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FIXED POINT APPROXIMATION VIA A NEW FASTER ITERATION PROCESS IN BANACH SPACES WITH AN APPLICATION

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Abstract. In this paper, we propose a new iterative process for approximating fixed points of mappings. First, we prove that our iterative scheme is faster than the iterative processes of Thakur and Piri for contractive mapping in Banach spaces. To support the analytical results, we give some numerical examples using the software program MATLAB. Afterwards, we give some weak and strong convergence theorems for monotone generalized α-nonexpansive mapping in uniformly convex ordered Banach spaces. To justify the utility of our main results, we presented an application regarding the approximation of the solution to an integral equation, supported by an illustrative example.

Keywords: fixed point; uniformly convex Banach space; generalized α-nonexpansive mapping; Opial property; ordered Banach space; fixed point approximation.

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1. INTRODUCTION

Throughout this paper, let E be a Banach space and C a nonempty subset of E. A mapping $T : C \longrightarrow C$ is said to be a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. The

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mapping T is called quasi-nonexpansive if the set $F(T)$ of fixed points of T is nonempty and $\|Tx-p\| \leq \|x-p\|$, for all $x \in C$ and $p \in F(T)$.

A number of extensions and generalizations of nonexpansive mappings have been considered by many authors. Indeed, in 2008, Suzuki [1] introduced the concept of generalized nonexpansive mappings, called also Suzuki generalized nonexpansive mappings, as a class of mapping satisfying a condition called condition (C). That is,

$$\frac{1}{2}\|x-Tx\| \leq \|x-y\|,$$

for all x, $y \in C$; and he obtained some existence and convergence results.

Recently, Ayoma and Kohska [2] introduced a new generalization of nonexpansive mappings known as α-nonexpansive mappings. A mapping $T : C \longrightarrow C$ is said to be α-nonexpansive mapping, if $0 < \alpha < 1$ and

$$\|Tx-Ty\|^2 \leq \alpha\|Tx-y\|^2+\alpha\|Ty-x\|^2+(1-2\alpha)\|x-y\|^2, \text{for all } x, y \in C.$$

In 2017, Pant and Shukla [3] introduced a new class of mapping called generalized α-nonexpansive mappings. A mapping $T : C \longrightarrow C$ is said to be generalized α-nonexpansive mapping if there exists $\alpha \in [0,1)$ such that

$$\frac{1}{2}\|x-Tx\| \leq \|x-y\| \Longrightarrow \|Tx-Ty\| \leq \alpha\|Tx-y\|+\alpha\|Ty-x\|+(1-2\alpha)\|x-y\|,$$

for all x, $y \in C$; and he obtained some existence and convergence theorems. This new class of mappings contains nonexpansive, Suzuki generalized nonexpansive and α-nonexpansive mappings.

On another side, fixed point theory in partially ordered metric spaces has been introduced by Ran and Reuring [4], Neito and Rodriguez Lopez [5]. Recently, Shukla et al [6] extended the generalized α-nonexpansive mapping to monotone generalized α-nonexpansive mapping in partially ordered Banach spaces and got some existence, weak and strong convergence results.

Many iteration processes have been introduced and developed to approximate the value of fixed point, and it is impossible to cover them all. See for example, Mann (1953) [7], Ishikawa (1974) [8], Noor (2000) [9], Agarwal (2007) [10], Abbas and Nazir (2014) [11].

In 2016, Thakur et *al.* [12] used the following iteration process

$$\begin{cases} x_1 \in C \\ x_{n+1} = T(y_n) \\ y_n = T((1-\alpha_n)x_n + \alpha_n z_n) \\ z_n = (1-\beta_n)x_n + \beta_n T(x_n), \ \ n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $(\alpha_n)_n$ and $(\beta_n)_n$ are in $(0,1)$. With the help of some numerical examples, they proved that their iteration converges faster than Picard, Mann, Agarwal, Noor and Abbas iterations processes for some classes of mappings.

Recently, Ullah and Arshad [13] introduced a new iteration scheme, named AK iteration, defined as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = T(y_n) \\ y_n = T((1-\alpha_n)z_n + \alpha_n T(z_n)) \\ z_n = T((1-\beta_n)x_n + \beta_n T(x_n)), \ \ n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $(\alpha_n)_n$ and $(\beta_n)_n$ are in $(0,1)$. The authors proved that AK iteration (1.2) converges faster than Vatan two-steps iteration, which is faster than all others algorithms.

Very recently, Piri et *al.* [14] proposed a new iteration process, faster than Thakur iteration (1.1), as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1-\alpha_n)T(z_n) + \alpha_n T(y_n) \\ y_n = T(z_n) \\ z_n = T((1-\beta_n)x_n + \beta_n T(x_n)), \ \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $(\alpha_n)_n$ and $(\beta_n)_n$ are in $(0,1)$.

Motivated by these considerations, we introduce and develop a new iteration process which rate of convergence for contractive mappings is faster than the iterations processes (1.1), (1.2) and (1.3).

The new iteration process is defined as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = T\left(\alpha_n T\left(y_n\right) + \left(1-\alpha_n\right) T\left(z_n\right)\right) \\ y_n = T\left(z_n\right) \\ z_n = T\left(\beta_n T\left(v_n\right) + \left(1-\beta_n\right) T\left(x_n\right)\right) \\ v_n = \left(1-\delta_n\right) x_n + \delta_n T\left(x_n\right) \end{cases} \tag{1.4}$$

where the sequences $(\alpha_n)_n$, $(\beta_n)_n$ and $(\delta_n)_n$ are in $(0,1)$.

In this paper, we prove, with the help of some numerical examples, that the new iteration process converges faster than the processes (1.1), (1.2) and (1.3). Afterwards, we give some strong and weak convergence theorems for monotone generalized α-nonexpansive mapping in partially uniformly convex partially ordered Banach spaces. In order to demonstrate the practical significance of our primary findings, we introduced an application focused on approximating the solution of an integral equation, substantiated by an illustrative example.

2. Preliminaries

In this section, we present some basic properties of generalized α-nonexpansive mappings. A subset K of a real Banach space E is said to be a closed convex cone if the following holds:

- K is nonempty closed and $K \neq \{0\}$,
- $ax+by \in K$ for $x, y \in K$ and $a, b \geq 0$,
- if $x \in K$ and $-x \in K$, then $x = 0$.

A partial order $\preceq$ in E with respect to the closed convex cone K is defined as follows:

$$x \preceq y \ (x \prec y) \Longleftrightarrow y - x \in K \ \left(y - x \in \overset{\circ}{K}\right).$$

for all $x, y \in E$, where $\overset{\circ}{K}$ is an interior of K.

A Banach space E is said uniformly convex if for each $\varepsilon > 0$, there exists $\eta > 0$, such that for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \varepsilon$ we have $\|x+y\| \leq 2(1-\eta)$.

Throughout, we denote the weak convergence and the strong convergence of a sequence $(x_n)_n$

to $x \in E$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. A Banach space E satisfy the Opial property [15], if for each $(x_n)_n$ in E such that $x_n \rightharpoonup x$ and $x \neq y$, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Let C be a closed convex nonempty subset of a Banach space E and let $(x_n)_n$ be a bounded sequence in E. For $x \in E$, we set $r(x,(x_n)) = \limsup_{n\to\infty} \|x - x_n\|$. The asymptotic radius of $(x_n)_n$ relative to C is given by

$$r(C,(x_n)) = \inf\{r(x,(x_n)) \,:\, x \in C\}.$$

The asymptotic center of $(x_n)_n$ relative to C is the set

$$A(C,(x_n)) = \{x \in C \,:\, r(x,(x_n)) = r(C,(x_n))\}.$$

If E is a uniformly convex Banach space, then $A(C,(x_n))$ consists exactly of one point. Moreover, if C is weakly compact and convex then $A(C,(x_n))$ is nonempty and convex (for more details, see [16]).

Throughout, we will assume that the order intervals are closed and convex subsets of an ordered Banach space $(E,\preceq)$. We denote as follows:

$$[a,\to) = \{x \in E \,:\, a \preceq x\} \quad \text{and} \quad (\leftarrow,b] = \{x \in E \,:\, x \preceq b\}\ , \text{ for any } a, b \in E.$$

In the sequel, $dist(x,A)$ will denote the distance from a point x to a set A.

A sequence $(s_n)_n \subset (0,1)$ is called bounded away from 0 if there exists $0 < a < 1$ such that $s_n \geq a$, for all $n \in \mathbb{N}$. Similarly, $(s_n)_n$ is called bounded away from 1 if there exists $0 < b < 1$ such that $s_n \leq b$, for every $n \in \mathbb{N}$.

Definition 2.1. *Let $(E,\preceq)$ be a partially ordered Banach space and a mapping $T : C \longrightarrow C$. The mapping T is said to be monotone if, for all x, $y \in E$,*

$$x \preceq y \quad \textit{implies} \quad Tx \preceq Ty.$$

Definition 2.2. [6] *Let C be a nonempty subset of an ordered Banach space $(E,\preceq)$. A mapping $T : C \longrightarrow C$ is said to be monotone generalized α-nonexpansive, if T is monotone and there* *exists* $\alpha \in [0,1)$ *such that*

$$\frac{1}{2}\|x-T(x)\| \leq \|x-y\| \text{ implies that } \|T(x)-T(y)\| \leq \alpha\|Tx-y\|+\alpha\|Ty-x\|+(1-2\alpha)\|x-y\|,$$

for all x, $y \in C$ *with* $x \preceq y$.

Lemma 2.1. [6] *Let* C *be a nonempty subset of an ordered Banach space* $(E,\preceq)$ *and* $T : C \longrightarrow C$ *a monotone generalized* α*-nonexpansive mapping with a fixed point* $p \in C$ *with* $x \preceq p$*. Then,* T *is monotone quasinonexpansive.*

Definition 2.3. [17, 3] *Let* C *be a subset of a Banach space* E*. A mapping* $T : C \longrightarrow C$ *is said to satisfy the condition (I) if there exists a nondecreasing function* $\varphi : [0,\infty) \longrightarrow [0,\infty)$ *satisfying* $\varphi(0) = 0$ *and* $\varphi(r) > 0$ *for all* $r \in (0,\infty)$ *such that* $\|x-T(x)\| \geq \varphi(dist(x, F(T)))$*, for all* $x \in C$.

Lemma 2.2. [6] *Let* C *be a nonempty subset of an ordered Banach space* $(E,\preceq)$*. Let* $T : C \to C$ *be a monotone generalized* α*-nonexpansive mapping. Then, for* x, $y \in C$ *with* $x \preceq y$,

$$\|x-T(y)\| \leq \frac{(3+\alpha)}{(1-\alpha)}\|T(x)-x\|+\|x-y\|.$$

Lemma 2.3. [23] *Let* C *be a nonempty subset of an ordered Banach space* $(E,\preceq)$*. Let* $T : C \to C$ *be a monotone generalized* α*-nonexpansive mapping. Then, for* x, $y \in C$ *with* $x \preceq y$,

$$\|Tx-Ty\| \leq \frac{2}{(1-\alpha)}\|Tx-x\|+\|x-y\|.$$

Lemma 2.4. [18] *Let* E *be a uniformly convex Banach space and* $0 < a \leq t_n \leq b < 1$*, for all* $n \in \mathbb{N}$*. Let* $(x_n)_n$ *and* $(y_n)_n$ *be two sequences such that* $\limsup_{n\to\infty}\|x_n\| \leq r$*,* $\limsup_{n\to\infty}\|y_n\| \leq r$ *and* $\lim_{n\to\infty}\|t_n x_n+(1-t_n)y_n\| = r$ *hold for some* $r \geq 0$*. Then,*

$$\lim_{n\to\infty}\|x_n-y_n\| = 0.$$

Lemma 2.5. [6] *Let* C *be a nonempty subset of an ordered Banach space* $(E,\preceq)$ *and* $T : C \longrightarrow C$ *a generalized* α*-nonexpansive mapping. Then,* $F(T)$ *is closed.*

Next, we give the definition of the rate of convergence (see [19] and [20]).

Definition 2.4. *Let $(x_n)_n$ and $(u_n)_n$ be two fixed point iteration processes that both converging to the same fixed point p and*

$$\|x_n - p\| \leq a_n\,,\ \|u_n - p\| \leq b_n,$$

for all $n \in \mathbb{N}$. If $(a_n)_n$ and $(b_n)_n$ be two sequences of real numbers converging to a and b, respectively, and

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0,$$

then we say that $(x_n)_n$ converges faster than $(u_n)_n$.

Suppose that $(x_n)_n$ be an iterative process, then it can be given by $x_1 \in C$ and $x_{n+1} = f(T, x_n)$, where f is a given function. For example, the iterative process (1.4) can be defined as: $x_1 \in C$ and

$$\begin{aligned}
&x_{n+1} = f(T, x_n) \\
&= T\left(\alpha_n T\left(T\left(T\left(\beta_n T\left((1-\delta_n)x_n + \delta_n T x_n\right) + (1-\beta_n) T x_n\right)\right) + (1-\alpha_n) T\left(T\left(\beta_n T\left((1-\delta_n)x_n + \delta_n T x_n\right)\right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. + (1-\beta_n) T x_n\right)\right)\right)\right),\ \text{for all } n \geq 1.
\end{aligned}$$

Definition 2.5. [21] *Let $(t_n)_n$ be an arbitrary sequence in C. Then, an iterative process $x_{n+1} = f(T, x_n)$ converging to a fixed point p, is said to be T-stable or stable with respect to T, if for $\varepsilon_n = \|t_{n+1} - f(T, t_n)\|$, $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \varepsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = p$*

Lemma 2.6. [22] *Let $(\psi_n)_n$ and $(\varphi_n)_n$ be two nonnegative real sequences satisfying $\psi_{n+1} \leq (1-\theta_n)\psi_n + \varphi_n$, where $\theta_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \theta_n = \infty$ and $\frac{\varphi_n}{\theta_n} \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} \psi_n = 0$.*

3. Main Results

3.1. Rate of convergence. In this subsection, we prove, analytically, that our new iteration process (1.4) converges faster than the iterative sequences generated by (1.1), (1.2) and (1.3). To support our theoretical results, we provide some numerical examples using MATLAB. First, let us prove that the iteration (1.4) converges to a unique fixed point of a contraction.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Banach space E and $T : C \longrightarrow C$ a contraction mapping with a factor $k \in (0,1)$. Let $(x_n)_n$ be an iterative sequence generated by* (1.4)*. Then, $(x_n)_n$ converges strongly to a unique fixed point of T.*

Proof. By the Banach's fixed point theorem, the mapping T has a unique fixed point $p \in C$. From (1.4), we have

$$\begin{aligned}\|v_n - p\| &= \|(1-\delta_n)x_n + \delta_n T(x_n) - p\|\\ &\leq (1-\delta_n)\|x_n - p\| + \delta_n \|T(x_n) - p\|\\ &\leq (1-\delta_n)\|x_n - p\| + k\delta_n \|x_n - p\|\\ &\leq (1-\delta_n(1-k))\|x_n - p\|.\end{aligned}$$

So that

$$\begin{aligned}\|z_n - p\| &= \|T(\beta_n T(v_n) + (1-\beta_n)T(x_n)) - p\|\\ &\leq k\|\beta_n T(v_n) + (1-\beta_n)T(x_n) - p\|\\ &\leq k^2\beta_n\|v_n - p\| + k^2(1-\beta_n)\|x_n - p\|\\ &\leq k^2\beta_n(1-\delta_n(1-k))\|x_n - p\| + k^2(1-\beta_n)\|x_n - p\|\\ &\leq k^2(\beta_n(1-\delta_n(1-k)) + 1 - \beta_n)\|x_n - p\|\\ &\leq k^2(1-\beta_n\delta_n(1-k))\|x_n - p\|\end{aligned}$$

Also, we have

$$\|y_n - p\| = \|T(z_n) - p\| \leq k\|z_n - p\|.$$

Hence,

$$\begin{aligned}\|x_{n+1} - p\| &= \|T(\alpha_n T(y_n) + (1-\alpha_n)T(z_n)) - p\|\\ &\leq k\|\alpha_n T(y_n) + (1-\alpha_n)T(z_n) - p\|\\ &\leq k\alpha_n\|T(y_n) - p\| + k(1-\alpha_n)\|T(z_n) - p\|\\ &\leq k^2\alpha_n\|y_n - p\| + k^2(1-\alpha_n)\|z_n - p\|\end{aligned}$$

$$\leq k^3\alpha_n\|z_n - p\| + k^2(1-\alpha_n)\|z_n - p\|$$
$$\leq k^2(1-\alpha_n + k\alpha_n)\|z_n - p\|$$
$$\leq k^4(1-\alpha_n(1-k))(1-\beta_n\delta_n(1-k))\|x_n - p\|.$$

Since $(1-\beta_n\delta_n(1-k)) < 1$,

$$\begin{aligned}\|x_{n+1} - p\| &\leq k^4(1-\alpha_n(1-k))\|x_n - p\| \\ &\leq k^4\|x_n - p\|\end{aligned}$$

By the above process, we get

$$\|x_{n+1} - p\| \leq k^{4n}\|x_1 - p\| \tag{3.1}$$

Taking the limit of the two sides of (3.1), we get $\lim_{n\to\infty}\|x_n - p\| = 0$. That is, $(x_n)_n$ converges to the unique fixed point p. □

Let $(x_n)_n$ be an iterative process which converges to a fixed point p of a mappings T. Numerically, we compute the sequence $(x_n)_n$ as follows:

(i) Choose an initial point $x_1 \in C$,

(ii) compute $x_2 = f(T, x_1)$. As a result of different errors due to machines, we do not obtain the exact value of x_2 but a value y_2 close enough to x_2, that is $y_2 \approx x_2$.

(iii) During the computation of the next term $x_3 = f(T, x_2)$, we expect an other value $y_3 = f(T, y_3) \approx x_3$.

At the end of the process we get a numerical approximative sequence $(y_n)_n$ of the theoretical sequence $(x_n)_n$. Then, the fixed point iterative process will be considered numerically stable or stable if and only if y_n is close enough to x_n at each iteration and the numerical sequence $(y_n)_n$ still converging to the same fixed point p of the mapping T.

To show the stability of the iterative process defined by the scheme (1.4), we give the following theorem:

Theorem 3.2. *Let C be a nonempty closed convex subset of a Banach space E and $T : C \longrightarrow C$ be a contraction. Let $(x_n)_n$ be an iterative sequence generated by* (1.4) *such that there exists $a > 0$ such that $0 < a \leq \alpha_n < 1$, for all $n \geq 1$. Then the iteration process* (1.4) *is T-stable.*

Proof. Let $(t_n)_n$ be an arbitrary sequence in C. Let the iterative sequence $(x_n)_n$ generated by (1.4) is $x_{n+1} = f(T, x_n)$ converging to the unique fixed point p of T and $\varepsilon_n = \|t_{n+1} - f(T, t_n)\|$. Let us prove that $\lim_{n\to\infty} \varepsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = p$.
Assume that $\lim_{n\to\infty} \varepsilon_n = 0$. Then,

$$\begin{aligned} \|t_{n+1} - p\| &\le \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - p\| \\ &= \varepsilon_n + \|f(T, t_n) - p\|. \end{aligned} \tag{3.2}$$

where

$$\begin{cases} f(T, t_n) = T(\alpha_n T(w_n) + (1 - \alpha_n) T(u_n)) \\ w_n = T(z_n) \\ u_n = T(\beta_n T(a_n) + (1 - \beta_n) T(t_n)) \\ a_n = (1 - \delta_n) t_n + \delta_n T(t_n) \end{cases}.$$

Then, similarly to the proof of Theorem 3.1,

$$\|t_{n+1} - p\| \le \varepsilon_n + \|f(T, t_n) - p\| \le k^4 (1 - \alpha_n (1 - k)) \|t_n - p\| + \varepsilon_n.$$

Set $\psi_n = \|t_n - p\|$, $\theta_n = \alpha_n (1 - k) \in (0, 1)$ and $\varphi_n = \varepsilon_n$.
We have $\lim_{n\to\infty} \dfrac{\varphi_n}{\theta_n} = 0$ and $\sum_{n=1}^{\infty} \theta_n = \infty$. Then, by Lemma 2.6, $\lim_{n\to\infty} \psi_n = \lim_{n\to\infty} \|t_n - p\| = 0$, that is $(t_n)_n$ converges to p.
Conversely, let $\lim_{n\to\infty} t_n = p$. Then,

$$\begin{aligned} \varepsilon_n &= \|t_{n+1} - f(T, t_n)\| \\ &\le \|t_{n+1} - p\| + \|f(T, t_n) - p\| \\ &\le \|t_{n+1} - p\| + k^4 (1 - \alpha_n (1 - k)) \|t_n - p\|. \end{aligned}$$

This implies that $\lim_{n\to\infty} \varepsilon_n = 0$. Therefore, the iterative process (1.4) is T-stable. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a Banach space E and $T : C \longrightarrow C$ be a contraction with a factor $k \in (0, 1)$ and $p \in C$ a fixed point of T. Consider the iteration processes generated by* (1.1), (1.2), (1.3) *and* (1.4), *where $(\alpha_n)_n$ and $(\beta_n)_n$ are in $(0, 1)$ bounded away from* 0 *and* 1. *Then the iteration* (1.4) *converges faster than* (1.1), (1.2) *and* (1.3).

Proof. Given the iteration process (1.4). Since the sequences $(\alpha_n)_n$ and $(\beta_n)_n$are assumed bounded away, we suppose that $0 < a \leq \alpha_n \leq b < 1$ and $0 < c \leq \beta_n \leq d < 1$. From the proof of Theorem 3.1, we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq k^4 (1 - \alpha_n (1-k)) \|x_n - p\| \\ &\leq k^{4n} (1 - a(1-k))^n \|x_1 - p\|.\end{aligned}$$

Therefore, let $a_n = k^{4n} (1 - a(1-k))^n \|x_1 - p\|$. Given the iteration process (1.1)

$$\begin{aligned}\|z_n - p\| &= \|(1-\beta_n) x_n + \beta_n T(x_n) - p\| \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n \|T(x_n) - p\| \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n k \|x_n - p\| \\ &\leq (1 - \beta_n (1-k)) \|x_n - p\|\end{aligned}$$

so that,

$$\begin{aligned}\|y_n - p\| &= \|T((1-\alpha_n) x_n + \alpha_n z_n) - p\| \\ &\leq k(1-\alpha_n) \|x_n - p\| + k\alpha_n \|z_n - p\| \\ &\leq k(1-\alpha_n) \|x_n - p\| + k\alpha_n (1 - \beta_n (1-k)) \|x_n - p\| \\ &\leq k(1 - \alpha_n + \alpha_n (1 - \beta_n (1-k))) \|x_n - p\| \\ &\leq k(1 - \alpha_n \beta_n (1-k)) \|x_n - p\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{n+1} - p\| = \|T(y_n) - p\| &\leq k\|y_n - p\| \\ &\leq k^2 (1 - \alpha_n \beta_n (1-k)) \|x_n - p\|.\end{aligned}$$

Then,

$$\|x_{n+1} - p\| \leq k^{2n} (1 - ac(1-k))^n \|x_1 - p\|.$$

Therefore, let $b_n = k^{2n} (1 - ac(1-k))^n \|x_1 - p\|$. Thus, we conclude that

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{k^{4n} (1 - a(1-k))^n \|x_1 - p\|}{k^{2n} (1 - ac(1-k))^n \|x_1 - p\|} = 0.$$

In fact, for all $n \geq 1$,

$$\frac{a_n}{b_n} = \frac{k^{4n}(1-a(1-k))^n \|x_1 - p\|}{k^{2n}(1-ac(1-k))^n \|x_1 - p\|} = \left(k^2 \frac{(1-a(1-k))}{(1-ac(1-k))}\right)^n.$$

Then, $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, since $\lim_{n\to\infty} k^{2n} = 0$ (because $0 < k < 1$) and $\frac{(1-a(1-k))}{(1-ac(1-k))} < 1$. Therefore, the iteration (1.4) converges faster than the iteration (1.1). Next, for the iteration process defined by (1.2), we have

$$\begin{aligned}
\|z_n - p\| &\leq k\|(1-\beta_n)x_n + \beta_n T x_n - p\| \\
&\leq k(1-\beta_n)\|x_n - p\| + k\beta_n \|T x_n - p\| \\
&\leq k(1-\beta_n)\|x_n - p\| + k^2\beta_n \|x_n - p\| \\
&\leq k(1-\beta_n(1-k))\|x_n - p\|,
\end{aligned}$$

So that

$$\begin{aligned}
\|y_n - p\| &\leq k\|(1-\alpha_n)z_n + \alpha_n T z_n - p\| \\
&\leq k(1-\alpha_n)\|z_n - p\| + k\alpha_n \|T z_n - p\| \\
&\leq k(1-\alpha_n)\|z_n - p\| + k^2\alpha_n \|z_n - p\| \\
&\leq k(1-\alpha_n(1-k))\|z_n - p\| \\
&\leq k^2(1-\alpha_n(1-k))(1-\beta_n(1-k))\|x_n - p\|.
\end{aligned}$$

Also,

$$\begin{aligned}
\|x_{n+1} - p\| &\leq k\|y_n - p\| \\
&\leq k^3(1-\alpha_n(1-k))(1-\beta_n(1-k))\|x_n - p\| \\
&\leq k^3(1-\alpha_n(1-k))\|x_n - p\|,
\end{aligned}$$

since $(1-\beta_n(1-k)) < 1$. Hence,

$$\begin{aligned}
\|x_{n+1} - p\| &\leq k^3(1-a(1-k))\|x_n - p\| \\
&\leq k^{3n}(1-a(1-k))^n \|x_1 - p\|.
\end{aligned}$$

Then, let $c_n = k^{3n}(1-a(1-k))^n\|x_1-p\|$. Then,

$$\lim_{n\to+\infty}\frac{a_n}{c_n} = \lim_{n\to+\infty}\frac{k^{4n}(1-a(1-k))^n\|x_1-p\|}{k^{3n}(1-a(1-k))^n\|x_1-p\|} = \lim_{n\to+\infty}k^n = 0.$$

Thus, the iteration (1.4) converge faster than the iteration (1.2). Now, given the iteration process (1.3), we have

$$\begin{aligned}
\|z_n-p\| &\le k\|(1-\beta_n)x_n+\beta_nTx_n-p\| \\
&\le k(1-\beta_n)\|x_n-p\|+k\beta_n\|Tx_n-p\| \\
&\le k(1-\beta_n)\|x_n-p\|+k^2\beta_n\|x_n-p\| \\
&\le k(1-\beta_n(1-k))\|x_n-p\|,
\end{aligned}$$

so that

$$\|y_n-p\| \le k\|z_n-p\|.$$

Also, we have

$$\begin{aligned}
\|x_{n+1}-p\| &\le (1-\alpha_n)\|Tz_n-p\|+\alpha_n\|Ty_n-p\| \\
&\le k(1-\alpha_n)\|z_n-p\|+k\alpha_n\|y_n-p\| \\
&\le k(1-\alpha_n)\|z_n-p\|+k^2\alpha_n\|z_n-p\| \\
&\le k(1-\alpha_n(1-k))\|z_n-p\| \\
&\le k^2(1-\alpha_n(1-k))(1-\beta_n(1-k))\|x_n-p\| \\
&\le k^2(1-\alpha_n(1-k))\|x_n-p\|,
\end{aligned} \tag{3.3}$$

since $(1-\beta_n(1-k))<1$. Hence,

$$\|x_{n+1}-p\| \le k^2(1-a(1-k))\|x_n-p\| \le k^{2n}(1-a(1-k))^n\|x_1-p\|.$$

Let $d_n = k^{2n}(1-a(1-k))^n\|x_1-p\|$. Then,

$$\lim_{n\to+\infty}\frac{a_n}{d_n} = \lim_{n\to+\infty}\frac{k^{4n}(1-a(1-k))^n\|x_1-p\|}{k^{2n}(1-a(1-k))^n\|x_1-p\|} = \lim_{n\to+\infty}k^{2n} = 0$$

Thus, the iteration (1.4) converge faster than the iteration (1.3). □

Next, we present some numerical examples which show that the iterative process (1.4) converges faster than the three other iteration processes (1.1), (1.2) and (1.3) in the case of contractive mappings.

Example 3.1. *Let $C = [0,15]$ and $T : C \longrightarrow C$ a mapping defined by $T(x) = \frac{2}{3}x + \frac{3}{2}$, for any $x \in C$. Choose $\alpha_n = \frac{n}{2n+1}$, $\beta_n = \frac{4n^2+3n}{(3n+2)^2}$ and $\delta_n = \frac{2n}{\sqrt{7n+9}}$. The initial point $x_1 = 9$ and let $\|x_n - x^*\| < 10^{-4}$ be the stop criterion.. It is clear that T is a contraction with $k = \frac{2}{3}$ and $x^* = 4.5$ is a fixed point of T. Table 1 shows the rate of convergence of the iterations* (1.1), (1.2), (1.3) *and* (1.4) *to the fixed point x^* of the mapping T.*

Steps	*Thakur* (1.1)	*Piri* (1.3)	*AK iteration* (1.2)	*new iteration* (1.4)
1	*9*	*9*	*9*	*9*
2	*6,4765*	*6,2151*	*5,6434*	*5,2803*
3	*5,3699*	*5,1445*	*4,7864*	*4,6322*
4	*4,8836*	*4,7410*	*4,5714*	*4,5222*
5	*4,6694*	*4,5899*	*4,5178*	*4,5037*
6	*4,5749*	*4,5335*	*4,5044*	*4,5006*
7	*4,5331*	*4,5125*	*4,5011*	*4,5001*
8	*4,5146*	*4,5046*	*4,5003*	*4.5*
9	*4,5065*	*4,5017*	*4,5001*	*4,5*
10	*4,5029*	*4,5006*	*4.5*	*4,5*
11	*4,5013*	*4,5002*	*4,5*	*4,5*
12	*4,5006*	*4,5001*	*4,5*	*4,5*
13	*4,5003*	*4,5*	*4,5*	*4,5*
14	*4,5001*	*4.5*	*4,5*	*4,5*
15	*4.5*	*4,5*	*4,5*	*4,5*

TABLE 1. Comparison of rate converges

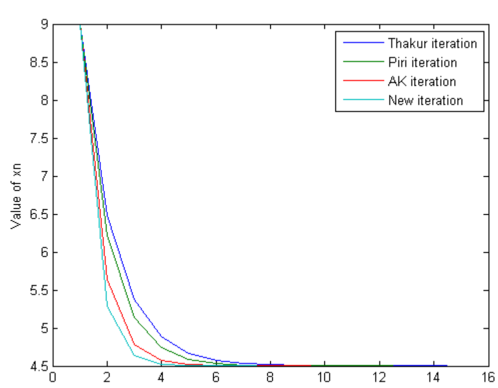

FIGURE 1. Comparison of converges iteration processes

Example 3.2. *Let $E=\mathbb{R}$ and $C=[1,+\infty)$. Let $T:C\longrightarrow C$ be an operator defined by $T(x)=\sqrt{x^2-8x+40}$, for all $x\in C$. Choose $\alpha_n=\frac{3n}{8n+4}$, $\beta_n=\frac{n}{4n+3}$, $\delta_n=1-\frac{n}{(6n+2)^2}$, the initial point $x_1=13$ and let $\|x_n-x^*\|<10^{-4}$ be the stop criterion.. It is clear that T is a contraction and $x^*=5$ is a fixed point of T. Table 2 shows the rate of convergence of the iterations* (1.1), (1.3), (1.2) *and* (1.4) *to fixed point of the mapping T in 10 iterations. In Table 3, we examine the influence of initial points for various iteration processes. Table 3 shows that the different*

Steps	*Thakur* (1.1)	*Piri* (1.3)	*AK iteration* (1.2)	*new iteration* (1.4)
1	*1*	*1*	*1*	*1*
2	*5,1719*	*5,0811*	*5,0165*	*5,0061*
3	*5,0072*	*5,0022*	*5,0001*	*5*
4	*5,0003*	*5,0001*	*5*	*5*
5	*5*	*5*	*5*	*5*

TABLE 2. Comparison of rate converges

Initial point	*Thakur* (1.1)	*Piri* (1.3)	*AK iteration* (1.2)	*ECM iteration* (1.4)
10	*5*	*4*	*3*	*3*
20	*6*	*6*	*4*	*3*
100	*17*	*14*	*10*	*8*
500	*65*	*53*	*38*	*30*
1000	*125*	*101*	*73*	*57*

TABLE 3. Influence of initial points for various iteration processes

values of the initial point x_1 have an effect on the rate of the convergence of scheme (1.4). *Numerically, we note that the sequence generated by* (1.4) *will converge more faster to a fixed point of T when the initial point x_1 become more bigger.*

Example 3.3. *Let $T:C=[0,\infty)\longrightarrow[0,\infty)$ be an operator defined by $T(x)=\frac{x+\sin(x)}{3}$, for all $x\in C$. It is clear that T is a contraction with a factor $k=\frac{2}{3}$ and $x^*=0$ is a fixed point of T. First, we show in Table 4 the convergence behaviour of the iteration* (1.4) *with a comparison to the iterations* (1.1), (1.2) *and* (1.3). *For this, we choose $\alpha_n=\frac{3n}{8n+4}$, $\beta_n=\frac{n}{4n+3}$, $\delta_n=1-\frac{n}{6n+2}$, the* *initial point* $x_1 = 10$ *and let* $\|x_n - x^*\| < 10^{-6}$ *be the stop criterion. Secondly, Table 5 shows the effect of different parameters* α_n*,* β_n *and* δ_n *on the rate of convergence for the iteration process* (1.4) *and other iterations with* $\|x_n - x^*\| < 10^{-6}$ *as a stop criterion.*

Table 5 shows that the different parameters α_n*,* β_n *and* δ_n *have a little effect on the number of iteration for the convergence of scheme* (1.4) *than the other processes.*

Steps	*Thakur* (1.1)	*Piri* (1.3)	*AK iteration* (1.2)	*ECM iteration* (1.4)
1	10	10	10	10
2	$1,047198$	$0,944833$	$0,633738$	$0,376306$
3	$0,404273$	$0,321011$	$0,150961$	$0,062179$
4	$0,172476$	$0,117441$	$0,037165$	$0,010339$
5	$0,074599$	$0,043069$	$0,009100$	$0,001708$
6	$0,032315$	$0,015734$	$0,002217$	$0,000281$
7	$0,013992$	$0,005728$	$0,000538$	$4,6.10^{-5}$
8	$0,006055$	$0,002079$	$0,000130$	$8,0.10^{-6}$
9	$0,002619$	$0,000753$	$3,1.10^{-5}$	$1,0.10^{-6}$
10	$0,001132$	$0,000272$	$7,0.10^{-6}$	0
11	$0,000489$	$9,8.10^{-5}$	$2,0.10^{-6}$	0
12	$0,000211$	$3,5.10^{-5}$	0	0
13	$9,1.10^{-5}$	$1,3.10^{-5}$	0	0
14	$3,9.10^{-5}$	$5,0.10^{-6}$	0	0
15	$1,7.10^{-5}$	$2,0.10^{-6}$	0	0
16	$7,0.10^{-6}$	$1,0.10^{-6}$	0	0
17	$3,0.10^{-6}$	0	0	0
18	$1,0.10^{-6}$	0	0	0
19	0	0	0	0

TABLE 4. Comparison of rate converges

Initial points	5	100	500	1000	3000	15000
Parameters 1: $\alpha_n = 0.7$, $\beta_n = 0.2$, $\delta_n = 0.64$						
Thakur (1.1)	*18*	*19*	*20*	*20*	*21*	*21*
Piri (1.3)	*14*	*15*	*16*	*16*	*16*	*17*
AK iteration (1.2)	*10*	*11*	*12*	*12*	*12*	*12*
ECM iteration (1.4)	*8*	*9*	*9*	*9*	*10*	*10*
Parameters 2: $\alpha_n = \frac{2n}{7n+10}$, $\beta_n = \frac{n}{5n-4}$, $\delta_n = \sqrt{\frac{2n}{6n+7}}$						
Thakur (1.1)	*18*	*19*	*20*	*20*	*21*	*22*
Piri (1.3)	*16*	*17*	*17*	*18*	*18*	*19*
AK iteration (1.2)	*11*	*12*	*13*	*13*	*13*	*14*
ECM iteration (1.4)	*9*	*10*	*10*	*10*	*11*	*11*
Parameters 3: $\alpha_n = \frac{n}{5n-4}$, $\beta_n = \frac{2n}{7n+10}$, $\delta_n = \sqrt{\frac{2n}{6n+7}}$						
Thakur (1.1)	*18*	*19*	*20*	*20*	*21*	*22*
Piri (1.3)	*16*	*17*	*17*	*18*	*18*	*19*
AK iteration (1.2)	*11*	*12*	*13*	*13*	*13*	*14*
ECM iteration (1.4)	*9*	*10*	*10*	*10*	*10*	*11*
Parameters 4: $\alpha_n = \frac{n}{2n+1}$, $\beta_n = \sqrt{\frac{n}{9n+1}}$, $\delta_n = \left(\frac{2n}{3n+5}\right)^{\frac{1}{4}}$						
Thakur (1.1)	*18*	*19*	*20*	*20*	*20*	*21*
Piri (1.3)	*14*	*15*	*16*	*16*	*17*	*17*
AK iteration (1.2)	*10*	*11*	*12*	*12*	*12*	*13*
ECM iteration (1.4)	*8*	*9*	*9*	*10*	*10*	*10*
Parameters 5: $\alpha_n = 1 - \frac{n}{\sqrt[4]{(8n+1)^5}}$, $\beta_n = 1 - \frac{3n}{(6n+2)^3}$, $\delta_n = 1 - \frac{n}{\sqrt{(7n+5)^3}}$						
Thakur (1.1)	*13*	*14*	*15*	*15*	*15*	*16*
Piri (1.3)	*10*	*11*	*11*	*11*	*11*	*12*
AK iteration (1.2)	*8*	*9*	*9*	*9*	*9*	*10*
ECM iteration (1.4)	*7*	*7*	*8*	*8*	*8*	*8*

TABLE 5. Influence of parameters and initial points

Remark 3.1. *From the results obtained for the parameters 2 and 3 in Table 5, we remark that the exchange of the role of α_n and β_n in the other iterations not in the iteration* (1.4)*, have no effect on the number of iterations.*

3.2. Convergence theorems in uniformly convex Banach spaces. In this subsection, we prove some weak and strong convergence results for a sequence generated by the iteration process (1.4) for monotone generalized α-nonexpansive mappings in the framework of uniformly convex partially ordered Banach spaces. In the sequel, we assume that the parameters sequences $(\alpha_n)_n$, $(\beta_n)_n$ and $(\delta_n)_n$ of the process (1.4) are in $(0,1)$ such that $(\beta_n)_n$ is bounded away from 0 and $(\delta_n)_n$ is bounded away from 0 and 1. That is, there exists a, b, $c \in (0,1)$ such that $c \leq \beta_n$ and $a \leq \delta_n \leq b$, for all n.

Lemma 3.1. *Let C be a nonempty closed convex subset of a partially ordered Banach space $(E, \preceq)$ and a monotone mapping $T : C \longrightarrow C$. Let $x_1 \in C$ such that $x_1 \preceq Tx_1$ (or $Tx_1 \preceq x_1$). Then, the sequence $(x_n)_n$ generated by* (1.4) *verify the following inequality*

$$x_n \preceq Tx_n \preceq x_{n+1}.$$

Proof. Let $x_1 \in C$ such that $x_1 \preceq Tx_1$.
By the convexity of the order interval $[x_1, Tx_1]$ and (1.4), we have

$$x_1 \preceq v_1 \preceq Tx_1. \tag{3.4}$$

As T is monotone, $Tx_1 \preceq Tv_1$. Again, by convexity of the order interval $[Tx_1, Tv_1]$, we get

$$x_1 \preceq v_1 \preceq Tx_1 \preceq (1-\beta_1)Tx_1 + \beta_1 Tv_1 \preceq Tv_1. \tag{3.5}$$

Using the monotonicity of T and (3.5), we have

$$x_1 \preceq v_1 \preceq Tx_1 \preceq (1-\beta_1)Tx_1 + \beta_1 Tv_1 \preceq Tv_1 \preceq T^2x_1 \preceq T\left((1-\beta_1)Tx_1 + \beta_1 Tv_1\right) = z_1 \preceq T^2v_1. \tag{3.6}$$

Again by the monotonicity of T, we have $T^2v_1 \leq Tz_1$. Hence, from (3.6)

$$x_1 \preceq v_1 \preceq Tx_1 \preceq Tv_1 \preceq z_1 \preceq T^2v_1 \preceq Tz_1 = y_1. \tag{3.7}$$

By the monotonicity of T and (3.7), we get $y_1 = Tz_1 \leq T^3v_1 \leq T(Tz_1) = Ty_1$. Thus, from (3.7) we have

$$x_1 \preceq v_1 \preceq Tx_1 \preceq Tv_1 \preceq z_1 \preceq T^2v_1 \preceq Tz_1 = y_1 \preceq Ty_1. \tag{3.8}$$

By the convexity of the order interval $[Tz_1, Ty_1]$, we get

$$y_1 = Tz_1 \preceq (1-\alpha_1)Tz_1 + \alpha_1 Ty_1 \preceq Ty_1. \tag{3.9}$$

As T is monotone, then

$$Ty_1 = T(Tz_1) \preceq T\left((1-\alpha_1)Tz_1 + \alpha_1 Ty_1\right) = x_2 \preceq T^2 y_1. \tag{3.10}$$

Hence, by (3.9) and (3.10) we have

$$y_1 = Tz_1 \preceq Ty_1 \preceq x_2. \tag{3.11}$$

Therefore, from (3.8) and (3.11)

$$x_1 \preceq Tx_1 \preceq x_2$$

Now, we suppose that it is true for n, that is

$$x_n \preceq Tx_n \preceq x_{n+1} \tag{3.12}$$

By the convexity of the order interval $[x_n, Tx_n]$ and (1.4), we have

$$x_n \preceq v_n \preceq Tx_n. \tag{3.13}$$

The monotonicity of T implies

$$x_n \preceq v_n \preceq Tx_n \preceq Tv_n. \tag{3.14}$$

By the convexity of the order interval $[Tx_n, Tv_n]$, we get

$$Tx_n \preceq (1-\beta_n)Tx_n + \beta_n Tv_n \preceq Tv_n. \tag{3.15}$$

From (3.14) and (3.15), we obtain

$$v_n \preceq Tx_n \preceq (1-\beta_n)Tx_n + \beta_n Tv_n \preceq Tv_n. \tag{3.16}$$

Since T is monotone, we have

$$Tv_n \preceq z_n = T\left((1-\beta_n)Tx_n + \beta_n Tv_n\right) \preceq T^2 v_n \tag{3.17}$$

and

$$T^2 v_n \preceq T^3 x_n \preceq Tz_n = y_n \preceq T^3 v_n. \tag{3.18}$$

Then, by (3.17) and (3.18)

$$Tv_n \preceq z_n \preceq T^2 v_n \preceq y_n = Tz_n.$$

Since T is monotone $y_n = Tz_n \preceq T^3 v_n \preceq Ty_n$. Thus,

$$y_n = Tz_n \preceq Ty_n. \tag{3.19}$$

By the convexity of the order interval $[Tz_n, Ty_n]$, we have

$$y_n = Tz_n \preceq (1-\alpha_n)Tz_n + \alpha_n Ty_n \preceq Ty_n. \tag{3.20}$$

Since T is monotone, then

$$Ty_n \preceq T\left((1-\alpha_n)Tz_n + \alpha_n Ty_n\right) \preceq T^2 y_n,$$

that is

$$Ty_n \preceq x_{n+1} \preceq T^2 y_n. \tag{3.21}$$

By the monotonicity of T and (3.21), we have $T^2 y_n \preceq Tx_{n+1}$. Thus, by (3.21)

$$x_{n+1} \preceq Tx_{n+1}. \tag{3.22}$$

By convexity of the order interval $[x_{n+1}, Tx_{n+1}]$, we have

$$x_{n+1} \preceq v_{n+1} \preceq Tx_{n+1}. \tag{3.23}$$

So, the monotony of T implies $Tx_{n+1} \preceq Tv_{n+1}$. Then, by convexity of the order interval $[Tx_{n+1}, Tv_{n+1}]$ we have

$$Tx_{n+1} \preceq (1-\beta_{n+1})Tx_{n+1} + \beta_{n+1}Tv_{n+1} \preceq Tv_{n+1}. \tag{3.24}$$

Since T is monotone, then from (3.23) and (3.24) we get

$$Tv_{n+1} \preceq T^2 x_{n+1} \preceq z_{n+1} = T\left((1-\beta_{n+1})Tx_{n+1} + \beta_{n+1}Tv_{n+1}\right) \preceq T^2 v_{n+1} \preceq Tz_{n+1}. \tag{3.25}$$

Hence, $z_{n+1} \preceq Tz_{n+1} = y_{n+1}$. Using the monotonicity of T, we obtain

$$y_{n+1} = Tz_{n+1} \preceq Ty_{n+1}. \tag{3.26}$$

By the convexity of the order interval $[Tz_{n+1}, Ty_{n+1}]$, we have

$$y_{n+1} = Tz_{n+1} \preceq (1-\alpha_{n+1})Tz_{n+1} + \alpha_{n+1}Ty_{n+1} \preceq Ty_{n+1}. \tag{3.27}$$

From (3.27) and the monotony of T, we get

$$y_{n+1} = Tz_{n+1} \preceq Ty_{n+1} \preceq x_{n+2} = T\left((1-\alpha_{n+1})Tz_{n+1} + \alpha_{n+1}Ty_{n+1}\right) \preceq T^2 y_{n+1}. \tag{3.28}$$

Finally, From (3.23), (3.24), (3.25) and (3.28) we conclude that

$$x_{n+1} \preceq Tx_{n+1} \preceq x_{n+2}.$$

□

Lemma 3.2. *Let C be a nonrmpty closed convex subset of a partially ordered Banach space $(E, \preceq)$ and $T : C \longrightarrow C$ a monotone generalized α-nonexpansive mapping. Assume that there exists $x_1 \in C$ such that $x_1 \preceq Tx_1$ (or $Tx_1 \preceq x_1$). Suppose that $F(T)$ is nonempty and $x_1 \preceq p$ for some $p \in F(T)$. Then, the sequence $(x_n)_n$ defined by* (1.4) *is bounded and $\lim\limits_{n\to+\infty} \|x_n - p\|$ exists.*

Proof. Let $p \in F(T)$ such that $x_1 \preceq p$. The monotonicity of T and convexity of order imply that: $x_n \preceq p$, $y_n \preceq p$, $z_n \preceq p$ and $v_n \preceq p$. In fact, we have $x_1 \preceq p$ and by the monotonicity of T we get $Tx_1 \preceq p$. Then,

$$\delta_1 Tx_1 + (1-\delta_1)x_1 = v_1 \preceq p.$$

Using the monotonicity of T, we have $Tv_1 \preceq p$. Hence, $\beta_1 Tv_1 + (1-\beta_1)Tx_1 \preceq p$. Thus,

$$T\left(\beta_1 Tv_1 + (1-\beta_1)Tx_1\right) = z_1 \preceq p.$$

Again by monotonicity of T,

$$Tz_1 = y_1 \preceq p.$$

So, $Ty_1 \preceq p$. Thus, $\alpha_1 Ty_1 + (1-\alpha_1)Tz_1 \preceq p$. Therefore,

$$x_2 = T\left(\alpha_1 Ty_1 + (1-\alpha_1)Tz_1\right) \preceq p.$$

Continuing in this way, we get

$$x_n \preceq p\ ,\ y_n \preceq p\ ,\ z_n \preceq p \ \text{ and } \ v_n \preceq p \ \text{ for all } \ n \geq 1.$$

By (1.4) and Lemma 2.1, we get

$$\begin{aligned}
\|v_n - p\| &= \|\,(1-\delta_n)\,x_n + \delta_n T\,(x_n) - p\| \\
&\leq (1-\delta_n)\,\|x_n - p\| + \delta_n \|T\,(x_n) - p\| \\
&\leq (1-\delta_n)\,\|x_n - p\| + \delta_n \|x_n - p\| \\
&\leq \|x_n - p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_n - p\| &= \|T\,((1-\beta_n)\,T(x_n) + \beta_n T\,(v_n)) - p\| \\
&\leq \|\,(1-\beta_n)\,T(x_n) + \beta_n T\,(v_n) - p\| \\
&\leq (1-\beta_n)\,\|T(x_n) - p\| + \beta_n \|T\,(v_n) - p\| \\
&\leq (1-\beta_n)\,\|x_n - p\| + \beta_n \|v_n - p\| \\
&\leq (1-\beta_n)\,\|x_n - p\| + \beta_n \|x_n - p\| \\
&\leq \|x_n - p\|.
\end{aligned}$$

Then,

$$\begin{aligned}
\|y_n - p\| &= \|T\,(z_n) - p\| \\
&\leq \|z_n - p\| \\
&\leq \|x_n - p\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|T\,((1-\alpha_n)\,T(z_n) + \alpha_n T\,(y_n)) - p\| \\
&\leq \|\,(1-\alpha_n)\,T(z_n) + \alpha_n T\,(y_n) - p\| \\
&\leq (1-\alpha_n)\,\|T(z_n) - p\| + \alpha_n \|T(y_n) - p\| \\
&\leq (1-\alpha_n)\,\|z_n - p\| + \alpha_n \|y_n - p\|
\end{aligned}$$

$$\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\|$$
$$\leq \|x_n - p\|.$$

Thus, the sequence $(\|x_n - p\|)_n$ is bounded and nonincreasing. Hence, $\lim_{n\to+\infty}\|x_n - p\|$ exists. □

Theorem 3.4. *Let C be a nonempty closed convex subset of a uniformly convex partially ordered Banach space $(E,\preceq)$ and $T : C \longrightarrow C$ a monotone generalized α-nonexpansive mapping. Assume that there exists $x_1 \in C$ such that $x_1 \preceq Tx_1$ (resp. $Tx_1 \preceq x_1$). Suppose that $F(T)$ is nonempty and $x_1 \preceq p$ (resp. $p \preceq x_1$) for some $p \in F(T)$. Let $(x_n)_n$ be a sequence defined by* (1.4) *and there exists $c \in (0,1)$ such that $0 < c \leq \beta_n < 1$. Then, $\lim_{n\to+\infty}\|x_n - T(x_n)\| = 0$.*

Proof. Suppose that $F(T)$ is nonempty and let $p \in F(T)$. By Lemma 3.2, the sequence $(x_n)_n$ is bounded and $\lim_{n\to\infty}\|x_n - p\|$ exists. Let

$$\lim_{n\to\infty}\|x_n - p\| = r. \tag{3.29}$$

We divide into two case

(i) if $r = 0$, we have

$$\|Tx_n - x_n\| \leq \|x_n - p\| + \|Tx_n - p\|.$$

By Lemma 2.1, we get $\|Tx_n - x_n\| \leq 2\|x_n - p\|$. Hence, $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.
(ii) If $r > 0$, then by Lemma 2.1 we have $\|T(x_n) - p\| \leq \|x_n - p\|$. Taking $\limsup$ as n goes to infinity in both sides of the above inequality, we have

$$\limsup_{n\to\infty}\|T(x_n) - p\| \leq r. \tag{3.30}$$

From the proof of Lemma 3.2,

$$\limsup_{n\to\infty}\|y_n - p\| \leq \limsup_{n\to\infty}\|x_n - p\| = r. \tag{3.31}$$

Otherwise,

$$\begin{aligned}\|x_{n+1}-p\| &\leq \|(1-\alpha_n)T(z_n)+\alpha_n T(y_n)-p\| \\ &= \|(1-\alpha_n)(y_n-p)+\alpha_n(T(y_n)-p)\| \\ &\leq (1-\alpha_n)\|y_n-p\|+\alpha_n\|y_n-p\| = \|y_n-p\|.\end{aligned}$$

Thus,

$$r = \liminf_{n\to\infty}\|x_{n+1}-p\| \leq \liminf_{n\to\infty}\|y_n-p\|. \tag{3.32}$$

Hence, by (3.31) and (3.32), we obtain $\lim_{n\to\infty}\|y_n-p\| = r$. Moreover,

$$\begin{aligned}\|y_n-p\| &\leq \|z_n-p\| \\ &\leq \beta_n\|v_n-p\|+(1-\beta_n)\|x_n-p\| \\ &\leq \beta_n\|(1-\delta_n)x_n+\delta_n T(x_n)-p\|+(1-\beta_n)\|x_n-p\| \\ &\leq \beta_n(1-\delta_n)\|x_n-p\|+\beta_n\delta_n\|T(x_n)-p\|+(1-\beta_n)\|x_n-p\| \\ &\leq \beta_n(1-\delta_n)\|x_n-p\|+\beta_n\delta_n\|x_n-p\|+(1-\beta_n)\|x_n-p\| \leq \|x_n-p\|.\end{aligned} \tag{3.33}$$

Hence, by (3.33) we have

$$\|y_n-p\| \leq \beta_n\|(1-\delta_n)x_n+\delta_n T(x_n)-p\|+(1-\beta_n)\|x_n-p\| \leq \|x_n-p\|.$$

Then,

$$\|y_n-p\|-(1-\beta_n)\|x_n-p\| \leq \beta_n\|(1-\delta_n)x_n+\delta_n T(x_n)-p\| \leq \|x_n-p\|-(1-\beta_n)\|x_n-p\|.$$

That is $(\|y_n-p\|-\|x_n-p\|)+\beta_n\|x_n-p\| \leq \beta_n\|(1-\delta_n)x_n+\delta_n T(x_n)-p\| \leq \beta_n\|x_n-p\|$. Since $\beta_n>0$, then

$$\frac{1}{\beta_n}(\|y_n-p\|-\|x_n-p\|)+\|x_n-p\| \leq \|(1-\delta_n)x_n+\delta_n T(x_n)-p\| \leq \|x_n-p\|. \tag{3.34}$$

Since $0<c\leq\beta_n<1$. Hence, $\dfrac{1}{\beta_n}\leq\dfrac{1}{c}$. From (3.33), $\|y_n-p\|-\|x_n-p\|\leq 0$. Thus,

$$\frac{1}{c}(\|y_n-p\|-\|x_n-p\|)+\|x_n-p\| \leq \frac{1}{\beta_n}(\|y_n-p\|-\|x_n-p\|)+\|x_n-p\|. \tag{3.35}$$

Then, by (3.34) and (3.35)

$$\frac{1}{c}(\|y_n-p\|-\|x_n-p\|)+\|x_n-p\| \leq \|(1-\delta_n)x_n+\delta_n T(x_n)-p\| \leq \|x_n-p\| \tag{3.36}$$

Taking the limit as n goes to infinity in the inequality (3.36), one has

$$r \leq \lim_{n} \| (1-\delta_n) x_n + \delta_n T(x_n) - p \| \leq r.$$

Hence,

$$\lim_{n} \| (1-\delta_n)(x_n - p) + \delta_n (T(x_n) - p) \| = r. \tag{3.37}$$

Therefore, from (3.29), (3.30), (3.37) and Lemma 2.4, we conclude that $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. □

Theorem 3.5. *Let C be a nonempty closed convex subset of a uniformly convex partially ordered Banach space E and $T : C \longrightarrow C$ a monotone generalized α-nonexpansive mapping. Assume that E satisfies the Opial property and $F(T)$ is nonempty and there exists $x_1 \in C$ such that $x_1 \preceq Tx_1$ and $x_1 \preceq p$ (resp. $Tx_1 \preceq x_1$ and $p \preceq x_1$), for some $z \in F(T)$. Let $(x_n)_n$ be a sequence defined by* (1.4) *such that there exists $c \in (0,1)$ such that* $0 < c \leq \beta_n < 1$. *Then, $(x_n)_n$ converges weakly to a fixed point of T.*

Proof. By Theorem 3.4, the sequence $(x_n)_n$ is bounded and $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. Since E is uniformly convex, then E is reflexive. Hence, there exists a subsequence $\left(x_{n_j}\right)_j$ of $(x_n)_n$ such that $\left(x_{n_j}\right)_j$ converges weakly to some $z \in C$. By Lemma 3.1, we have

$$x_1 \preceq x_{n_j} \preceq z \quad \text{for all} \quad j \geq 1.$$

Using Lemma 2.3,

$$\|Tx_{n_j} - Tz\| \leq \frac{2}{(1-\alpha)} \|x_{n_j} - Tx_{n_j}\| + \|x_{n_j} - z\|.$$

Then,

$$\limsup_{j\to\infty} \|Tx_{n_j} - Tz\| \leq \limsup_{j\to\infty} \|x_{n_j} - z\|. \tag{3.38}$$

Now, we prove that $Tz = z$. In fact, suppose that $z \neq Tz$. Then, by Opial's condition, we have

$$\begin{aligned}\limsup_{j\to\infty} \|x_{n_j} - z\| &< \limsup_{j\to\infty} \|x_{n_j} - Tz\| \\ &\leq \limsup_{j\to\infty} \left(\|x_{n_j} - Tx_{n_j}\| + \|Tx_{n_j} - Tz\|\right)\end{aligned}$$

$$\leq \limsup_{j\to\infty} \|x_{n_j} - z\|$$

which is a contradiction. This implies that $z = Tz$ and $x_n \preceq z$, for all $n \geq 1$. In order to complete the proof, we show that $(x_n)_n$ converges weakly to the point z. Arguing by contradiction, that is $(x_n)_n$ does not converges weakly to z. Then, we suppose that there exists another subsequence $(x_{n_k})_k$ of (x_n) which converges weakly to a point $y \neq z$ in C. Similarly as above, we get $y \in F(T)$ and by Lemma 3.2 $\lim_{n\to\infty} \|x_n - y\|$ exists. Using Opial property,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z\| &= \lim_{j\to\infty} \|x_{n_j} - z\| \\
&< \lim_{j\to\infty} \|x_{n_j} - y\| = \lim_{n\to\infty} \|x_n - y\| = \lim_{k\to\infty} \|x_{n_k} - y\| \\
&< \lim_{k\to\infty} \|x_{n_k} - z\| = \lim_{n\to\infty} \|x_n - z\|,
\end{aligned}$$

contradiction, then $z = y$. Therefore, $(x_n)_n$ converges weakly to a fixed point of T. □

Theorem 3.6. *Let C be a compact convex nonempty subset of a uniformly convex ordered Banach space $(E, \preceq)$ and $T : C \longrightarrow C$ be a monotone generalized α-nonexpansive mapping. Assume that $F(T)$ is nonempty and there exists $x_1 \in C$ such that $x_1 \preceq Tx_1$ and $x_1 \preceq p$ (resp. $Tx_1 \preceq x_1$ and $p \preceq x_1$), for some $p \in F(T)$. Let $(x_n)_n$ be a sequence defined by* (1.4) *such that there exists $c \in (0,1)$ such that $0 < c \leq \beta_n < 1$. Then, (x_n) converges strongly to a fixed point of T.*

Proof. From Theorem 3.4, the sequence $(x_n)_n$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. By the compactness of C, there exists a subsequence $\left(x_{n_j}\right)_j$ of $(x_n)_n$ converges strongly to a point $z \in C$. Using Lemma 2.2, we have

$$\|x_{n_j} - Tz\| \leq \frac{(3+\alpha)}{(1-\alpha)} \|x_{n_j} - Tx_{n_j}\| + \|x_{n_j} - z\|.$$

Taking limit as j goes to infinity, then $\left(x_{n_j}\right)_j$ converges strongly to $T(z)$. So, $z = T(z)$. Therefore, $(x_n)_n$ converges strongly to fixed point z of T. □

In the sequel, we will use the fixed point set with the partial orders $\mathscr{F}_{x_1}^{\succeq}(T)$ and $\mathscr{F}_{x_1}^{\preceq}(T)$ given by

$$\mathscr{F}_{x_1}^{\succeq}(T) = \{p \in C : Tp = p, x_1 \leq p\} \quad \text{and} \quad \mathscr{F}_{x_1}^{\preceq}(T) = \{p \in C : Tp = p, p \leq x_1\}$$

Note that, since the partial order $\preceq$ is defined by the closed convex cone K and by Lemma (2.5), it is obvious that both $\mathscr{F}_{x_1}^{\succeq}(T)$ and $\mathscr{F}_{x_1}^{\preceq}(T)$ are closed.

Theorem 3.7. *Let C be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \preceq)$ and $T : C \longrightarrow C$ be a monotone generalized α-nonexpansive Assume that there exists $x_1 \in C$ such that $x_1 \preceq Tx_1$ and $\mathscr{F}_{x_1}^{\succeq}(T) \neq \emptyset$ (resp. $Tx_1 \preceq x_1$ and $\mathscr{F}_{x_1}^{\preceq}(T) \neq \emptyset$). Then, the sequence $(x_n)_n$ generated by* (1.4) *converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right) = 0$.*

Proof. Suppose that $\liminf_{n\to\infty} dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right) = 0$. By the proof of Lemma 3.2, the sequence $(\|x_n - p\|)_n$ is bounded and decreasing, for all $p \in \mathscr{F}_{x_1}^{\succeq}(T)$. That is $\|x_{n+1} - p\| \leq \|x_n - p\|$, for all $n \geq 1$. Taking the infimum over all $p \in \mathscr{F}_{x_1}^{\succeq}(T)$, we get

$$dist\left(x_{n+1}, \mathscr{F}_{x_1}^{\succeq}(T)\right) \leq dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right),$$

for all $n \geq 1$. So, the sequence $\left(dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right)\right)_n$ is bounded and deceasing. Hence,

$$\lim_{n\to\infty} dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right) \quad \text{exists.}$$

So,

$$\lim_{n\to\infty} d\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right) = 0. \tag{3.39}$$

In view of (3.39), let $\left(x_{n_j}\right)_j$ be a subsequence of $(x_n)_n$ such that

$$\|x_{n_j} - z_j\| \leq \frac{1}{2^j}, \text{ for all } j \geq 1,$$

where $\left(z_j\right)_j$ is a sequence of $\mathscr{F}_{x_1}^{\succeq}(T)$. Moreover,

$$\|x_{n_{j+1}} - z_j\| \leq \|x_{n_j} - z_j\| \leq \frac{1}{2^j}.$$

Hence,

$$\begin{aligned}\|z_{j+1} - z_j\| &\leq \|z_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - z_j\| \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} < \frac{1}{2^{j-1}}.\end{aligned}$$

Therefore, $\left(z_j\right)_j$ is a Cauchy sequence in the closed subset $\mathscr{F}_{x_1}^{\succeq}(T)$ of the Banach space E. Thus, $\left(z_j\right)_j$ converges strongly to a fixed point z of T. Otherwise,

$$\|x_{n_j} - z\| \leq \|x_{n_j} - z_j\| + \|z_j - z\|.$$

Then $\left(x_{n_j}\right)_j$ converges strongly to z. By Lemma 3.2, $\lim_{n\to\infty} \|x_n - z\|$ exists. Hence, $(x_n)_n$ converges strongly to the fixed point z. The necessity is obvious. $\square$

Now, we extend the definition of condition (I) to the case of monotone mappings.

Definition 3.1. *Let C be a subset of a partially ordered Banach space E and $a \in C$. A mapping $T : C \longrightarrow C$ is said to satisfy the condition ($I^{\succeq}$) (resp. condition ($I^{\preceq}$)) at the point a, if there exists a nondecreasing function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ satisfying $\varphi(0) = 0$ and $\varphi(r) > 0$, for all $r \in (0,\infty)$, such that*

$$\|x - T(x)\| \geq \varphi\left(dist\left(x, \mathscr{F}_a^{\succeq}(T)\right)\right) \ \left(resp.\ \|x - T(x)\| \geq \varphi\left(dist\left(x, \mathscr{F}_a^{\preceq}(T)\right)\right)\right),$$

for all $x \in C$ such that $\mathscr{F}_a^{\succeq}(T)$ (resp. $\mathscr{F}_a^{\preceq}(T)$) nonempty.

Example 3.4. *Let $C = [0,1]$ be equipped with the norm $|\cdot|$ and the order defined by the closed convex cone $\mathbb{R}_+$. Consider the mapping $T : C \longrightarrow C$ defined by $T(x) = \dfrac{2x+1}{4}$, for all $x \in C$.*
We have $\mathscr{F}(T) = \left\{\dfrac{1}{2}\right\}$.
For all $0 \leq a \leq \dfrac{1}{2}$, we have $\mathscr{F}_a^{\succeq}(T) = \left\{\dfrac{1}{2}\right\}$.
Consider the function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ defined by $\varphi(r) = \dfrac{ln(1+r)}{2}$, for all $r \in [0,\infty)$. We have $\varphi(0) = 0$ and $\varphi(r) > 0$, for all $r > 0$. Moreover,

$$\begin{aligned}
\varphi\left(x, d\left(x, \mathscr{F}_a^{\succeq}(T)\right)\right) &= \varphi\left(\left|x - \frac{1}{2}\right|\right) \\
&= \frac{1}{2} ln\left(1 + \left|x - \frac{1}{2}\right|\right) \\
&\leq \frac{1}{2}\left|x - \frac{1}{2}\right| \\
&= \left|\frac{2x-1}{2}\right| = |x - Tx|,
\end{aligned}$$

for all $x \in C$. Therefore T satisfies condition ($I^{\succeq}$) at the point a.

Theorem 3.8. *Let C be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E,\preceq)$ and $T: C \longrightarrow C$ be a monotone generalized α-nonexpansive mapping. Assume that there exists $x_1 \in C$ such that $x_1 \preceq Tx_1$, $\mathscr{F}_{x_1}^{\succeq}(T)$ nonempty and T satisfies condition ($I^{\succeq}$) at the point x_1 (resp. $Tx_1 \preceq x_1$, $\mathscr{F}_{x_1}^{\preceq}(T)$ nonempty and T satisfies condition ($I^{\preceq}$) at the point x_1). Let $(x_n)_n$ be a sequence generated by* (1.4) *such that there exists $c \in (0,1)$ such that $0 < c \leq \beta_n < 1$. Then, $(x_n)_n$ converges strongly to a fixed point of T.*

Proof. By the proof of Lemma 3.2, the sequence $(\|x_n - p\|)_n$ is bounded and decreasing for all $p \in \mathscr{F}_{x_1}^{\succeq}(T)$. That is $\|x_{n+1} - p\| \leq \|x_n - p\|$, for all $n \geq 1$. Taking the infimum over all $p \in \mathscr{F}_{x_1}^{\succeq}(T)$, we get $dist\left(x_{n+1}, \mathscr{F}_{x_1}^{\succeq}(T)\right) \leq dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right)$, for all $n \geq 1$. So, the sequence $\left(dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right)\right)_n$ is bounded and deceasing. Hence, $\lim_{n\to\infty} dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right)$ exists. As T satisfies the condition ($\mathrm{I}^{\succeq}$) at the point x_1, then there exists a nondecreasing function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(u) > 0$ for all $u \in (0,\infty)$ and

$$\|u - T(u)\| \geq \varphi\left(dist\left(u, \mathscr{F}_{x_1}^{\succeq}(T)\right)\right) \quad \forall u \in C$$

Hence, for all $n \geq 1$

$$\|x_n - T(x_n)\| \geq \varphi\left(dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right)\right). \tag{3.40}$$

By Theorem 3.4,

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = 0. \tag{3.41}$$

From (3.40) and (3.41), we get

$$\lim_{n\to\infty} \varphi\left(dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right)\right) = 0.$$

Since φ is nondecreasing, $\varphi(0) = 0$ and $\varphi(r) > 0$ for all $r > 0$. Then,

$$\lim_{n\to\infty} dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right) = 0. \tag{3.42}$$

$(x_n)_n$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$. By (3.42), there exists $n_0 \geq 0$ such that, for all $n \geq n_0$, $dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right) < \frac{\varepsilon}{2}$. Thus, there exists $p \in \mathscr{F}_{x_1}^{\succeq}(T)$ such that $\|x_{n_0} - p\| < \frac{\varepsilon}{2}$. Hence, for all m, $n > n_0$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq \|x_{n_0} - p\| + \|x_{n_0} - p\| \leq \varepsilon.$$

Therefore, $(x_n)_n$ is a Cauchy sequence in the closed subset C of the Banach space E. Then, $(x_n)_n$ converges strongly to a point $z \in C$. Moreover, $dist\left(z, \mathscr{F}_{x_1}^{\succeq}(T)\right) = \lim_{n\to\infty} dist\left(x_n, \mathscr{F}_{x_1}^{\succeq}(T)\right) = 0$. Since $\mathscr{F}_{x_1}^{\succeq}(T)$ is closed, then $z \in \mathscr{F}_{x_1}^{\succeq}(T)$. Thus, $(x_n)_n$ converges strongly to a fixed point z of T. □

In order to compare the behaviour of the iteration process (1.4) to the other iterative processes, we give two numerical examples 3.15 and 3.16 with the same mappings as in the article [14] and [23], respectively.

Example 3.5. *Let $C = [0,\infty)$ be equipped with the usual norm $|\cdot|$ and the order $\leq$ defined by the closed convex cone $\mathbb{R}_+$. Let $T : C \longrightarrow C$ be defined as:*

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x > 2, \\ 0, & \text{if } x \in [0,2]. \end{cases}$$

The mappings T is monotone generalized α-nonexpansive. we show in Table 5 the convergence behaviour of the new iterative process (1.4) *with different initial points and we choose the parameters $\alpha_n = \dfrac{3n}{8n+4}$, $\beta_n = \dfrac{n^2}{(2n+6)^2}$ and $\delta_n = \dfrac{n}{\sqrt{2n^2+9}}$. As a stop criterion, we set $\|x_n - x^*\| < 10^{-15}$.*

Initial points	10	20	100	500	3000	15000
Thakur (1.1)	*2*	*3*	*4*	*5*	*6*	*7*
Piri (1.3)	*2*	*3*	*4*	*5*	*6*	*7*
AK iteration (1.2)	*2*	*2*	*3*	*3*	*4*	*5*
ECM iteration (1.4)	*1*	*2*	*2*	*3*	*3*	*4*

TABLE 6. Influence of parameters and initial points

Example 3.6. *Let $C = [0,1]$ be equipped with the usual norm $|\cdot|$ and the order $\leq$ defined by the closed convex cone $\mathbb{R}_+$. Consider the mapping $T : C \longrightarrow C$ given by:*

$$Tx = \begin{cases} x + \dfrac{3}{5}, & \text{if } x \in \left[0, \dfrac{1}{5}\right) \\ \dfrac{x+4}{5}, & \text{if } x \in \left[\dfrac{1}{5}, 1\right] \end{cases}$$

Note that T is not continuous and not nonexpansive. However, T is monotone generalized α-nonexpansive mapping with $\alpha = \dfrac{3}{8}$ and $x^* = 1$ as a fixed point (for more details see [23]*).*
Next, we choose different parameters α_n, β_n and δ_n and we set $\|x_n - x^*\| < 10^{-15}$. Then, we examine the influence of parameters on the number of iterations needed to achieve the fixed point $x^* = 1$.

Initial points	0.0004	0.001	0.05	0.1	0.5	0.87
Parameters 1: $\alpha_n = 0.7$, $\beta_n = 0.2$, $\delta_n = 0.64$						
Thakur (1.1)	11	11	11	11	11	10
Piri (1.3)	9	9	9	9	9	8
AK iteration (1.2)	7	7	7	7	6	6
ECM iteration (1.4)	5	5	5	5	5	5
Parameters 2: $\alpha_n = \frac{n}{n+1}$, $\beta_n = \frac{n}{n+5}$, $\delta_n = \frac{\sqrt{n}}{\sqrt{2n+7}}$						
Thakur (1.1)	10	10	10	10	10	10
Piri (1.3)	8	8	8	8	8	8
AK iteration (1.2)	6	6	6	6	6	6
ECM iteration (1.4)	5	5	5	5	5	5
Parameters 3: $\alpha_n = \frac{n}{\sqrt{2n^2+7}}$, $\beta_n = \frac{1}{2} - \frac{ln(n+1)}{n+1}$, $\delta_n = \frac{2n}{5n+2}$						
Thakur (1.1)	11	11	11	11	11	10
Piri (1.3)	9	9	9	9	9	9
AK iteration (1.2)	7	7	7	7	7	6
ECM iteration (1.4)	6	6	6	5	5	5
Parameters 4: $\alpha_n = \frac{2n}{3n+2}$, $\beta_n = \frac{n}{\sqrt{49n^2+1}}$, $\delta_n = \frac{\sqrt{2n}}{\sqrt{3n+5}}$						
Thakur (1.1)	11	11	11	11	11	11
Piri (1.3)	10	10	9	9	9	9
AK iteration (1.2)	7	7	7	7	7	6
ECM iteration (1.4)	6	6	6	6	5	5
Parameters 5: $\alpha_n = 1 - \frac{n}{\sqrt[4]{(8n+1)^5}}$, $\beta_n = 1 - \frac{3n}{(6n+2)^3}$, $\delta_n = 1 - \frac{n\sqrt{n}}{\sqrt{(7n+5)^3}}$						
Thakur (1.1)	8	8	8	8	8	8
Piri (1.3)	6	6	6	6	6	6
AK iteration (1.2)	5	5	5	5	5	5
ECM iteration (1.4)	4	4	4	4	4	4

TABLE 7. Influence of parameters and initial points

4. APPLICATION

Integral equations have an impact, in fields like mathematics, engineering and economics. They offer a framework for solving problems within these disciplines. In the last few years, there has been considerable advancement in fixed point theory resulting in creative solutions, for different types of integral equations.

The main aim of this section is to demonstrate the practical application of our theoretical results for approximating the solutions for integral equations. Through this approach, we highlight the utility and significance of our research findings.

For a sufficiently large chosen value of $r>0$ and $u\in L^2([0,1],\mathbb{R})$ such that $\|u\|_2\leq \frac{r}{2}$, define a closed convex subset C, within in the uniformly convex Banach space $\left(L^2([0,1],\mathbb{R}),\|\cdot\|_2\right)$, as follows :

$$C=\left\{h\in L^2([0,1],\mathbb{R})\ :\ \|h-u\|_2\leq r\ \text{ and }\ u(t)\leq h(t)\ a.e.\ t\in[0,1]\right\}.$$

Examine the subsequent integral equation :

$$\varphi(x)=u(x)+\lambda\int_{[0,1]}\psi(x,s)f(s,\varphi(s))ds, \tag{4.1}$$

where $x\in[0,1]$ and $\lambda\geq 0$.

Consider the partial order $\preceq_K$, generated by the cone $K=\left\{h\in L^2([0,1],\mathbb{R})\ :\ h(t)\geq 0\ a.e.\ t\in[0,1]\right\}$, defined as follows :

$$h,k\in L^2([0,1],\mathbb{R})\ ,\ h\preceq_K k\Longleftrightarrow h(t)\leq k(t)\ a.e.\ t\in[0,1].$$

Assume the subsequent hypothesis :

(H1) $f:[0,1]\times C\longrightarrow C$ continuous, $f(s,u(s))\geq 0$ for all $s\in[0,1]$, and satisfying the following condition :

(H1-a) for any ω, $\nu\in C$ such that $\nu\preceq_K\omega$

$$0\leq f(x,\omega)-f(x,\nu)\leq|\omega(x)-\nu(x)|, \tag{4.2}$$

(H1-b) $\int_0^1|f(t,\omega(t))|^2dt\leq r^2$;

(H2) $\psi:[0,1]\times[0,1]\longrightarrow\mathbb{R}$ a continuous function so that $\psi(x,s)\geq 0$ and $\lambda^2\int_0^1|\psi(x,s)|^2ds\leq 1$.

Theorem 4.1. *Under the assumptions (H1) and (H2), if the integral equation has a solution then the iterative process* (1.4) *converges to a solution of the integral equation* (4.1).

Proof. From (4.1), define the subsequent mapping Υ as follows

$$\Upsilon(\varphi(x)) = u(x) + \lambda \int_{[0,1]} \psi(x,s) f(s,\varphi(s)) ds. \tag{4.3}$$

Following the hypothesis (H1-a), it is quite easy to prove that Υ is a monotone mapping according the order $\preceq_K$, and (H1-b) implies that $T(C) \subset C$. Moreover, $u \preceq_K \Upsilon(\varphi)$, for all $\varphi \in C$.

Consider ω, $\nu \in C$ so that $\omega \preceq_K \nu$, then

$$\begin{aligned}
\|\Upsilon(\omega) - \Upsilon(\nu)\|_2^2 &= \int_0^1 |\Upsilon(\omega(x)) - \Upsilon(\nu(x))|^2 dx \\
&= \int_0^1 \left| \lambda \int_{[0,1]} \psi(x,s) f(s,\omega(s)) ds - \lambda \int_{[0,1]} \psi(x,s) f(s,\nu(s)) ds \right|^2 dx \\
&\le \int_0^1 \lambda^2 \left| \int_{[0,1]} \psi(x,s) \left(f(s,\omega(s)) - f(s,\nu(s)) \right) \right|^2 ds\, dx \\
&\le \int_0^1 \lambda^2 \int_{[0,1]} |\psi(x,s)|^2 |f(s,\omega(s)) - f(s,\nu(s))|^2 ds dx \\
&\le \int_0^1 \lambda^2 |\omega(s) - \nu(s)|^2 \int_{[0,1]} |\psi(x,s)|^2 ds\, dx \\
&\le \int_0^1 |\omega(s) - \nu(s)|^2 \left(\lambda^2 \int_{[0,1]} |\psi(x,s)|^2 dx \right) ds \\
&\le \int_0^1 |\omega(s) - \nu(s)|^2 ds = \|\omega - \nu\|_2^2.
\end{aligned}$$

Therefore, $\|\Upsilon(\omega) - \Upsilon(\nu)\|_2 \le \|\omega - \nu\|_2$, for every ω, $\nu \in C$ so that $\omega \preceq_K \nu$. Thus, T is monotone nonexpansive (i.e. monotone generalized 0-nonexpansive).

As a consequence, theorem 3.5 entails that the iterative process (1.4) converges to a solution of the equation (4.1). $\square$

Example 4.1. *Consider the functions* $u(x) = 2(1-x^2)$, $\psi(x,s) = x^2 s$ *and* $f(s,\varphi(s)) = \dfrac{\varphi(s)}{2}$, *With the parameters* $\lambda = \dfrac{1}{2}$ *and* $r = 7$. *Consequently, we can express the integral equation* (4.1) *as follows :*

$$\varphi(x) = 2(1-x^2) + \frac{1}{2} \int_0^1 x^2 s \frac{\varphi(s)}{2} ds, \tag{4.4}$$

for $x \in [0,1]$. *This equation is a special case of the class of Fredholm equation. The exact solution of the equation* (4.4) *is given by* $\phi(x) = 2(1 - \frac{14}{15}x^2)$, *where* $x \in [0,1]$.
One can easily verify that all the assumptions (H1) and (H2) are fulfilled.

Next, consider the operator $\Upsilon : C \longrightarrow C$ *given by*

$$\Upsilon(\varphi(x)) = 2(1-x^2) + \frac{1}{2}\int_0^1 x^2 s \frac{\varphi(s)}{2} ds, \tag{4.5}$$

where $x \in [0,1]$.

By employing distinct initial functions $\varphi_1(x) = 2(1-x^2)$ *and* $\varphi_2(x) = 2$, *for* $x \in [0,1]$, *with control sequences* $\alpha_n = \frac{1}{1+n}$, $\beta_n = \frac{n}{1+n^2}$ *and* $\delta_n = \frac{n^2}{1+n^2}$, *for all* $n \geq 1$, *we derive the subsequent approximate values of* $\varphi(x)$, *as illustrated in Figure 2 and Table 8.*

Iteration number	*For* $\varphi_1(x) = 2(1-x^2)$	*For* $\varphi_2(x) = 2$
1	1.9712	2
2	1.97311536652147	1.97311822697207
3	1.97311501007693	1.97311501041302
4	1.97311478477883	1.97311478477888
5	1.97311470230586	1.97311470230586
6	1.97311466709605	1.97311466709605
7	1.97311464989585	1.97311464989585
8	1.97311464058236	1.97311464058236
9	1.97311463512602	1.97311463512602
10	1.97311463172652	1.97311463172652

TABLE 8. Approximative values of $\phi(0,12)$

Value of x	*Error in case* $\varphi_1(x) = 2(1-x^2)$	*Error in case* $\varphi_2(x) = 2$
0	0	0
0.12	$6.30825192082796.10^{-7}$	$6.30835192082897.10^{-7}$
0.5	$1.09496358424721.10^{-7}$	$1.09596339224721.10^{-7}$
1	$4.37985432921728.10^{-7}$	$4.37995531921728.10^{-7}$

TABLE 9. Error for approximated values and the exact solution just for the first 10 iterations

Table 8 displays the approximate values of the image for $x = 0.12$ calculated using the function ϕ, with $\phi(0.12) = 1.97312$. Notably, as the number of iterations increases, the approximated values gradually converge toward the exact value, ultimately approaching the minimum absolute error value shown in Table 9 for the first 10 iterations.

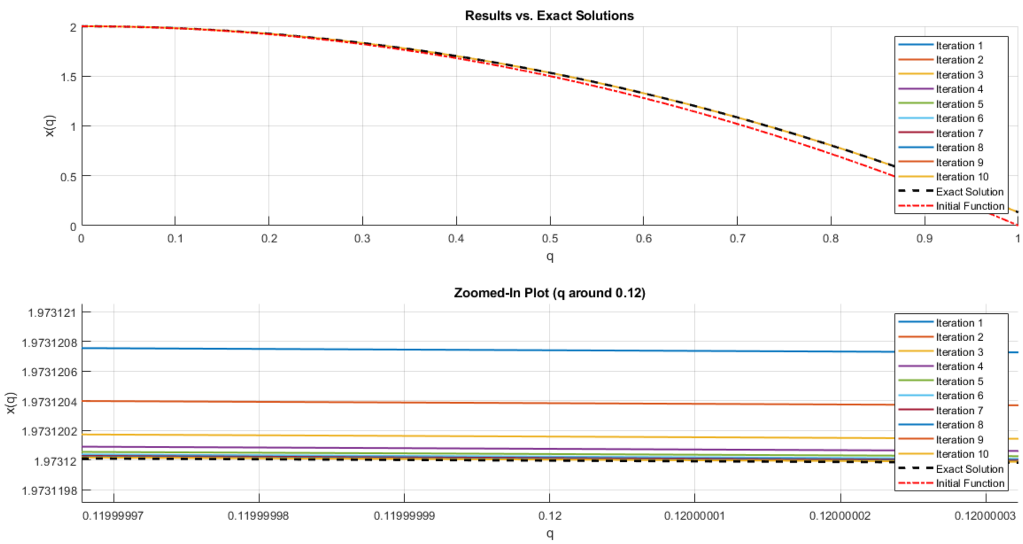

FIGURE 2. Case $\varphi_1(x) = 2(1-x^2)$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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