\psi(p_\alpha) - \psi\left(p_\beta\right) \leq \dfrac{\lambda}{\nu}$ and

$$\begin{aligned} F_{p_\alpha, p_\beta}(\varepsilon) &\geq H\left(\varepsilon - \nu\left(\psi(p_\alpha) - \psi\left(p_\beta\right)\right)\right) \\ &\geq H(\varepsilon - \lambda) = 1 \\ &> 1 - \lambda. \end{aligned}$$

By the same argument, there exists $\alpha_1 \in I$ such that for all $\alpha, \beta \in I$,

$$\alpha_1 \leqslant \alpha \leqslant \beta \Longrightarrow G_{q_\alpha, q_\beta}(\varepsilon) > 1 - \lambda.$$

Let $\alpha_2 = \alpha_0 \vee \alpha_1$. For all $\alpha, \beta \in I$ such that $\alpha_2 \leqslant \alpha \leqslant \beta$, we have $F_{p_\alpha, p_\beta}(\varepsilon) > 1 - \lambda$ and $G_{q_\alpha, q_\beta}(\varepsilon) > 1 - \lambda$. Then $F^\infty_{(p_\alpha, q_\alpha), \left(p_\beta, q_\beta\right)}(\varepsilon) > 1 - \lambda$. Hence $((p_\alpha, q_\alpha))_{\alpha \in I}$ is a Cauchy net in $X \times Y$. From Lemma 1.15, the sequence $((p_\alpha, q_\alpha))_{\alpha \in I}$ is convergent in $X \times Y$. Let $(\overline{p}, \overline{q}) \in X \times Y$ such that $(p_\alpha, q_\alpha) \longrightarrow (\overline{p}, \overline{q})$. Thus, $p_\alpha \longrightarrow \overline{p}$ and $q_\alpha \longrightarrow \overline{q}$. In view of the lower semicontinuity of $\phi$ and $\psi$, it follows that for all $\alpha \in I$,

$$\phi(\overline{q}) \leq \liminf_\beta \phi\left(q_\beta\right) = \lim_\beta \phi\left(q_\beta\right) = l_2 \leq \phi(q_\alpha).$$

By the same argument, we obtain $\psi(\overline{p}) \leq \psi(p_\alpha)$, for all $\alpha \in I$.

Now, we prove that $(\overline{p}, \overline{q})$ is an upper bound of $((p_\alpha, q_\alpha))_{\alpha \in I}$. In fact, let $\alpha \in I$ and $t > 0$ is a continuous point of $F_{p_\alpha, \overline{p}}$. By Theorem 1.14, we have

$$\begin{aligned} F_{p_\alpha, \overline{p}}(t) &= \lim_\beta F_{p_\alpha, p_\beta}(t) \\ &\geq \liminf_\alpha H\left(t - \nu\left(\psi(p_\alpha) - \psi\left(p_\beta\right)\right)\right) \\ &\geq H(t - \nu(\psi(p_\alpha) - \psi(\overline{p}))). \end{aligned}$$

Let $t$ be any positive number. Since the distribution function $F_{p_\alpha, \overline{p}}$ is nondecreasing, there exists a sequence of continuous points $t_1 < t_2 < ... < t_n < ..$ of $F_{p_\alpha, \overline{p}}$ such that $t_n \longrightarrow t$. Letting $n \longrightarrow +\infty$ in the inequality

$$F_{p_\alpha, \overline{p}}(t_n) \geq H(t_n - \nu(\psi(p_\alpha) - \psi(\overline{p})))$$

and using the left continuity de $F_{p_\alpha,\overline{p}}$ and $H$, we get

$$F_{p_\alpha,\overline{p}}(t) \geq H(t-\nu(\psi(p_\alpha)-\psi(\overline{p}))).$$

By the same argument, one can show that, for all $t>0$ and $\alpha \in I$,

$$G_{q_\alpha,\overline{q}}(t) \geq H(t-\nu(\phi(q_\alpha)-\phi(\overline{q}))).$$

Applying (2.2), we obtain $(p_\alpha,q_\alpha) \preceq (\overline{p},\overline{q})$, for all $\alpha \in I$, i.e., $(\overline{p},\overline{q})$ is an upper bound of $((p_\alpha,q_\alpha))_{\alpha\in I}$. By Zorn's lemma, $(X\times Y,\preceq)$ has a maximal element $(p_1,q_1)$ proving the claim.

By (2.1), we obtain:

$$\begin{cases} F_{p_1,STp_1}(t) \geq H(t-\nu(\psi(p_1)-\psi(STp_1))) \\ G_{q_1,TSq_1}(t) \geq H(t-\nu(\phi(q_1)-\phi(TSq_1))) \end{cases}$$

According to (2.2), we have $(p_1,q_1)\preceq (STp_1,TSq_1)$. Then $(p_1,q_1)= (STp_1,TSq_1)$, that is, $STp_1=p_1$ and $TSq_1=q_1$. If we put $p^*=p_1$ and $q^*=Tp_1$, the desired result holds.

□

**Example 2.3.** *Consider the sets* $X=[1;+\infty[$ *and* $Y=]0;1]$ *respectively endowed with the metrics* $d$ *and* $\delta$ *defined, for all* $(p_1,p_2)\in X^2$ *and* $(q_1,q_2)\in Y^2$*, by*

$$d(p_1,p_2)=2|p_1-p_2| \text{ and } \delta(q_1,q_2)=\left|\frac{1}{q_1}-\frac{1}{q_2}\right|$$

*and define the functions*

$$\begin{array}{rccc} \mathscr{F}: & X\times X & \longrightarrow & \mathscr{D} \\ & (p_1,p_2) & \longmapsto & F_{p_1,p_2} \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathscr{G}: & Y\times Y & \longrightarrow & \mathscr{D} \\ & (q_1,q_2) & \longmapsto & G_{q_1,q_2} \end{array}$$

*as follows:*

$$F_{p_1,p_2}(t)=H(t-2|p_1-p_2|) \text{ and } G_{q_1,q_2}(t)=H\left(t-\left|\frac{1}{q_1}-\frac{1}{q_2}\right|\right), \text{ for all } t\in\mathbb{R}.$$

*Since the metric spaces* $(X,d)$ *and* $(Y,\delta)$ *are complete,* $(X,\mathscr{F},\tau_M)$ *and* $(Y,\mathscr{G},\tau_M)$ *are complete Menger spaces.*

*Consider the two mappings $T : X \longrightarrow Y$ and $S : Y \longrightarrow X$ defined by*

$$Tp = \begin{cases} \frac{1}{p} & ; \quad p \in [1,2[ \\ 1 & ; \quad p \in [2,+\infty[ \end{cases} \quad and \quad Sq = \begin{cases} \frac{1}{q} & ; \quad q \in ]0,\frac{1}{2}[ \\ 1 & ; \quad q \in [\frac{1}{2},1] \end{cases}$$

*and the lower semi-continuous functions $\psi : X \longrightarrow \mathbb{R}_+$, $\phi : Y \longrightarrow \mathbb{R}_+$ defined by:*

$$\psi(p) = \begin{cases} 2p & ; \quad p \in [1,2[ \\ p+1 & ; \quad p \in [2,+\infty[ \end{cases} \quad and \quad \phi(q) = \begin{cases} \frac{2}{q} & ; \quad q \in ]0,\frac{1}{2}[ \\ \frac{1}{q}+1 & ; \quad q \in [\frac{1}{2},1] \end{cases}$$

*Let us show that, for all $(p,q) \in X \times Y$ and $t > 0$,*

$$\begin{cases} F_{p,STp}(t) \geq H(t - 2(\psi(p) - \phi(Tp))), \\ G_{q,TSq}(t) \geq H(t - (\phi(q) - \psi(Sq))). \end{cases}$$

*Since the function $H$ is nondecreasing, it suffices to show that, for all $(p,q) \in X \times Y$,*

$$(S) : \begin{cases} |p - STp| \leq \psi(p) - \phi(Tp), \\ \left|\frac{1}{q} - \frac{1}{TSq}\right| \leq \phi(q) - \psi(Sq). \end{cases}$$

*Note that for all $(p,q) \in X \times Y$, $STp = 1$ and $TSq = 1$.*

- *If $p \in [1,2[$, we have $|p - STp| = p - 1$ and $\psi(p) - \phi(Tp) = 2p - p - 1 = p - 1$.*
- *If $p \in [2,+\infty[$,we have $|p - STp| = p - 1$ and $\psi(p) - \phi(Tp) = p + 1 - 2 = p - 1$.*
- *If $q \in \left]0, \frac{1}{2}\right[$, we have $\left|\frac{1}{q} - \frac{1}{TSq}\right| = \frac{1}{q} - 1$ and $\phi(q) - \psi(Sq) = \frac{2}{q} - \frac{1}{q} - 1 = \frac{1}{q} - 1$.*
- *If $q = \frac{1}{2}$, we have $\left|\frac{1}{q} - \frac{1}{TSq}\right| = 1 = \phi\left(\frac{1}{2}\right) - \psi\left(S\frac{1}{2}\right)$.*
- *if $q \in \left]\frac{1}{2}, 1\right]$, we have $\left|\frac{1}{q} - \frac{1}{TSq}\right| = \frac{1}{q} - 1$ and $\phi(q) - \psi(Sq) = \frac{1}{q} + 1 - 2 = \frac{1}{q} - 1$.*

*Then, for all $(p,q) \in X \times Y$, the system $(S)$ is satisfied. Hence, we have, for all $(p,q) \in X \times Y$ and $t > 0$,*

$$\begin{cases} F_{p,STp}(t) \geq H(t - 2(\psi(p) - \phi(Tp))), \\ G_{q,TSq}(t) \geq H(t - (\phi(q) - \psi(Sq))). \end{cases}$$

*All conditions of Theorem 2.2 are satisfied and $p^* = q^* = 1$.*

If we take $X = Y$, $\mathscr{F} = \mathscr{G}$, $\psi = \phi$, $S = id_X$ and $a = b = 1$, we obtain [21, Theorem 3] as a corollary:

**Corollary 2.4.** *Let $(X,\mathscr{F},\tau)$ be a complete Menger space with $\tau$ satisfying the condition $(\mathscr{P})$. Let $T : X \longrightarrow X$ be a mapping such that, for all $p \in X$ and $t > 0$,*

$$F_{p,Tp}(t) \geq H(t-\phi(p)+\phi(Tp)),$$

*where $\phi : X \longrightarrow \mathbb{R}_+$ is a lower semi-continuous function. Then the mapping $T$ admits a fixed point $p^*$ in $X$.*

## 3. Application

Let $(\Omega,\mathscr{A},P)$ be a probability measure space. Denote by $\mathscr{E}$ the set of all the equivalence classes of measurable mappings $p : \Omega \longrightarrow \mathbb{R}_+$. For every $p,q \in \mathscr{E}$ the function $F_{p,q}$, defined for every $t \in \mathbb{R}$ by $F_{p,q}(t) = P(\{\omega : |p(\omega)-q(\omega)| < t\})$, is a distribution function. Consider the family defined by $\mathscr{F}(p,q) = F_{p,q}$, for every $(p,q) \in \mathscr{E}^2$ and the Lukasiewicz t-norm defined by $\tau_L(x,y) = max\{x+y-1,0\}$ for every $x,y \in [0,1]$. From [6, Proposition 2.84], the triplet $(\mathscr{E},\mathscr{F},\tau_L)$ is a complete Menger space. For $k > 0$ and $\alpha > 0$, we denote by $\mathscr{RL}_k(\mathscr{E})$ the set of functions $f : \mathscr{E} \longrightarrow \mathscr{E}$ such that, for all $p,q \in \mathscr{E}$, $t \in \mathbb{R}$ and $\omega \in \Omega$,

$$F_{fp,fq}(t) \geq F_{p,q}\left(\frac{t}{k}\right) \text{ and } fp(\omega) \leq \alpha$$

Let $f$ and $g$ be two elements of $\mathscr{RL}_k(\mathscr{E})$. The following application

$$\begin{array}{rccc} d_\infty(f,g): & (\Omega,\mathscr{A}) & \longrightarrow & \left([0,1],\mathscr{B}_{[0,1]}\right) \\ & \omega & \longmapsto & \sup\limits_{p\in\mathscr{E}} d(fp(\omega),gp(\omega)) \end{array}$$

is a random variable, where $d(fp(\omega),gp(\omega)) = |fp(\omega)-gp(\omega)|$.
Let $\mathscr{F}^\infty$ be the family of distribution functions defined by:

$$F^\infty_{f,g}(t) = P(\{\omega : d_\infty(f,g)(\omega) < t\}), \text{ for all } f,g \in \mathscr{RL}_k(\mathscr{E}) \text{ and } t > 0.$$

**Lemma 3.1.** *The ordered triple $(\mathscr{RL}_k(\mathscr{E}),\mathscr{F}^\infty,\tau_L)$ is a complete Menger space.*

Proof.

(1) $(\mathscr{RL}_k(\mathscr{E}),\mathscr{F}^\infty,\tau_L)$ is a Menger space. Indeed, let $f,g,h \in \mathscr{RL}_k(\mathscr{E})$.

- $(\forall t > 0 : F^\infty_{f,g}(t) = 1) \iff \forall p \in \mathscr{E}\colon fp = gp \iff f = g.$
- $F^\infty_{f,g}(0) = P(\{\omega \in \Omega : d_\infty(f,g)(\omega) < 0\}) = P(\emptyset) = 0.$
- $F^\infty_{f,g} = F^\infty_{g,f}.$

- Let $t_1, t_2 > 0$ such that $F^{\infty}_{f,g}(t_1) = 1$ and $F^{\infty}_{g,h}(t_2) = 1$. Consider the two sets

  $A = \{\omega \in \Omega : d_{\infty}(f,g)(\omega) < t_1\}$ and $B = \{\omega \in \Omega : d_{\infty}(g,h)(\omega) < t_2\}$.

  Since for all $\omega \in \Omega$

  $$d_{\infty}(f,h)(\omega) \leq d_{\infty}(f,g)(\omega) + d_{\infty}(g,h)(\omega),$$

  it follows that

  $$A \cap B \subset \{\omega \in \Omega : d_{\infty}(f,h)(\omega) < t_1 + t_2\}.$$

  Therefore,

  $$P(A \cap B) \leq P(\{\omega \in \Omega : d_{\infty}(f,h)(\omega) < t_1 + t_2\}).$$

  Since

  $$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 = 1,$$

  then $P(A) = F^{\infty}_{f,g}(t_1) = 1$ and $P(B) = F^{\infty}_{g,h}(t_2) = 1$. Hence

  $$F^{\infty}_{f,h}(t_1 + t_2) = P(\{\omega \in \Omega : d_{\infty}(f,h)(\omega) < t_1 + t_2\}) = 1.$$

  From the above calculation we have

  $$F^{\infty}_{f,h}(t_1 + t_2) \geq F^{\infty}_{f,g}(t_1) + F^{\infty}_{g,h}(t_2) - 1.$$

  Since $F^{\infty}_{f,h}(t_1 + t_2) \geq 0$, then

  $$F^{\infty}_{f,h}(t_1 + t_2) \geq \max\left(F^{\infty}_{f,g}(t_1) + F^{\infty}_{g,h}(t_2) - 1, 0\right).$$

  Hence the triangular inequality holds.

(2) $(\mathscr{RL}_k(\mathscr{E}), \mathscr{F}^{\infty}, \tau_L)$ is complete. Indeed, let $(f_n)_n$ be a Cauchy sequence in $(\mathscr{RL}_k(\mathscr{E}), \mathscr{F}^{\infty}, \tau_L)$. Let $\varepsilon, \lambda > 0$. Then there exists $N \in \mathbb{N}$ such that, for all $n, m \geq N$, we have

$$F^{\infty}_{f_n, f_m}(\varepsilon) \geq 1 - \lambda.$$

Let $p \in \mathscr{E}$. Since

$$\{\omega \in \Omega : d_{\infty}(f_n, f_m)(\omega) < \varepsilon\} \subset \{\omega \in \Omega : d_{\infty}(f_n p, f_m p)(\omega) < \varepsilon\},$$

then $F_{f_np,f_mp}(\varepsilon) \geq F^{\infty}_{f_n,f_m}(\varepsilon) \geq 1-\lambda$ and $(f_np)_n$ is a Cauchy sequence in $(\mathscr{E},\mathscr{F},\tau_L)$. As the space $(\mathscr{E},\mathscr{F},\tau_L)$ is a complete Menger space [20, Theorem 3.4], the sequence $(f_np)_n$ converges in $\mathscr{E}$. The mapping $f:\mathscr{E}\longrightarrow\mathscr{E}$, $p\longmapsto fp:=\lim\limits_{n\to\infty} f_np$ is well defined. To prove that $f\in\mathscr{R}\mathscr{L}_k(\mathscr{E})$, let us fix $p,q\in\mathscr{E}$ and $t>0$ and proceed in two steps:

(i) Suppose that $t$ is a continuous point of $F_{fp,fq}$. Since $(f_np)_n$ converges to $fp$ and $F_{f_np,f_nq}(t)\geq F_{p,q}\left(\frac{t}{k}\right)$ for all $n\in\mathbb{N}$, we get, according to Theorem 1.14, that $F_{fp,fq}(t)\geq F_{p,q}\left(\frac{t}{k}\right)$.

(ii) If $t$ is not a continuous point of $F_{fp,fq}$, there exists a nondecreasing sequence of continuous points $(t_n)_n$ in $F_{fp,fq}$ such that $t_n$ converges to $t$. As proved in the above step, $F_{fp,fq}(t_n)\geq F_{fp,fq}\left(\frac{t_n}{k}\right)$, for all $n\in\mathbb{N}$. Using the left continuity of $F_{fp,fq}$, we obtain that $F_{fp,fq}(t)\geq F_{p,q}\left(\frac{t}{k}\right)$.

Let us prove that $(f_n)_n$ converges to $f$ with respect to $\mathscr{F}^{\infty}$. Let $\varepsilon,\lambda>0$, there exists $N\in\mathbb{N}$, such that for all $m>n\geq N$, $F^{\infty}_{f_n,f_m}(\varepsilon)\geq 1-\lambda$. Using the same argument above depending on whether $\varepsilon$ is a continuous point of $F^{\infty}_{f_n,f}$ or not, we prove that

$$F^{\infty}_{f_n,f}(\varepsilon)\geq 1-\lambda, \text{ for all } n\in\mathbb{N}.$$

It remains to prove that $(fp)(\omega)\leq\alpha$, for all $p$ in $\mathscr{E}$ and all $\omega$ in $\Omega$. Let $p$ be an arbitrary element in $\mathscr{E}$. Since $(f_n)_n$ converges to $f$ with respect to $\mathscr{F}^{\infty}$, the sequence $(f_np)_n$ converges to $fp$ with respect to $\mathscr{F}$. This implies that $(f_np)_n$ converges in law to $fp$. Consider the two distribution functions:

$$F_n(t)=P(\{\omega\in\Omega:\ (f_np)(\omega)\leq t\}) \text{ and } F(t)=P(\{\omega\in\Omega:\ (fp)(\omega)\leq t\}).$$

Suppose $\alpha$ is any continuous point of $F$. Since $(f_np)_n$ converges in law to $fp$, then $\lim\limits_{n\to\infty}F_n(\alpha)=F(\alpha)$. Since $F_n(\alpha)=1$ for all $n\in\mathbb{N}$, then

$$P(\{\omega\in\Omega:\ fp(\omega)\leq\alpha\})=F(\alpha)=1,$$

which implies that $(fp)(\omega)\leq\alpha$ for all $p$ in $\mathscr{E}$ and all $\omega$ in $\Omega$. Let $\alpha$ be an arbitrary positive real. Since $F$ is nondecreasing, then there exists a nonincreasing sequence of continuous points $(t_m)_m$ of $F$ converging to $\alpha$ and satisfying

$$F(t_m)=\lim_{n\to\infty}F_n(t_m)\geq\lim_{n\to\infty}F_n(\alpha)=1, \text{ for every } m\in\mathbb{N}.$$

Using the right continuity of $F$ and letting $m \longrightarrow \infty$, we get $F(\alpha) = 1$. Therefore, $(fp)(\omega) \leq \alpha$ for all $p$ in $\mathscr{E}$ and all $\omega$ in $\Omega$, proving that $(\mathscr{RL}_k(\mathscr{E}), \mathscr{F}^\infty, \tau_L)$ is a complete Menger space.

□

Let $\mathscr{H}$ be the set of the pairs $(u,v)$ of continuous functions defined from $[a,b] \times [0,\alpha]$ to $\mathbb{R}_+$ satisfying the three following conditions:

(i) For all $s \in [a,b]$ and all $x, y \in [0,\alpha]$:

$$\begin{cases} |u(s,x) - u(s,y)| \leq |x-y|; \\ |v(s,x) - v(s,y)| \leq |x-y|. \end{cases}$$

(ii) For all $s \in [a,b]$: $\quad u(s,0) = 0$ and $v(s,0) = 0$.

(iii) for each $(s,x) \in [a,b] \times [0,\alpha]$,

$$\begin{cases} v(s,\mu_u(x)) \geq 2u(s,x) - x \\ u(s,\mu_v(x)) \geq 2v(s,x) - x, \end{cases}$$

where $\mu_u(x) = \dfrac{1}{b-a}\displaystyle\int_a^b u(s,x)\,ds$ and $\mu_v(x) = \dfrac{1}{b-a}\displaystyle\int_a^b v(s,x)\,ds$

**Remark 3.2.** If $(u,v) \in \mathscr{H}$, then $0 \leq u(s,x) \leq x$ and $0 \leq v(s,x) \leq x$ , for all $(s,x) \in [a,b] \times [0,\alpha]$.

Let $(u,v) \in \mathscr{H}$. Consider the two mappings $T, S : \mathscr{RL}_k(\mathscr{E}) \to \mathscr{RL}_k(\mathscr{E})$ defined by

$$(Tf)\,p(\omega) = \frac{1}{b-a}\int_a^b u(s, fp(\omega))\,ds \text{ and } (Sg)\,p(\omega) = \frac{1}{b-a}\int_a^b v(s, gp(\omega))\,ds,$$

for all $f, g \in \mathscr{RL}_k(\mathscr{E})$, $p \in \mathscr{E}$ and $\omega \in \Omega$.

Denote by $\mathfrak{S}$ the set of all permutations on $\mathbb{N}$.

**Theorem 3.3.** *Let $(u,v) \in \mathscr{H}$. If there exists $\beta > 0$ and $f_0 \in \mathscr{RL}_k(\mathscr{E})$ such that*

$$T^{\sigma_1(n)} \circ S^{\sigma_2(n)} \circ \ldots \circ T^{\sigma_{2l-1}(n)} \circ S^{\sigma_{2l}(n)} f_0 p(\omega) \geq \beta,$$

*for all $n \in \mathbb{N}$, $l \in \mathbb{N}$, $p \in \mathscr{E}$, $\omega \in \Omega$ and all $\sigma_1, ..., \sigma_{2l} \in \mathfrak{S}$.*

*then the integral equation system*

$$(IES)\ \begin{cases} gp(\omega) = \dfrac{1}{b-a}\displaystyle\int_a^b u(s, fp(\omega))\,ds \\ fp(\omega) = \dfrac{1}{b-a}\displaystyle\int_a^b v(s, gp(\omega))\,ds \end{cases}$$

*admits a non trivial solution* $(f^*, g^*)$ *in* $(\mathscr{RL}_k(\mathscr{E}))^2$.

Proof. 1) First we show that $T$ and $S$ are well defined. Let $f \in \mathscr{RL}_k(\mathscr{E})$, $(p,q) \in \mathscr{E}^2$ and $t > 0$ such that $d(fp(\omega), fq(\omega)) < t$. Using the condition (i), we have $d(u(s, fp(\omega)), u(s, fq(\omega))) < t$, for all $s \in [a,b]$. Applying the integral properties, we get

$$d\left(\frac{1}{b-a}\int_a^b u(s, fp(\omega))\,ds, \frac{1}{b-a}\int_a^b u(s, fq(\omega))\,ds\right) < t.$$

This implies that $d((Tf)p(\omega), (Tf)q(\omega)) < t$. Thus,

$$\{\omega : d(fp(\omega), fq(\omega)) < t\} \subset \{\omega : d((Tf)p(\omega), (Tf)q(\omega)) < t\}.$$

Hence

$$F_{(Tf)p,(Tf)q}(t) \geq F_{fp,fq}(t) \geq F_{p,q}\left(\frac{t}{k}\right).$$

On the other hand, one can easily show that $0 \leq (Tf)p(\omega) \leq fp(\omega)$, for all $f \in \mathscr{RL}_k(\mathscr{E})$ and $(p,\omega) \in \mathscr{E} \times \Omega$. It follows that $Tf \in \mathscr{RL}_k(\mathscr{E})$. Similary, we prove that $Sf \in \mathscr{RL}_k(\mathscr{E})$ for all $f \in \mathscr{RL}_k(\mathscr{E})$.

2) We show that $T$ is continuous. Let $(f_n)_n$ be a sequence in $\mathscr{RL}_k(\mathscr{E})$ converging in $\mathscr{RL}_k(\mathscr{E})$ to $f$ with respect to $\mathscr{F}^\infty$. Let $n \in \mathbb{N}$, $p \in \mathscr{E}$ and $\omega \in \Omega$.

$$\begin{aligned} |(Tf_n p)(\omega) - (Tfp)(\omega)| &\leq \frac{1}{b-a}\int_a^b |u(s, (f_n p)(\omega)) - u(s, (fp)(\omega))| \\ &\leq |(f_n p)(\omega) - (fp)(\omega)| \end{aligned}$$

Then $d_\infty(f_n, f)(\omega) \geq d_\infty(Tf_n, Tf)(\omega)$, and $F^\infty_{f_n,f}(t) \leq F^\infty_{Tf_n,Tf}(t)$ for all $t$ in $\mathbb{R}$. Hence for all $\varepsilon, \lambda > 0$, there exists $N$ in $\mathbb{N}$ such $F^\infty_{Tf_n,Tf}(\varepsilon) \geq 1 - \lambda$.

3) Let $[\mathscr{RL}_k(\mathscr{E})]_\beta$ be the set of the elements $f \in \mathscr{RL}_k$ such that for all $l, n \in \mathbb{N}, p \in \mathscr{E}, \omega \in \Omega$ and $\{\sigma_i\}_{1 \leq i \leq 2l} \in \mathfrak{S}$

$$T^{\sigma_1(n)} \circ S^{\sigma_2(n)} \circ \ldots \circ T^{\sigma_{2l-1}(n)} \circ S^{\sigma_{2l}(n)} fp(\omega) \geq \beta$$

The non empty set $[\mathscr{RL}_k(\mathscr{E})]_\beta$ is a closed. Indeed, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of $[\mathscr{RL}_k(\mathscr{E})]_\beta$ converging to $f$ with respect to $\mathscr{F}^\infty$. Let $l,m\in\mathbb{N}$, $\{\sigma_i\}_{1\leq i\leq 2l}\in\mathfrak{S}$, $p\in\mathscr{E}$ and $\omega\in\Omega$, Since $T$ and $S$ are continuous functions then

$$T^{\sigma_1(m)}\circ S^{\sigma_2(m)}\circ\ldots\circ T^{\sigma_{2l-1}(m)}\circ S^{\sigma_{2l}(m)}f_n$$

converges to $T^{\sigma_1(m)}\circ S^{\sigma_2(m)}\circ\ldots\circ T^{\sigma_{2l-1}(m)}\circ S^{\sigma_{2l}(m)}f$ with respect to $\mathscr{F}^\infty$. Since

$$T^{\sigma_1(m)}\circ S^{\sigma_2(m)}\circ\ldots\circ T^{\sigma_{2l-1}(m)}\circ S^{\sigma_{2l}(m)}f_n p(\omega)\geq\beta$$

for all $p\in\mathscr{E}$ and all $\omega\in\Omega$. Using the same argument based in the convergence in law, we can show that

$$T^{\sigma_1(m)}\circ S^{\sigma_2(m)}\circ\ldots\circ T^{\sigma_{2l-1}(m)}\circ S^{\sigma_{2l}(m)}f p(\omega)\geq\beta,$$

proving that $f\in[\mathscr{RL}_k(\mathscr{E})]_\beta$.

Now, we prove that $[\mathscr{RL}_k(\mathscr{E})]_\beta$ is sable by $T$ and $S$. Let $f\in[\mathscr{RL}_k(\mathscr{E})]_\beta$. Let $l,n\in\mathbb{N}$, $p\in\mathscr{E}$, $\omega\in\Omega$ and $\sigma_1,\ldots,\sigma_{2l}\in\mathfrak{S}$.

$$\begin{aligned}&T^{\sigma_1(n)}\circ S^{\sigma_2(n)}\circ\ldots\circ T^{\sigma_{2l-1}(n)}\circ S^{\sigma_{2l}(n)}(Tf)\,p(\omega)\\&\qquad=T^{\sigma_1(n)}\circ S^{\sigma_2(n)}\circ\ldots\circ T^{\sigma_{2l+1}(n)}\circ S^{\sigma_{2l+2}(n)}f p(\omega)\end{aligned}$$

where $\sigma_{2l+1},\sigma_{2l+2}\in\mathfrak{S}$ such that $\sigma_{2l+1}(n)=1$ and $\sigma_{2l+2}(n)=0$. Since $f\in[\mathscr{RL}_k(\mathscr{E})]_\beta$, we get

$$T^{\sigma_1(n)}\circ S^{\sigma_2(n)}\circ\ldots\circ T^{\sigma_{2l-1}(n)}\circ S^{\sigma_{2l}(n)}(Tf)\,p(\omega)\geq\beta.$$

By the same we obtain

$$T^{\sigma_1(n)}\circ S^{\sigma_2(n)}\circ\ldots\circ T^{\sigma_{2l-1}(n)}\circ S^{\sigma_{2l}(n)}(Sf)\,p(\omega)\geq\beta,$$

where $\sigma_{2l+1},\sigma_{2l+2}\in\mathfrak{S}$ such that $\sigma_{2l+1}(n)=0$ and $\sigma_{2l+2}(n)=1$.

4) We conclude by checking that all conditions of Theorem 2.2 are satisfied. Let $f\in[\mathscr{RL}_k(\mathscr{E})]_\beta$, $t>0$, $s\in[a,b]$, $p\in\mathscr{E}$ and $\omega\in\Omega$, we have

$$v(s,(Tf)p(\omega))\geq 2u(s,fp(\omega))-fp(\omega).$$

By integrating, we get

$$(STf)p(\omega))\geq 2(Tf)p(\omega)-fp(\omega).$$

Then

$$(Tf)p(\omega) - (STf)p(\omega) \leq fp(\omega) - (Tf)p(\omega).$$

Using this fact,

$$\begin{aligned}|fp(\omega) - (STf)p(\omega)| &= (fp(\omega) - (Tf)p(\omega)) + ((Tf)p(\omega) - (STf)p(\omega)) \\ &\leq 2\left(fp(\omega) - (Tf)p(\omega)\right).\end{aligned}$$

Then

$$\sup_{p\in\mathscr{E}} (fp)(\omega) - \sup_{p\in\mathscr{E}} (Tfp)(\omega) \geq \frac{1}{2} \sup_{p\in\mathscr{E}} |(STf)p(\omega) - fp(\omega)|\,..$$

If we consider the mapping $\phi : [\mathscr{RL}_k(\mathscr{E})]_\beta \to \mathbb{R}_+$ defined by

$$\phi(f) = \int_\Omega \sup_{p\in\mathscr{E}} (fp)(\omega)\, dP,$$

we obtain

$$2(\phi(f) - \phi(Tf)) \geq \int_\Omega d_\infty(STf, f)(\omega)\ dP.$$

The map $\phi$ is continuous on $[\mathscr{RL}_k(\mathscr{E})]_\beta$. Indeed, let us fix a sequence $(f_n)_n \subset \mathscr{RL}_k(\mathscr{E})$ and $f \in [\mathscr{RL}_k(\mathscr{E})]_\beta$ such that $f_n \underset{n\to\infty}{\longrightarrow} f$ with respect to $\mathscr{F}^\infty$. Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$ we have $F_{f_n,f}(\varepsilon) \geq 1 - \varepsilon$. Then, for all $n \geq N$, we get $P(\{\omega :\, d(f_n(\omega), f(\omega)) \geq \varepsilon\}) \leq \varepsilon$ and

$$\begin{aligned}|\phi(f_n) - \phi(f)| &= \left|\int_\Omega d_\infty(f_n, 0)(\omega) - d_\infty(f, 0)(\omega)\, dP\right| \\ &\leq \int_\Omega d_\infty(f_n, f)(\omega)\, dP\end{aligned}$$

If we set $E_{n,\varepsilon} = \{\omega : d_\infty(f_n, f)(\omega) < \varepsilon\}$, then

$$\begin{aligned}|\phi(f_n) - \phi(f)| &\leq \int_{E_{n,\varepsilon}} d_\infty(f_n, f)(\omega)\, dP + \int_{\Omega - E_{n,\varepsilon}} d_\infty(f_n, f)(\omega)\, dP \\ &\leq \varepsilon + P(\{\omega :\, d_\infty(f_n, f)(\omega) \geq \varepsilon\}) \\ &\leq 2\varepsilon.\end{aligned}$$

If $t > 2(\phi(f) - \phi(Tf))$, then $t > \int_\Omega d_\infty(f, STf)(\omega)\ dP$ and $t > d_\infty(f, STf)(\omega)$ p.p.

Thus

$$F^{\infty}_{f,STf}(t) = P\{\omega : d_{\infty}(f,STf)(\omega) < t\} = 1 \geq H(t - 2(\phi(f) - \phi(Tf))).$$

If $t \leq 2(\phi(f) - \phi(Tf))$, the above inequality is obvious.

By the same argument we show that, for all $t > 0$,

$$F^{\infty}_{g,TSg}(t) = P\{\omega : d_{\infty}(g,TSg)(\omega) < t\} \geq H(t - 2(\phi(g) - \phi(Sg))).$$

Thus all the assumptions of Theorem 2.2 are satisfied, therefore the system $(IES)$ admits at least a solution $(f^*, g^*) \in ([\mathscr{R}\mathscr{L}_k(\mathscr{E})]_{\beta})^2$.

□

## 4. Conclusion

This work is part of a fixed point research project for mappings verifying Caristi-type contractions in the context of Menger spaces. This is a continuation of the study published in [3]. Our main result improves, extends, and generalizes some classical results:

(i) It improves and generalizes [21, Theorem 3] as mentioned in Corollary 2.4.
(ii) It extends [3, Corollary 2.2] from standard metric spaces to a Menger spaces.

The objective of the application is to prove the existence, under certain conditions, of a solution of a system of random nonlinear integral equations. It's an extension and a generalization of many existing classical ones.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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