Available online at http://scik.org
Adv. Fixed Point Theory, 2024, 14:28
https://doi.org/10.28919/afpt/8531
ISSN: 1927-6303

# CERTAIN APPLICATIONS VIA $b$-SIMULATION FUNCTION OF $(\alpha,\beta)$-$Z_b$-GERAGHTY TYPE CONTRACTION IN $C^*$-ALGEBRA VALUED $S_b$-METRIC SPACES

N. NARSIMHA[1,2], G. UPENDER REDDY[2,*], P. NARESH[3], B. SRINUVASA RAO[4]

[1] Department of Mathematics, Nagarjuna Govt.College (A), Nalgonda,508001, Telangana, India

[2] Department of Mathematics, Mahatma Gandhi University, Nalgonda,508254, Telangana, India

[3] Department of Mathematics, Sreenidhi Institute of Science and Technology, Ghatkesar, Hyderabad, 501301, Telangana, India

[4] Department of Mathematics, Dr. B. R. Ambedkar University, Srikakulam, Etcherla, 532410, Andhra Pradesh, India



**Abstract.** The purpose of this paper is to develop a common fixed point theorem for a pair of self-mappings in the setting of $(\alpha,\beta)$-$Z_b$-Geraghty type contraction via $b$-simulation function in $C^*$-algebra valued $S_b$-metric spaces. We also discuss various applications for integral equations and homotopy. Furthermore, we present instances to support our key findings.

**Keywords:** $(\alpha,\beta)$-$Z_b$-Geraghty type mapping; common fixed point; $b$-simulation function; $C^*$-algebra valued $S_b$-metric spaces.

**2020 AMS Subject Classification:** 55M20, 54E50, 47H09, 14F35, 55P10.

## 1. INTRODUCTION

A metric space is suitable for those interested in analysis, mathematical physics, and applied sciences. Several extensions of metric spaces have been explored, and several results about the

*Corresponding author
E-mail address: upendermathsmgu@gmail.com
Received March 08, 2024

existence of fixed points were obtained (see [1]-[5]).

In 2014, Ma *et al.* introduced $C^*$-algebra-valued metric spaces [6]. In 2015, they introduced the idea of $C^*$-algebra-valued $b$-metric spaces and examined some findings [7]. Razavi and Masiha also studied $C^*$-algebra-valued $b$-metric spaces [8] to identify common principles.

Sedghi et al. [9] created $S_b$-metric spaces by combining the concepts of $S$ and $b$-metric spaces, and demonstrated common fixed point findings in these spaces. To improve, numerous authors developed many results on $S_b$-metric spaces (e.g., [10]-[14]).

Inspired by the work of Souayah and Mlaiki in [10], in 2023, Razavi *et al.* presented the idea of $C^*$-algebra valued $S_b$-metric space [15], and established some common fixed point results in this space [16].

In 2015, Khojasteh *et al.* [17] proposed the use of simulation functions to expand the class of mappings that give a fixed point and present a novel approach for proving fixed point theorems. Demma *et al.* [18] developed a $b$-simulation function and demonstrated fixed point outcomes in $b$-metric space. Several academics have studied simulation functions in broader contexts (e.g., [19]-[21]).

Geraghty, on the other hand, generalised the Banach contraction principle, coining the term "Geraghty contraction". S. Chandok [22] proposed the notion of $(\alpha,\beta)$-admissible Geraghty type contractive mapping in metric space. Several authors later used this strategy, giving fascinating results (e.g., [23]-[25] and their references).

Inspired and motivated by the results of Demma *et al.* [18] and S.Chandok [22], S.Negi *et al.* [26] introduce the notion of $(\alpha,\beta)$-$Z_b$-Geraghty type contraction for a pair of mappings using $b$-simulation function and construct a common fixed point theorem in the setting of $b$-metric-like spaces.

The current study aims to offer common fixed point theorems by employing $(\alpha,\beta)$-$Z_b$-Geraghty type contraction mappings via $b$-simulation function in the context of $C^*$-algebra valued $S_b$-metric spaces. We can also present examples that are acceptable and relevant to both integral equations and homotopy.

First we recall some basic results.

## 2. PRELIMINARIES

This section provides a short introduction to some realities about the theory of $C^*$- algebras [27]. First, suppose that $\mathfrak{A}$ is a unital $C^*$ algebra with the unit $1_{\mathfrak{A}}$. Set $\mathfrak{A}_h = \{s \in \mathfrak{A} : s = s^*\}$. The element $s \in \mathfrak{A}$ is said to be positive, and we write $s \succeq 0_{\mathfrak{A}}$ if and only if $s = s^*$ and $\sigma(s) \subseteq [0,\infty)$, in which $0_{\mathfrak{A}}$ in $\mathfrak{A}$ is the zero element and the spectrum of $s$ is $\sigma(s)$. On $\mathfrak{A}_h$, we can find a natural partial ordering given by $\ell \preceq \wp$ if and only if $\wp - \ell \succeq 0_{\mathfrak{A}}$. We denote with $\mathfrak{A}_+ = \{s \in \mathfrak{A} : s \succeq 0_{\mathfrak{A}}\}$ and $\mathfrak{A}' = \{s \in \mathfrak{A} : st = ts\ \forall t \in \mathfrak{A}\}$.

**Definition 2.1.** *([15]) Let $\mathscr{G}$ be a non-empty set and $\kappa \in \mathfrak{A}'$ with $||\kappa|| \geq 1$. Suppose that a mapping $S_b : \mathscr{G}^3 \to \mathfrak{A}$ be a function satisfying the following properties :*

$(S_{b_1})$ $S_b(\ell,\beta,\gamma) \succeq 0_{\mathfrak{A}}$ *for all* $\ell,\beta,\gamma \in \mathscr{G}$,

$(S_{b_2})$ $S_b(\ell,\beta,\gamma) = 0_{\mathfrak{A}} \Leftrightarrow \ell = \beta = \gamma$,

$(S_{b_3})$ $S_b(\ell,\beta,\gamma) \preceq \kappa(S_b(\ell,\ell,\theta) + S_b(\beta,\beta,\theta) + S_b(\gamma,\gamma,\theta))$ *for all* $\ell,\beta,\gamma,\theta \in \mathscr{G}$.

*Then the function $S_b$ is called a $C^*$-algebra valued $S_b$-metric on $\mathscr{G}$ and the pair $(\mathscr{G},\mathfrak{A},S_b)$ is called a $C^*$-algebra valued $S_b$-metric space ($C^*$-AV-$S_b$MS) with a coefficient $\kappa$.*

**Example 2.2.** *([15]) . Let $\mathscr{G} = \mathbb{R}$ and $\mathfrak{A} = M_2(\mathbb{R})$ be all $2\times 2$ matrices with the usual operations of addition, scalar multiplication, and matrix multiplication. It is clear that $||P|| = \sqrt{\sum_{i,j=1}^{2} |p_{ij}|^2}$ defines a norm on $\mathfrak{A}$, where $P = (p_{ij}) \in \mathfrak{A}$. $* : \mathfrak{A} \to \mathfrak{A}$ defines an involution on $\mathfrak{A}$ and where $\mathfrak{A}^* = \mathfrak{A}$. Then, $\mathfrak{A}$ is a $C^*$-algebra. For $P = (p_{ij})$ and $Q = (q_{ij})$ in $\mathfrak{A}$, a partial order on $\mathfrak{A}$ can be given as follows:*

$$P \preceq Q \iff p_{ij} - q_{ij} \leq 0\ \forall\ i,j = 1,2.$$

*Let $(\mathscr{G},d)$ be a b-metric space where, $||\kappa|| \geq 1$ and $S_b : \mathscr{G}^3 \to M_2(\mathbb{R})$, fulfilling*

$$S_b(p,q,r) = \begin{bmatrix} d(p,q)+d(q,r)+d(r,p) & 0 \\ 0 & d(p,q)+d(q,r)+d(r,p) \end{bmatrix}.$$

*Then, clearly $(\mathscr{G},\mathfrak{A},S_b)$ is a $C^*$-AV-$S_b$MS.*

**Definition 2.3.** *([15]) A $C^*$-AV-$S_b$MS $S_b$ is symmetric if*

$$S_b(\ell,\ell,\beta) = S_b(\beta,\beta,\ell)\ \forall\ \ell,\beta \in \mathscr{G}$$

**Definition 2.4.** *([15]) Let $(\mathscr{G},\mathfrak{A},S_b)$ be a $C^*$-AV-$S_b$MS and $\{\chi_n\}$ be a sequence in $\mathscr{G}$:*

(1) *If for all* $p \in \mathbb{N}$, $||S_b(\chi_{n+p}, \chi_{n+p}, \chi_n)|| \to 0$, *where* $n \to \infty$, *then* $\{\chi_n\}$ *is a Cauchy sequence in* $\mathscr{G}$.

(2) *If* $||S_b(\chi_n, \chi_n, \chi)|| \to 0$, *where* $n \to \infty$, *then* $\{\chi_n\}$ *converges to* $\chi$, *and we present it with* $\lim_{n\to\infty} \chi_n = \chi$.

(3) *If every Cauchy sequence is convergent in* $\mathscr{G}$ , *then* $(\mathscr{G}, \mathfrak{A}, S_b)$ *is a complete* $C^*$*-AV-*$S_b$*MS.*

**Definition 2.5.** *(*[15]*) Suppose that* $(\mathscr{G}_1, \mathfrak{A}_1, S_{b1})$ *and* $(\mathscr{G}_2, \mathfrak{A}_2, S_{b2})$ *are* $C^*$*-AV-*$S_b$*MS, and let* $\Gamma : (\mathscr{G}_1, \mathfrak{A}_1, S_{b1}) \to (\mathscr{G}_2, \mathfrak{A}_2, S_{b2})$ *be a function. Then,* $\Gamma$ *is continuous at a point* $\chi \in \mathscr{G}_1$ *if, for every sequence,* $\{\chi_n\}$ *in* $\mathscr{G}_1$ , $S_b(\chi_n, \chi_n, \chi) \to 0_{\mathfrak{A}}$, $(n \to \infty)$ *implies* $S_b(\Gamma(\chi_n), \Gamma(\chi_n), \gamma(\chi)) \to 0_{\mathfrak{A}'}$ *where* $n \to \infty$. *A function* $\Gamma$ *is continuous at* $\mathscr{G}_1$ *if and only if it is continuous at all* $\chi \in \mathscr{G}_1$ .

**Definition 2.6.** *(*[29]*) Let* $\mathscr{G}$ *be a non-empty set,* $\Gamma, \Lambda : \mathscr{G} \to \mathscr{G}$ *be two mappings and* $\alpha, \beta : \mathscr{G} \times \mathscr{G} \to \mathfrak{A}_+$ *be two functions then* $(\Gamma, \Lambda)$ *is called a pair of* $(\alpha, \beta)$*-admissible mappings, if for all* $\ell, æ \in \mathscr{G}$

$$\alpha(\ell, æ) \succeq 1_{\mathfrak{A}},\ \beta(\ell, æ) \succeq 1_{\mathfrak{A}} \text{ implies } \alpha(\Gamma\ell, \Lambda æ) \succeq 1_{\mathfrak{A}}, \alpha(\Lambda\ell, \Gamma æ) \succeq 1_{\mathfrak{A}}$$

$$\text{and } \beta(\Gamma\ell, \Lambda æ) \succeq 1_{\mathfrak{A}}, \beta(\Lambda\ell, \Gamma æ) \succeq 1_{\mathfrak{A}}$$

**Lemma 2.7.** *(*[28]*) Suppose that* $\mathfrak{A}$ *is a unital* $C^*$*-algebra with a unit* $1_{\mathfrak{A}}$*:*

(1) *If* $\{\chi_n\}_{n=1}^{\infty} \subseteq \mathfrak{A}$ *and* $\lim_{n\to\infty} \chi_n = 0_{\mathfrak{A}}$, *then for any* $\chi \in \mathfrak{A}$, $\lim_{n\to\infty} \chi^* \chi_n \chi = 0_{\mathfrak{A}}$

(2) *If* $\chi, \xi \in \mathfrak{A}_h$ *and* $s \in \mathfrak{A}'_+$ *then* $\chi \preceq \xi$ *yields* $s\chi \preceq s\xi$ *in which* $\mathfrak{A}'_+ = \mathfrak{A}_+ \cap \mathfrak{A}'$.

(3) *If* $\chi \in \mathfrak{A}_+$ *with* $||\chi|| < \frac{1}{2}$ *then* $1_{\mathfrak{A}} - \chi$ *is invertible,and* $||\chi(1_{\mathfrak{A}} - \chi)^{-1}|| < 1$.

(4) *If* $\chi, \xi \in \mathfrak{A}_+$ *such that* $\chi\xi = \xi\chi$, *then* $\chi\xi \succeq 0_{\mathfrak{A}}$.

## 3. MAIN RESULTS

In this section, first we introduce $(\alpha, \beta)$-$Z_b$-Geraghty type contraction and then prove our main result.

**Definition 3.1.** *Let* $(\mathscr{G}, \mathfrak{A}, S_b)$ *be a* $C^*$*-AV-*$S_b$*MS,* $\Gamma, \Lambda : \mathscr{G} \to \mathscr{G}$ *be two mappings and* $\alpha, \beta : \mathscr{G} \times \mathscr{G} \times \mathscr{G} \to \mathfrak{A}_+$ *be two functions then* $(\Gamma, \Lambda)$ *is called a pair of* $(\alpha, \beta)$*-admissible mappings, if for all* $\ell, æ \in \mathscr{G}$

$$\alpha(\ell, \ell, æ) \succeq 1_{\mathfrak{A}},\ \beta(\ell, \ell, æ) \succeq 1_{\mathfrak{A}} \text{ implies } \alpha(\Gamma\ell, \Gamma\ell, \Lambda æ) \succeq 1_{\mathfrak{A}}, \alpha(\Lambda\ell, \Lambda\ell, \Gamma æ) \succeq 1_{\mathfrak{A}}$$

$$\text{and } \beta(\Gamma\ell,\Gamma\ell,\Lambda æ) \succeq 1_{\mathfrak{A}}, \beta(\Lambda\ell,\Lambda\ell,\Gamma æ) \succeq 1_{\mathfrak{A}}$$

**Definition 3.2.** *Let $(\mathscr{G},\mathfrak{A},S_b)$ be a $C^*$-AV-$S_b$MS, $\Gamma,\Lambda:\mathscr{G}\to\mathscr{G}$ be two mappings and $\alpha,\beta:\mathscr{G}\times\mathscr{G}\times\mathscr{G}\to\mathfrak{A}_+$ be two functions then $\mathscr{G}$ is said to be $(\alpha,\beta)$-regular, if $\{\chi_n\}$ is a sequence in $\mathscr{G}$ such that $\chi_n\to\chi\in\mathscr{G}$ and $\alpha(\chi_n,\chi_n,\chi_{n+1})\succeq 1_{\mathfrak{A}}$, $\beta(\chi_n,\chi_n,\chi_{n+1})\succeq 1_{\mathfrak{A}}$, $\forall\, n\in N$, then there exists a subsequence $\{\chi_{n_k}\}$ of $\{\chi_n\}$ such that $\alpha(\chi_{n_k},\chi_{n_k},\chi_{n_{k+1}})\succeq 1_{\mathfrak{A}}$, $\beta(\chi_{n_k},\chi_{n_k},\chi_{n_{k+1}})\succeq 1_{\mathfrak{A}}\ \forall\, k\in N$. Also $\alpha(\chi,\chi,\Gamma\chi)\succeq 1_{\mathfrak{A}}$ and $\beta(\chi,\chi,\Lambda\chi)\succeq 1_{\mathfrak{A}}$.*

**Definition 3.3.** *Let $(\mathscr{G},\mathfrak{A},S_b)$ be a $C^*$-AV-$S_b$MS. A b-simulation function is a function $\eta_b:\mathfrak{A}_+\times\mathfrak{A}_+\to\mathfrak{A}$ satisfying the following conditions:*

$(\eta_{b_1})$ *$\eta_b(\chi,\xi)\prec\xi-\chi$ for all $\chi,\xi\succ 0_{\mathfrak{A}}$;*

$(\eta_{b_2})$ *if $\{\chi_n\}$ and $\{\xi_n\}$ are sequences in $\mathfrak{A}_+$ such that*

$$0<\lim_{n\to\infty}||\chi_n||\le\lim_{n\to\infty}\inf||\xi_n||\le\lim_{n\to\infty}\sup||\xi_n||\le\kappa\lim_{n\to\infty}||\chi_n||<\infty$$

*then $\lim\limits_{n\to\infty}\sup||\eta_b(\kappa\chi_n,\xi_n)||<0$.*

*Throughout the paper $Z_b$ will represent the family of all b-simulation functions.*

**Definition 3.4.** *Let $(\mathscr{G},\mathfrak{A},S_b)$ be a $C^*$-AV-$S_b$MS with coefficient $||\kappa||>1$, $\Gamma,\Lambda:\mathscr{G}\to\mathscr{G}$ be two mappings and $\alpha,\beta:\mathscr{G}\times\mathscr{G}\times\mathscr{G}\to\mathfrak{A}_+$ be two functions such that for all $\ell,\varkappa\in\mathscr{G}$ with $r=1,2,3,4$, then we say that the pair of mappings $(\Gamma,\Lambda)$ is $(\alpha,\beta)$-$Z_b$-Geraghty type generalized contraction if for $\eta_b\in Z_b$, we have*

$$\eta_b(\kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Lambda\varkappa),\varphi(M^r(\ell,\varkappa))a^*M^r(\ell,\varkappa)a)\succeq 0_{\mathfrak{A}} \tag{3.1}$$

*where $\varphi:\mathfrak{A}_+\to[0,1)$ is a Geraghty function, $a\in\mathfrak{A}$ in which $||a||<1$ and*

$$M^1(\ell,\varkappa)=S_b(\ell,\ell,\varkappa)$$

$$M^2(\ell,\varkappa)=\max\left\{\ S_b(\ell,\ell,\varkappa),S_b(\Lambda\ell,\Lambda\ell,\Gamma\varkappa),\ \right\}$$

$$M^3(\ell,\varkappa)=\max\left\{\begin{array}{c} S_b(\ell,\ell,\varkappa),\frac{S_b(\ell,\ell,\Gamma\ell)+S_b(\varkappa,\varkappa,\Lambda\varkappa)}{2\kappa^4},\\ \frac{S_b(\ell,\ell,\Lambda\varkappa)+S_b(\varkappa,\varkappa,\Gamma\ell)}{2\kappa^4}\end{array}\right\}$$

$$M^4(\ell,\varkappa)=\max\left\{\begin{array}{c} S_b(\ell,\ell,\varkappa),S_b(\ell,\ell,\Gamma\ell),\\ S_b(\varkappa,\varkappa,\Lambda\varkappa),\frac{S_b(\ell,\ell,\Lambda\varkappa)+S_b(\varkappa,\varkappa,\Gamma\ell)}{2\kappa^4}\end{array}\right\}$$

**Theorem 3.5.** *Let $(\mathscr{G},\mathfrak{A},S_b)$ be a complete $C^*$-AV-$S_b$MS, $\ell,\beta:\mathscr{G}\times\mathscr{G}\to\mathfrak{A}_+$ be two functions and $\Gamma,\Lambda:\mathscr{G}\to\mathscr{G}$ be two mappings with the following assumptions:*

- $(i)$ $(\Gamma,\Lambda)$ *is a pair of* $(\alpha,\beta)$*-admissible mappings;*
- $(ii)$ $(\Gamma,\Lambda)$ *is a pair of* $(\alpha,\beta)$*-*$Z_b$*-Geraghty type generalized contraction mappings with* $r=4$*;*
- $(iii)$ *there exist* $\ell_0\in\mathscr{G}$ *such that* $\alpha(\ell_0,\ell_0,\Gamma\ell_0)\succeq 1_{\mathfrak{A}}$, $\beta(\ell_0,\ell_0,\Lambda\ell_0)\succeq 1_{\mathfrak{A}}$*;*
- $(iv)$ *either* $\Gamma$ *and* $\Lambda$ *are continuous or* $\mathscr{G}$ *is* $(\alpha,\beta)$*-regular space.*

*Then* $\Gamma$ *and* $\Lambda$ *have a unique common fixed point in* $\mathscr{G}$*.*

*Proof.* From (iii) hypotheses, there exists $\varkappa_0\in\mathscr{G}$ such that $\alpha(\varkappa_0,\varkappa_0,\Gamma\varkappa_0)\geq 1_{\mathfrak{A}}$, $\beta(\varkappa_0,\varkappa_0,\Gamma\varkappa_0)\geq 1_{\mathfrak{A}}$. We construct a sequence $\{\varkappa_p\}$ by setting $\varkappa_{p+1}=\Gamma\varkappa_p$ and $\varkappa_{p+2}=\Lambda\varkappa_{p+1}$ $\forall$ $p\in N\cup\{0\}$. Since $(\Gamma,\Lambda)$ is a pair of $(\alpha,\beta)$-admissible mappings, then $\alpha(\varkappa_0,\varkappa_0,\varkappa_1)\succeq 1_{\mathfrak{A}}$ implies $\alpha(\varkappa_1,\varkappa_1,\varkappa_2)\succeq 1_{\mathfrak{A}}$ and $\beta(\varkappa_0,\varkappa_0,\varkappa_1)\succeq 1_{\mathfrak{A}}$ implies $\beta(\varkappa_1,\varkappa_1,\varkappa_2)\succeq 1_{\mathfrak{A}}$. By repeating similar process, we obtain $\alpha(\varkappa_p,\varkappa_p,\varkappa_{p+1})\succeq 1_{\mathfrak{A}}$ and $\beta(\varkappa_p,\varkappa_p,\varkappa_{p+1})\succeq 1_{\mathfrak{A}}$ for every $p\in N$. From Equation 3.1, we have

$$\eta_b\left(\kappa\alpha(\varkappa_p,\varkappa_p,\Gamma\varkappa_p)\beta(\varkappa_{p+1},\varkappa_{p+1},\Lambda\varkappa_{p+1})S_b(\Gamma\varkappa_p,\Gamma\varkappa_p,\Lambda\varkappa_{p+1}),\varphi(M^4(\varkappa_p,\varkappa_{p+1}))a^*M^4(\varkappa_p,\varkappa_{p+1})a\right)\succeq 0_{\mathfrak{A}}$$

From $(\eta_{b_1})$, we have

$$\kappa\alpha(\varkappa_p,\varkappa_p,\Gamma\varkappa_p)\beta(\varkappa_{p+1},\varkappa_{p+1},\Lambda\varkappa_{p+1})S_b(\Gamma\varkappa_p,\Gamma\varkappa_p,\Lambda\varkappa_{p+1})\prec\varphi(M^4(\varkappa_p,\varkappa_{p+1}))a^*M^4(\varkappa_p,\varkappa_{p+1})a$$

Now from above inequality, we see that

$$\begin{aligned}
S_b(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+2}) &= S_b(\Gamma\varkappa_p,\Gamma\varkappa_p,\Lambda\varkappa_{p+1})\\
&\preceq \kappa\alpha(\varkappa_p,\varkappa_p,\Gamma\varkappa_p)\beta(\varkappa_{p+1},\varkappa_{p+1},\Lambda\varkappa_{p+1})S_b(\Gamma\varkappa_p,\Gamma\varkappa_p,\Lambda\varkappa_{p+1})\\
&\prec \varphi(M^4(\varkappa_p,\varkappa_{p+1}))a^*M^4(\varkappa_p,\varkappa_{p+1})a\\
&\preceq a^*M^4(\varkappa_p,\varkappa_{p+1})a \qquad (3.2)
\end{aligned}$$

Now, by simple computations, we have

$$\begin{aligned}
M^4\left(\varkappa_p,\varkappa_{p+1}\right) &= =\max\left\{\begin{array}{c} S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+1}\right),S_b\left(\varkappa_p,\varkappa_p,\Gamma\varkappa_p\right),\\ S_b\left(\varkappa_{p+1},\varkappa_{p+1},\Lambda\varkappa_{p+1}\right),\frac{S_b\left(\varkappa_p,\varkappa_p,\Lambda\varkappa_{p+1}\right)+S_b\left(\varkappa_{p+1},\varkappa_{p+1},\Gamma\varkappa_p\right)}{2\kappa^4}\end{array}\right\}\\
&= \max\left\{\begin{array}{c} S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+1}\right),S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+1}\right),\\ S_b\left(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+2}\right),\frac{S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+2}\right)+S_b\left(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+1}\right)}{2\kappa^4}\end{array}\right\}
\end{aligned}$$

$$= \max\left\{\begin{array}{c} S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+1}\right), \\ S_b\left(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+2}\right), \frac{S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+2}\right)+S_b\left(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+1}\right)}{2\kappa^4} \end{array}\right\}.$$

Notice that

$$\begin{aligned} \frac{S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+2}\right)+S_b\left(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+1}\right)}{2\kappa^4} &\leq \frac{2\kappa S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+1}\right)+\kappa S_b\left(\varkappa_{p+2},\varkappa_{p+2},\varkappa_{p+1}\right)}{2\kappa^4} \\ &\leq \max\left\{\begin{array}{c} S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+1}\right), \\ S_b\left(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+2}\right) \end{array}\right\} \end{aligned}$$

From (3.2), we obtain that

$$\begin{aligned} S_b(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+2}) &\preceq a^*M^4(\varkappa_p,\varkappa_{p+1})a \\ &\preceq a^*\max\left\{\begin{array}{c} S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+1}\right), \\ S_b\left(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+2}\right) \end{array}\right\}a \end{aligned} \tag{3.3}$$

If we take $\max\left\{\begin{array}{c} S_b\left(\varkappa_p,\varkappa_p,\varkappa_{p+1}\right), \\ S_b\left(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+2}\right) \end{array}\right\}=S_b\left(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+2}\right)$, since $||a||<1$ then (3.3) gives a contradiction. Thus, we obtain

$$\begin{aligned} S_b(\varkappa_{p+1},\varkappa_{p+1},\varkappa_{p+2}) &\preceq a^*S_b(\varkappa_p,\varkappa_p,\varkappa_{p+1})a \\ &\preceq (a^*)^2S_b(\varkappa_{p-1},\varkappa_{p-1},\varkappa_p)(a)^2 \\ &\vdots \\ &\preceq (a^*)^{p+1}S_b(\varkappa_0,\varkappa_0,\varkappa_1)(a)^{p+1} \end{aligned}$$

By remembering the property where if $a,b\in\mathfrak{A}_h$ then $a\preceq b$ yields $u^*au\preceq u^*bu$, we see the following for each $p\in N$,

$$S_b(\varkappa_p,\varkappa_p,\varkappa_{p+1})\preceq(a^*)^pS_b(\varkappa_0,\varkappa_0,\varkappa_1)(a)^p$$

Let $S_b(\varkappa_0,\varkappa_0,\varkappa_1)=X_0$ for some $X_0\in\mathfrak{A}_+$. For any $l\in N$, we achieve

$$\begin{aligned} S_b(\varkappa_{p+l},\varkappa_{p+l},\varkappa_p) &\preceq \kappa\left(S_b(\varkappa_{p+l},\varkappa_{p+l},\varkappa_{p+l-1})+S_b(\varkappa_{p+l},\varkappa_{p+l},\varkappa_{p+l-1})+S_b(\varkappa_p,\varkappa_p,\varkappa_{p+l-1})\right) \\ &\preceq 2\kappa S_b(\varkappa_{p+l},\varkappa_{p+l},\varkappa_{p+l-1})+\kappa S_b(\varkappa_p,\varkappa_p,\varkappa_{p+l-1}) \\ &= 2\kappa S_b(\varkappa_{p+l},\varkappa_{p+l},\varkappa_{p+l-1})+\kappa S_b(\varkappa_{p+l-1},\varkappa_{p+l-1},\varkappa_p) \\ &\preceq 2\kappa S_b(\varkappa_{p+l},\varkappa_{p+l},\varkappa_{p+l-1})+2\kappa^2S_b(\varkappa_{p+l-1},\varkappa_{p+l-1},\varkappa_{p+l-2}) \end{aligned}$$

$$
\begin{aligned}
& +\kappa^2 S_b(\varkappa_{p+l-2},\varkappa_{p+l-2},\varkappa_p) \\
& \vdots \\
\preceq\ & 2\kappa S_b(\varkappa_{p+l},\varkappa_{p+l},\varkappa_{p+l-1})+2\kappa^2 S_b(\varkappa_{p+l-1},\varkappa_{p+l-1},\varkappa_{p+l-2}) \\
& +2\kappa^3 S_b(\varkappa_{p+l-2},\varkappa_{p+l-2},\varkappa_{p+l-3})+\ldots+2\kappa^l S_b(\varkappa_{p+1},\varkappa_{p+1},\varkappa_p) \\
\preceq\ & 2\kappa(a^*)^{p+l-1}S_b(\varkappa_0,\varkappa_0,\varkappa_1)(a)^{p+l-1}+2\kappa^2(a^*)^{p+l-2}S_b(\varkappa_0,\varkappa_0,\varkappa_1)(a)^{p+l-2} \\
& +2\kappa^3(a^*)^{p+l-3}S_b(\varkappa_0,\varkappa_0,\varkappa_1)(a)^{p+l-3}+\ldots++2\kappa^l(a^*)^p S_b(\varkappa_0,\varkappa_0,\varkappa_1)(a)^p \\
\preceq\ & 2\sum_{i=1}^{l-1}\kappa^i(a^*)^{p+l-i}S_b(\varkappa_0,\varkappa_0,\varkappa_1)(a)^{p+l-i} \\
=\ & 2\sum_{i=1}^{l-1}\kappa^i(a^*)^{p+l-i}X_0(a)^{p+l-i} \\
=\ & 2\sum_{i=1}^{l-1}\left((a^*)^{p+l-i}\kappa^{\frac{i}{2}}X_0^{\frac{1}{2}}\right)\left(X_0^{\frac{1}{2}}\kappa^{\frac{i}{2}}(a)^{p+l-i}\right) \\
\preceq\ & 2\sum_{i=1}^{l-1}\left(X_0^{\frac{1}{2}}\kappa^{\frac{i}{2}}a^{p+l-i}\right)^*\left(X_0^{\frac{1}{2}}\kappa^{\frac{i}{2}}(a)^{p+l-i}\right) \\
\leq\ & 2\sum_{i=1}^{l-1}||X_0^{\frac{1}{2}}\kappa^{\frac{i}{2}}a^{p+l-i}||^2 1_{\mathfrak{A}} \\
\leq\ & 2||X_0||\sum_{i=1}^{l-1}||a||^{2(p+l-i)}||\kappa||^i 1_{\mathfrak{A}} \\
\leq\ & 2||X_0||\frac{||\kappa||^l||a||^{2(p+1)}}{||\kappa||-||a||^2}1_{\mathfrak{A}}\to 0 \text{ as } p\to\infty.
\end{aligned}
$$

in which $1_{\mathfrak{A}}$ is the unit element in $\mathfrak{A}$. As $\{\varkappa_p\}$ is a Cauchy sequence in $\mathscr{G}$ , and $\mathscr{G}$ is complete, there exists $\varkappa\in\mathscr{G}$ such that $\lim_{p\to\infty}\varkappa_p=\varkappa=\lim_{p\to\infty}\varkappa_{p+1}$ and

$$\lim_{p\to\infty}S_b(\varkappa_p,\varkappa_p,\varkappa)=S_b(\varkappa,\varkappa,\varkappa)=\lim_{p,l\to\infty}S_b(\varkappa_p,\varkappa_p,\varkappa_l)=0. \tag{3.4}$$

Now, we shall show that $\Gamma\varkappa=\varkappa=\Lambda\varkappa$

By hypotheses (iv), First we assume that $\Gamma$ and $\Lambda$ are continuous, then using (3.4), we have

$$\lim_{p\to\infty}S_b(\varkappa_{p+1},\varkappa_{p+1},\varkappa)=\lim_{p\to\infty}S_b(\Gamma\varkappa_p,\Gamma\varkappa_p,\varkappa)=S_b(\Gamma\varkappa,\Gamma\varkappa,\varkappa)=0.$$

Similarly,

$$\lim_{p\to\infty}S_b(\varkappa_{p+2},\varkappa_{p+2},\varkappa)=\lim_{p\to\infty}S_b(\Lambda\varkappa_{p+1},\Lambda\varkappa_{p+1},\varkappa)=S_b(\Lambda\varkappa,\Lambda\varkappa,\varkappa)=0.$$

This implies that $\Gamma\varkappa = \Lambda\varkappa = \varkappa$. Hence, the pair $(\Gamma,\Lambda)$ has a common fixed point $\varkappa \in \mathscr{G}$.

Now, consider that $\mathscr{G}$ is $(\alpha,\beta)$-regular space then there exists a subsequence $\{\varkappa_{p_z}\}$ of $\{\varkappa_p\}$ such that $\alpha\left(\varkappa_{p_z},\varkappa_{p_z},\varkappa_{p_{z+1}}\right) \succeq 1_{\mathfrak{A}}$ and $\beta\left(\varkappa_{p_z},\varkappa_{p_z},\varkappa_{p_{z+1}}\right) \succeq 1_{\mathfrak{A}}$ for each $z \in N$ and $\alpha(\varkappa,\varkappa,\Gamma\varkappa) \succeq 1_{\mathfrak{A}}$, $\beta(\varkappa,\varkappa,\Lambda\varkappa) \succeq 1_{\mathfrak{A}}$.

From (3.1), we have

$$\eta_b\left(\kappa\alpha(\varkappa_{p_z},\varkappa_{p_z},\Gamma\varkappa_{p_z})\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\varkappa_{p_z},\Gamma\varkappa_{p_z},\Lambda\varkappa),\varphi(M^4(\varkappa_{p_z},\varkappa))a^*M^4(\varkappa_{p_z},\varkappa)a\right) \succeq 0_{\mathfrak{A}}$$

From $(\eta_{b_1})$ and $(\eta_{b_2})$, we have

$$\kappa\alpha(\varkappa_{p_z},\varkappa_{p_z},\Gamma\varkappa_{p_z})\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\varkappa_{p_z},\Gamma\varkappa_{p_z},\Lambda\varkappa) \prec \varphi(M^4(\varkappa_{p_z},\varkappa))a^*M^4(\varkappa_{p_z},\varkappa)a$$

This further implies that

$$\begin{aligned} S_b(\varkappa_{p_{z+1}},\varkappa_{p_{z+1}},\Lambda\varkappa) &= S_b(\Gamma\varkappa_{p_z},\Gamma\varkappa_{p_z},\Lambda\varkappa) \\ &\preceq \kappa\alpha(\varkappa_{p_z},\varkappa_{p_z},\Gamma\varkappa_{p_z})\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\varkappa_{p_z},\Gamma\varkappa_{p_z},\Lambda\varkappa) \\ &\prec \varphi(M^4(\varkappa_{p_z},\varkappa))a^*M^4(\varkappa_{p_z},\varkappa)a \\ &\preceq a^*M^4(\varkappa_{p_z},\varkappa)a \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} M^4\left(\varkappa_{p_z},\varkappa\right) &= \max\left\{\begin{array}{c} S_b\left(\varkappa_{p_z},\varkappa_{p_z},\varkappa\right),S_b\left(\varkappa_{p_z},\varkappa_{p_z},\Gamma\varkappa_{p_z}\right), \\ S_b(\varkappa,\varkappa,\Lambda\varkappa),\frac{S_b\left(\varkappa_{p_z},\varkappa_{p_z},\Lambda\varkappa\right)+S_b\left(\varkappa,\varkappa,\Gamma\varkappa_{p_z}\right)}{2\kappa^4} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} S_b\left(\varkappa_{p_z},\varkappa_{p_z},\varkappa\right),S_b\left(\varkappa_{p_z},\varkappa_{p_z},\varkappa_{p_{z+1}}\right), \\ S_b(\varkappa,\varkappa,\Lambda\varkappa),\frac{S_b\left(\varkappa_{p_z},\varkappa_{p_z},\Lambda\varkappa\right)+S_b\left(\varkappa,\varkappa,\varkappa_{p_{z+1}}\right)}{2\kappa^4} \end{array}\right\} \end{aligned}$$

Taking limit as $z \to \infty$ in above expression, we get

$$\lim_{n\to\infty} M^4\left(\varkappa_{p_z},\varkappa\right) = \max\left\{\begin{array}{c} S_b(\varkappa,\varkappa,\Lambda\varkappa), \\ \frac{S_b(\varkappa,\varkappa,\Lambda\varkappa)}{2\kappa^4} \end{array}\right\} = S_b(\varkappa,\varkappa,\Lambda\varkappa) \tag{3.6}$$

Therefore, taking limit as $z \to \infty$ in (3.5) and using (3.6), we get

$$\begin{aligned} ||S_b(\varkappa,\varkappa,\Lambda\varkappa)|| = \lim_{n\to\infty} ||S_b(\varkappa_{p_{z+1}},\varkappa_{p_{z+1}},\Lambda\varkappa)|| &\leq \lim_{n\to\infty} ||\varphi(M^4(\varkappa_{p_z},\varkappa))a^*M^4(\varkappa_{p_z},\varkappa)a|| \\ &\leq ||a||^2||S_b(\varkappa,\varkappa,\Lambda\varkappa)|| \end{aligned}$$

since $||a|| < 1$,this implies $\lim_{n\to\infty} ||\varphi(M^4(\varkappa_{p_z},\varkappa))|| = 1$ and therefore, $\lim_{n\to\infty} ||M^4(\varkappa_{p_z},\varkappa)|| = 0$. Thus, we obtain $||S_b(\varkappa,\varkappa,\Lambda\varkappa)|| = 0$ implies that $\Lambda\varkappa = \varkappa$. Similarly, we get $\Gamma\varkappa = \varkappa$ and we conclude that $\varkappa$ is a common fixed point of $\Gamma$ and $\Lambda$.

Now, for the uniqueness part, let $\varkappa$ and $\ell$ are two common fixed points of $\Gamma$ and $\Lambda$ and $\varkappa \neq \ell$. Also $\alpha(\varkappa,\varkappa,\Gamma\varkappa) \succeq 1_{\mathfrak{A}}$, $\alpha(\ell,\ell,\Gamma\ell) \succeq 1_{\mathfrak{A}}$ and $\beta(\varkappa,\varkappa,\Lambda\varkappa) \succeq 1_{\mathfrak{A}}$, $\beta(\ell,\ell,\Lambda\ell) \succeq 1_{\mathfrak{A}}$. By (3.1), we have

$$\eta_b\left(\kappa\alpha(\varkappa,\varkappa,\Gamma\varkappa)\beta(\ell,\ell,\Lambda\ell)S_b(\Gamma\varkappa,\Gamma\varkappa,\Lambda\ell),\varphi(M^4(\varkappa,\ell))a^*M^4(\varkappa,\ell)a\right) \succeq 0_{\mathfrak{A}}.$$

From $(\eta_{b_1})$, we have

$$\kappa\alpha(\varkappa,\varkappa,\Gamma\varkappa)\beta(\ell,\ell,\Lambda\ell)S_b(\Gamma\varkappa,\Gamma\varkappa,\Lambda\ell) \prec \varphi(M^4(\varkappa,\ell))a^*M^4(\varkappa,\ell)a \tag{3.7}$$

where

$$\begin{aligned} M^4(\varkappa,\ell) &= \max\left\{\begin{array}{c} S_b(\varkappa,\varkappa,\ell), S_b(\varkappa,\varkappa,\Gamma\varkappa), \\ S_b(\ell,\ell,\Lambda\ell), \frac{S_b(\varkappa,\varkappa,\Lambda\ell)+S_b(\ell,\ell,\Gamma\varkappa)}{2\kappa^4} \end{array}\right\} \\ &= S_b(\varkappa,\varkappa,\ell) \end{aligned}$$

Thus, by using (3.7), we have

$$\begin{aligned} S_b(\varkappa,\varkappa,\ell) &= S_b(\Gamma\varkappa,\Gamma\varkappa,\Lambda\ell) \preceq \kappa\alpha(\varkappa,\varkappa,\Gamma\varkappa)\beta(\ell,\ell,\Lambda\ell)S_b(\Gamma\varkappa,\Gamma\varkappa,\Lambda\ell) \\ &\prec \varphi(M^4(\varkappa,\ell))a^*M^4(\varkappa,\ell)a \\ &\preceq a^*M^4(\varkappa,\ell)a \\ &\preceq a^*S_b(\varkappa,\varkappa,\ell)a. \end{aligned} \tag{3.8}$$

Therefore,

$$||S_b(\varkappa,\varkappa,\ell)|| \leq ||a||^2||S_b(\varkappa,\varkappa,\ell)||$$

This is incongruous. Consequently, $\vartheta = \ell$. Therefore, the UCFP of $\Gamma$ is $\vartheta$. □

**Theorem 3.6.** *A complete $C^*$-AV-$S_b$MS with the coefficient $||\kappa|| > 1$ is defined as $(\mathscr{G},\mathfrak{A},S_b)$. Let $\Gamma,\Lambda : \mathscr{G} \to \mathscr{G}$ be a $(\alpha,\beta)$-$Z_b$-Geraghty type generalized contraction mappings with $r = 3,2,1$, assuming that all the requirements in Theorem (3.5) are true. Then, in $\mathscr{G}$, there is a UCFP of $\Gamma$ and $\Lambda$.*

*Proof.* If we substitute $M^3(\ell,\varkappa)$, $M^2(\ell,\varkappa)$ or $M^1(\ell,\varkappa)$ for $M^4(\ell,\varkappa)$ in Theorem (3.5), then follows in a manner similar to Theorem (3.5). □

**Theorem 3.7.** *A complete $C^*$-AV-$S_b$MS with the coefficient $||\kappa|| > 1$ is defined as $(\mathscr{G},\mathfrak{A},S_b)$. Let $\Gamma:\mathscr{G}\to\mathscr{G}$ be a $(\alpha,\beta)$-$Z_b$-Geraghty type generalized contraction mappings with $r = 3,2,1$, assuming that all the requirements in Theorem (3.5) are true. Then, in $\mathscr{G}$, there is a UCFP of $\Gamma$.*

*Proof.* If we substitute $M^3(\ell,\varkappa)$, $M^2(\ell,\varkappa)$ or $M^1(\ell,\varkappa)$ for $M^4(\ell,\varkappa)$ in Theorem (3.5), and take $\Lambda = I_{\mathscr{G}}$ then follows in a manner similar to Theorem (3.5). □

**Corollary 3.8.** *Let $(\mathscr{G},\mathfrak{A},S_b)$ be a complete $C^*$-AV-$S_b$MS, $\ell,\beta:\mathscr{G}\times\mathscr{G}\to\mathfrak{A}_+$ be two functions and $\Gamma:\mathscr{G}\to\mathscr{G}$ be a mapping with the following assumptions:*

*(i)* *$\Gamma$ is a pair of $(\alpha,\beta)$-admissible mappings;*

*(ii)* *there exist $\ell,\varkappa\in\mathscr{G}$ and $a\in\mathfrak{A}$ with $||a|| < 1$ such that*

$$\eta_b\left(\kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Gamma\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Gamma\varkappa),\varphi(S_b(\ell,\ell,\varkappa))a^*S_b(\ell,\ell,\varkappa)a\right)\succeq 0_{\mathfrak{A}}$$

*where, $\varphi$ is a Geraghty function and $\eta_b\in\mathscr{Z}_b$;*

*(iii)* *there exist $\ell_0\in\mathscr{G}$ such that $\alpha(\ell_0,\ell_0,\Gamma\ell_0)\succeq 1_{\mathfrak{A}}$, $\beta(\ell_0,\ell_0,\Gamma\ell_0)\succeq 1_{\mathfrak{A}}$;*

*(iv)* *either $\Gamma$ is continuous or $\mathscr{G}$ is $(\alpha,\beta)$-regular space.*

*Then $\Gamma$ has a unique fixed point in $\mathscr{G}$ .*

*Proof.* The proof follows from Theorem (3.5) by taking $\Gamma = \Lambda$. □

**Example 3.9.** *Let $\mathscr{G} = [0,\infty)$ and $\mathfrak{A} = M_2(\mathbb{R})$ be all $2\times 2$ matrices whose norm is defined in Example (2.2) and define the mapping $d:\mathscr{G}^2\to[0,\infty)$ as*
*$d(\ell,\varkappa) = (\ell-\varkappa)^2$ for all $\ell,\varkappa\in\mathscr{G}$. Then clearly, $(\mathscr{G},d)$ is b-metric space with $\kappa = 2$. Let $S_b:\mathscr{G}^3\to M_2(\mathbb{R})$ be as $S_b(p,q,r) = \begin{bmatrix} d(p,q)+d(q,r)+d(r,p) & 0 \\ 0 & d(p,q)+d(q,r)+d(r,p)\end{bmatrix}$.*
*Then, clearly $(\mathscr{G},\mathfrak{A},S_b)$ is a complete $C^*$-AV-$S_b$MS with $||\kappa|| = 2\geq 1$.*
*Let $\eta_b:\mathfrak{A}_+\times\mathfrak{A}_+\to\mathfrak{A}$ defined by $\eta_b(\aleph,\wp) = \frac{\wp}{1+\wp}-\aleph$ and $a\in\mathfrak{A}$ with $||a|| = \frac{1}{\sqrt{6}} < 1$. We define mappings $\Gamma,\Lambda:\mathscr{G}\to\mathscr{G}$, and Geraghty function $\varphi:\mathfrak{A}_+\to[0,1)$ as follows $\Gamma(\ell) = \frac{b\ell}{3+\ell}$, $\Lambda(\ell) = \frac{b\ell}{2+\ell}$ for all $\ell\in\mathscr{G}$ with $b\in(0,\frac{1}{6}]$*

*and* $\varphi(\ell) = \begin{cases} \frac{1}{1+||\ell||} & \ell > 0_{\mathfrak{A}} \\ 0 & \ell = 0_{\mathfrak{A}} \end{cases}$. *Also, we define* $\alpha, \beta : \mathscr{G} \times \mathscr{G} \times \mathscr{G} \to \mathfrak{A}_+$ *as* $\alpha(\aleph, \aleph, \wp) =$ $\beta(\aleph, \aleph, \wp) = \begin{cases} 1_{\mathfrak{A}} & \text{if } \aleph, \wp \in [0,1] \\ 0_{\mathfrak{A}} & \text{Otherwise} \end{cases}$.

*Now for* $\alpha(\aleph, \aleph, \wp) \succeq 1_{\mathfrak{A}}$ *and* $\beta(\aleph, \aleph, \wp) \succeq 1_{\mathfrak{A}}$, *then we have* $\aleph, \wp \in [0,1]$, *implies it follows that* $\alpha(\Gamma\aleph, \Gamma\aleph, \Lambda\wp) \succeq 1_{\mathfrak{A}}$, $\alpha(\Gamma\wp, \Gamma\wp, \Lambda\aleph) \succeq 1_{\mathfrak{A}}$,*and* $\beta(\Gamma\aleph, \Gamma\aleph, \Lambda\wp) \succeq 1_{\mathfrak{A}}$, $\beta(\Gamma\wp, \Gamma\wp, \Lambda\aleph) \succeq 1_{\mathfrak{A}}$. *Therefore,* $(\Gamma, \Lambda)$ *is a pair of* $(\alpha, \beta)$*-admissible mapping.*

*Furthermore, if* $\{\wp_p\}$ *is a sequence in* $\mathscr{G}$ *such that* $\alpha(\wp_p, \wp_p, \wp_{p+1}) \succeq 1_{\mathfrak{A}}$ *and* $\beta(\wp_p, \wp_p, \wp_{p+1}) \succeq 1_{\mathfrak{A}}$ *then* $\{\wp_p\} \subseteq [0,1]$. *Suppose* $\wp_p \to \wp$ *then* $\wp \in [0,1]$ *and implies that* $\alpha(\wp, \wp, \Gamma\wp) = 1_{\mathfrak{A}}$ *and* $\beta(\wp, \wp, \Lambda\wp) = 1_{\mathfrak{A}}$.

*Indeed, we obtain*

$$\begin{aligned}
& \eta_b\left(\kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Lambda\varkappa), \varphi(M^4(\ell,\varkappa))a^*M^4(\ell,\varkappa)a\right) \\
= \; & \frac{\varphi(M^4(\ell,\varkappa))a^*M^4(\ell,\varkappa)a}{1+\varphi(M^4(\ell,\varkappa))a^*M^4(\ell,\varkappa)a} - \kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Lambda\varkappa) \\
= \; & \frac{\frac{a^*M^4(\ell,\varkappa)a}{1+M^4(\ell,\varkappa)}}{1+\frac{a^*M^4(\ell,\varkappa)a}{1+M^4(\ell,\varkappa)}} - \kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Lambda\varkappa) \\
= \; & \frac{a^*M^4(\ell,\varkappa)a}{1+M^4(\ell,\varkappa)+a^*M^4(\ell,\varkappa)a} - \kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Lambda\varkappa) \\
\succeq \; & \frac{a^*S_b(\ell,\ell,\varkappa)a}{1+S_b(\ell,\ell,\varkappa)+a^*S_b(\ell,\ell,\varkappa)a} - \kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Lambda\varkappa)
\end{aligned}$$

*Therefore, we have*

$$\begin{aligned}
& ||\eta_b\left(\kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Lambda\varkappa), \varphi(M^4(\ell,\varkappa))a^*M^4(\ell,\varkappa)a\right)|| \\
\geq \; & ||\frac{a^*S_b(\ell,\ell,\varkappa)a}{1+S_b(\ell,\ell,\varkappa)+a^*S_b(\ell,\ell,\varkappa)a} - \kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Lambda\varkappa)|| \\
\geq \; & ||\frac{a^*S_b(\ell,\ell,\varkappa)a}{1+S_b(\ell,\ell,\varkappa)+a^*S_b(\ell,\ell,\varkappa)a}|| - ||\kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Lambda\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Lambda\varkappa)|| \\
\geq \; & \frac{||a||^2||S_b(\ell,\ell,\varkappa)||}{1+||S_b(\ell,\ell,\varkappa)||+||a||^2||S_b(\ell,\ell,\varkappa)||} - 2||S_b(\Gamma\ell,\Gamma\ell,\Lambda\varkappa)|| \\
\geq \; & \frac{\sqrt{2}(\ell-\varkappa)^2}{1+3\sqrt{2}(\ell-\varkappa)^2} - 36\sqrt{2}b^2\frac{(\ell-\varkappa)^2}{[(3+\ell)(2+\varkappa)]^2}
\end{aligned}$$

$$\geq \frac{\sqrt{2}(\ell-\varkappa)^2}{1+3\sqrt{2}(\ell-\varkappa)^2} - 36\sqrt{2}b^2\frac{(\ell-\varkappa)^2}{1+3\sqrt{2}(\ell-\varkappa)^2}$$
$$\geq \frac{(1-36b^2)\sqrt{2}(\ell-\varkappa)^2}{1+3\sqrt{2}(\ell-\varkappa)^2} \geq 0 \ \forall\ \ell,\varkappa \in \mathscr{G}$$

*Thus all the conditions of Theorem 3.5 are satisfied. Hence $\Gamma$ and $\Lambda$ have a common fixed point (at $\ell = 0$).*

## 4. Application to Integral Equations

Suppose that $\mathscr{E} = [0,1]$ be a Lebesgue measurable set with $m(\mathscr{E}) < \infty$ such that $\mathscr{G} = L^\infty(\mathscr{E})$ and $B(L^2(\mathscr{E}))$ is a set of bounded linear operators on a Hilbert space $L^2(\mathscr{E})$. We equip $\mathscr{G}$ with $S_b : \mathscr{G}\times\mathscr{G}\times\mathscr{G} \to B(L^2(\mathscr{E}))$, which is ascertained by $S_b(æ, ß, œ) = \mathbb{M}_{(|æ-œ|+|ß-œ|)^p}$, where $\mathbb{M}_{(|æ-œ|+|ß-œ|)^p}$ is the multiplication operator on $L^2(\mathscr{E})$ ascertained by $\mathbb{M}_h(\alpha) = h.\alpha$, $\alpha \in L^2(\mathscr{E})$. Therefore, $(\mathscr{G}, B(L^2(\mathscr{E})), S_b)$ is a complete $C^*$-AV-$S_b$MS with $\kappa = 2^{2(p-1)}$ where $p > 1$. Let us consider the two-point boundary value problem of the second order differential equation:

$$\frac{d^2\ell(s)}{dt^2} = \mathfrak{f}(s,t,\ell(t)) \ \forall\ s \in [0,1] \text{ and } \ell(0) = 1 = \ell(1) \tag{4.1}$$

where $\mathfrak{f} : \mathscr{E}\times\mathscr{E}\times\mathbb{R} \to \mathbb{R}$ is integrable.

The inequality (4.1) is equivalent to the following integral equation:

$$\ell(s) = \int_0^1 \mathscr{K}(s,t)\mathfrak{f}(s,t,\ell(t))dt \text{ for all } s \in [0,1].$$

where Green function associated to (4.1) is defined by $\mathscr{K}(s,t) = \begin{cases} s(1-t) & \text{if } 0 \leq s \leq t \\ t(1-s) & \text{if } t \leq s \leq 1 \end{cases}$

Assume that the following conditions hold:

$(i_0)$ $\Phi, \Psi : \mathbb{R}^3 \to \mathfrak{A}_+$ and $\lambda \in (0,1)$ such that $|\mathfrak{f}(s,t,a)| + |\mathfrak{f}(s,t,b)| \leq \lambda||a-b||$ for all $s,t \in \mathscr{E}$ and $a,b \in \mathbb{R}$ with $\Phi(a,a,b) \succ 0_{\mathfrak{A}}$ and $\Psi(a,a,b) \succ 0_{\mathfrak{A}}$

$(i_1)$ $\exists\ \ell_0 \in \mathscr{G} \ni \Phi(\ell_0(s), \ell_0(s), \Gamma\ell_0(s)) \succeq 0_{\mathfrak{A}}$ and $\Psi(\ell_0(s), \ell_0(s), \Gamma\ell_0(s)) \succeq 0_{\mathfrak{A}}$ for all $s \in \mathscr{E}$ where $\Gamma : \mathscr{G} \to \mathscr{G}$ is defined as

$$\Gamma\ell(s) = \int_0^1 \mathscr{K}(s,t)\mathfrak{f}(s,t,\ell(t))dt \text{ for all } s \in [0,1].$$

$(i_2)$ for each $s \in \mathscr{E}$ and $\ell, \varkappa \in \mathscr{G}$, $\Phi(\ell(s), \ell(s), \varkappa(s)) \succ 0_{\mathfrak{A}} \Rightarrow \Phi(\Gamma\ell(s), \Gamma\ell(s), \Gamma\varkappa(s)) \succ 0_{\mathfrak{A}}$ and $\Psi(\ell(s), \ell(s), \varkappa(s)) \succ 0_{\mathfrak{A}} \Rightarrow \Psi(\Gamma\ell(s), \Gamma\ell(s), \Gamma\varkappa(s)) \succ 0_{\mathfrak{A}}$

$(i_3)$ for each $s \in \mathscr{E}$ and $\{\ell_p\} \subseteq \mathscr{G}$ be a sequence such that $\ell_p \to \ell$ in $\mathscr{G}$ and $\Phi(\ell_p(s), \ell_p(s), \ell_{p+1}(s)) \succ 0_{\mathfrak{A}}$, $\Psi(\ell_p(s), \ell_p(s), \ell_{p+1}(s)) \succ 0_{\mathfrak{A}}$ for all $p \in N$ then $\Phi(\ell_p(s), \ell_p(s), \ell(s)) \succ 0_{\mathfrak{A}}$, $\Psi(\ell_p(s), \ell_p(s), \ell(s)) \succ 0_{\mathfrak{A}}$ for all $p \in N$

**Theorem 4.1.** *Under the assumption $(i_0)$-$(i_4)$, the equation (4.1) has a solution in in $L^\infty(\mathscr{E})$.*

*Proof.* Define $\Gamma : \mathscr{G} \to \mathscr{G}$ by $\Gamma\ell(s) = \int_0^1 \mathscr{K}(s,t)\mathfrak{f}(s,t,\ell(t))dt$ for all $s \in [0,1]$.
Now for all $\ell, \varkappa \in \mathscr{G}$, $\Phi(\ell(s), \ell(s), \varkappa(s)) \succ 0_{\mathfrak{A}}$ and $\Psi(\ell(s), \ell(s), \varkappa(s)) \succ 0_{\mathfrak{A}}$ $\forall\ s \in [0,1]$. we have

$$S_b(\Gamma\ell, \Gamma\ell, \Gamma\varkappa) = \mathbb{M}_{(2|\Gamma\ell - \Gamma\varkappa|)^p}$$

We obtain that

$$\begin{aligned}
||S_b(\Gamma\ell, \Gamma\ell, \Gamma\varkappa)|| &= \sup_{||h||=1} \left\langle \mathbb{M}_{(2|\Gamma\ell-\Gamma\varkappa|)^p} h, h \right\rangle \\
&= \sup_{||h||=1} \left\langle 2^p \mathbb{M}_{|\Gamma\ell-\Gamma\varkappa|^p} h, h \right\rangle \\
&= \sup_{||h||=1} \int_0^1 (2^p |\Gamma\ell(t) - \Gamma\varkappa(t)|^p) h(t)\overline{h(t)} dt \\
&\leq 2^p \sup_{||h||=1} \int_0^1 \left[ |\int_0^1 \mathscr{K}(s,t)\mathfrak{f}(s,t,\ell(t))| - |\int_0^1 \mathscr{K}(s,t)\mathfrak{f}(s,t,\varkappa(t))| \right]^p |h(t)|^2 dt \\
&\leq 2^p \sup_{||h||=1} \int_0^1 \left[ \int_0^1 \mathscr{K}(s,t) \left( |\mathfrak{f}(s,t,\ell(t))| + |\mathfrak{f}(s,t,\varkappa(t))| \right) \right]^p |h(t)|^2 dt \\
&\leq 2^p \sup_{||h||=1} \int_0^1 \left[ \int_0^1 \mathscr{K}(s,t) dt \right]^p |h(t)|^2 \lambda^p ||\ell - \varkappa||_\infty^p \\
&\leq \lambda \sup_{s\in[0,1]} \left[ \int_0^1 \mathscr{K}(s,t) dt \right]^p \sup_{||h||=1} \int_0^1 |h(t)|^2 ||2(\ell - \varkappa)||_\infty^p
\end{aligned}$$

$$\leq \lambda \sup_{s\in[0,1]} \left[\int_0^1 \mathscr{K}(s,t)dt\right]^p ||\mathbb{M}_{(|\ell-\varkappa|+|\ell-\varkappa|)^p}||$$
$$\leq ||a||^2 \sup_{s\in[0,1]} \left[\int_0^1 \mathscr{K}(s,t)dt\right]^p ||S_b(\ell,\ell,\varkappa)||$$

By setting $a=\lambda 1_{B(L^2(\mathscr{E}))}$, then $a\in B(L^2(\mathscr{E}))$ and $||a||=\lambda<1$,
since $\int_0^1 \mathscr{K}(s,t)dt = -\frac{s^2}{2}+\frac{s}{2}$ for all $s\in[0,1]$, we have $\sup_{s\in[0,1]}\left[\int_0^1 \mathscr{K}(s,t)dt\right]^p = (\frac{1}{8})^p$, then it follows that

$$||S_b(\Gamma\ell,\Gamma\ell,\Gamma\varkappa)|| \leq ||a||^2(\tfrac{1}{8})^p||S_b(\ell,\ell,\varkappa)|| \tag{4.2}$$

Let $\eta_b:\mathfrak{A}_+\times\mathfrak{A}_+\to\mathfrak{A}$ as $\eta_b(x,y)=\frac{5}{6}y-x\;\forall\, x,y\in\mathfrak{A}_+$ and Geraghty function $\varphi:\mathfrak{A}_+\to[0,1)$ is defined by $||\varphi(x)||=\frac{3}{4}$, for all $x\in\mathfrak{A}_+$. For $t\in[0,1]$ the following is defined: $\alpha,\beta:\mathscr{G}\times\mathscr{G}\times\mathscr{G}\to\mathfrak{A}_+$ as $\alpha(\ell,\ell,\varkappa)=\begin{cases}1_{\mathfrak{A}} & \text{if } \Phi(\ell(t),\ell(t),\varkappa(t))\succ 0_{\mathfrak{A}}\\ 0_{\mathfrak{A}} & \text{Otherwise}\end{cases}$
and $\beta(\ell,\ell,\varkappa)=\begin{cases}1_{\mathfrak{A}} & \text{if } \Psi(\ell(t),\ell(t),\varkappa(t))\succ 0_{\mathfrak{A}}\\ 0_{\mathfrak{A}} & \text{Otherwise}\end{cases}$.
From (4.2), we have

$$\begin{aligned}||\kappa S_b(\Gamma\ell,\Gamma\ell,\Gamma\varkappa)|| &= ||2^{2(p-1)}S_b(\Gamma\ell,\Gamma\ell,\Gamma\varkappa)|| \leq 2^{2(p-1)}||a||^2(\frac{1}{8})^p||S_b(\ell,\ell,\varkappa)||\\ &\leq \frac{5}{8}||a||^2||S_b(\ell,\ell,\varkappa)||\end{aligned} \tag{4.3}$$

Now, using (4.3), we get

$$\frac{5}{6}||\varphi(S_b(\ell,\ell,\varkappa))||||a||^2||S_b(\ell,\ell,\varkappa)|| - ||\alpha(\ell,\ell,\Gamma\ell)||||\beta(\varkappa,\varkappa,\Gamma\varkappa)||||\kappa S_b(\Gamma\ell,\Gamma\ell,\Gamma\varkappa)||$$
$$=\frac{5}{8}||a||^2|||S_b(\ell,\ell,\varkappa)|| - ||\kappa S_b(\Gamma\ell,\Gamma\ell,\Gamma\varkappa)|| \geq 0$$

Hence

$$\eta_b\left(\kappa\alpha(\ell,\ell,\Gamma\ell)\beta(\varkappa,\varkappa,\Gamma\varkappa)S_b(\Gamma\ell,\Gamma\ell,\Gamma\varkappa),\varphi(S_b(\ell,\ell,\varkappa))a^*S_b(\ell,\ell,\varkappa)a\right)\succeq 0_{\mathfrak{A}}.$$

Therefore the mapping $\Gamma$ is $(\alpha,\beta)$-$Z_b$-Geraghty type contraction. From $(i_1)$, there exists $\ell_0\in\mathscr{G}$ such that $\alpha(\ell_0(s),\ell_0(s),\Gamma\ell_0(s))\succeq 0_{\mathfrak{A}}$ and $\beta(\ell_0(s),\ell_0(s),\Gamma\ell_0(s))\succeq 0_{\mathfrak{A}}$.

Now using $(i_2)$, we get

$$\begin{aligned}\alpha(\ell,\ell,\varkappa)\succeq 1_{\mathfrak{A}} &\Rightarrow \Phi(\ell(t),\ell(t),\varkappa(t))\succ 0_{\mathfrak{A}}\\ &\Rightarrow \Phi(\Gamma\ell(t),\Gamma\ell(t),\Gamma\varkappa(t))\succ 0_{\mathfrak{A}}\\ &\Rightarrow \alpha(\Gamma\ell,\Gamma\ell,\Gamma\varkappa)\succ 1_{\mathfrak{A}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\beta(\ell,\ell,\varkappa)\succeq 1_{\mathfrak{A}} &\Rightarrow \Psi(\ell(t),\ell(t),\varkappa(t))\succ 0_{\mathfrak{A}}\\ &\Rightarrow \Psi(\Gamma\ell(t),\Gamma\ell(t),\Gamma\varkappa(t))\succ 0_{\mathfrak{A}}\\ &\Rightarrow \beta(\Gamma\ell,\Gamma\ell,\Gamma\varkappa)\succ 1_{\mathfrak{A}}.\end{aligned}$$

So, the mapping $\Gamma$ is $(\alpha,\beta)$-admissible. Therefore, all the hypotheses of Corollary 3.8 are satisfied. Hence, $\Gamma$ must have a fixed point in $L^\infty(\mathscr{E})$ (say $\ell$), which is a solution of (4.1). □

## 5. Application to Homotopy

In this section, we investigate whether homotopy could have a unique solution.

**Theorem 5.1.** *If $(\mathscr{G},\mathfrak{A},S_b)$ is a complete $C^*$-AV-$S_b$MS, then $\mathfrak{U}$ and $\overline{\mathfrak{U}}$ are open and closed subsets of $\mathscr{G}$, respectively, such that $\mathfrak{U}\subseteq\overline{\mathfrak{U}}$. Let $\mathfrak{H}_b:\overline{\mathfrak{U}}\times[0,1]\to\mathscr{G}$ be an homotopy operator meeting the requirements listed below.*

- $(\tau_0)$ *$\ell\neq\mathfrak{H}_b(\ell,s)$, for each $\ell\in\partial\mathfrak{U}$ and $s\in[0,1]$ (here $\partial\mathfrak{U}$ is boundary of $\mathfrak{U}$ in $\mathscr{G}$)*
- $(\tau_1)$ *there exist $\ell,\varkappa\in\overline{\mathfrak{U}}$ and $a\in\mathfrak{A}$ with $||a||<1$ such that*

$$\eta_b\left(2\kappa^2\alpha(\ell,\ell,\mathfrak{H}_b(\ell,s))\beta(\varkappa,\varkappa,\mathfrak{H}_b(\varkappa,s))S_b(\mathfrak{H}_b(\ell,s),\mathfrak{H}_b(\ell,s),\mathfrak{H}_b(\varkappa,s)),\varphi(S_b(\ell,\ell,\varkappa))a^*S_b(\ell,\ell,\varkappa)a\right)\succeq 0_{\mathfrak{A}}$$

  *where, $\varphi$ is a Geraghty function and $\eta_b\in\mathscr{Z}_b$ and $s\in[0,1]$;*
- $(\tau_2)$ *$\exists\, M_b\succeq 0_{\mathfrak{A}}\ni S_b(\mathfrak{H}_b(\ell,s),\mathfrak{H}_b(\ell,s),\mathfrak{H}_b(\ell,t))\leq ||M_b|||s-t|$ for every $\ell\in\overline{\mathfrak{U}}$ and $s,t\in[0,1]$;*
- $(\tau_3)$ *$\mathfrak{H}_b$ is a pair of $(\alpha,\beta)$-admissible mappings and $\mathscr{G}$ is $(\alpha,\beta)$ regular space;*
- $(\tau_4)$ *$\exists\,\ell_0\in\mathscr{G}$ such that $\alpha(\ell_0,\ell_0,\mathfrak{H}_b(\ell_0,s))\succeq 1_{\mathfrak{A}}$, $\beta(\ell_0,\ell_0,\mathfrak{H}_b(\ell_0,s))\succeq 1_{\mathfrak{A}}$.*

*Then $\mathfrak{H}_b(.,0)$ has a fixed point $\iff$ $\mathfrak{H}_b(.,1)$ has a fixed point.*

*Proof.* Take into account the set $\mathfrak{B}=\{s\in[0,1]:\ell=\mathfrak{H}_b(\ell,s)$ for some $\ell\in\mathfrak{U}\}$.

Due to the fact that $\mathfrak{H}_b(,0)$ has a FP in $\Delta$, we have that $0 \in \mathfrak{B}$. The set $\mathfrak{B}$ is not empty as a result. We will prove that $\mathfrak{B} = [0,1]$ by establishing that $\mathfrak{B}$ is both open and closed in $[0,1]$. $\mathfrak{H}_b(,1)$ has a FP in $\mathfrak{U}$ as a result. The first thing we do is show that $\mathfrak{B}$ is closed in $[0,1]$. To observe this, assign $s_p \to s \in [0,1]$ as $p \to \infty$ and let $\{s_p\}_{p=1}^{\infty} \subseteq \mathfrak{B}$. We must show that $s$ is in $\mathfrak{B}$. Given that $s_p$ in $\mathfrak{B}$ for $p = 0,1,2,...$, there is $\ell_p$ in $\mathfrak{U}$ with $\ell_{p+1} = \mathfrak{H}_b(\ell_p, s_p)$. The proof is successful if $p \in N$ exist such that $S_b(\ell_p, \ell_p, \mathfrak{H}_b(\ell_p, s_p)) = 0$.

Since $\mathfrak{H}_b$ is a pair of $(\alpha,\beta)$-admissible mappings, then $\alpha(\ell_0,\ell_0,\ell_1) \succeq 1_{\mathfrak{A}}$ implies $\alpha(\mathfrak{H}_b(\ell_0,s_0),\mathfrak{H}_b(\ell_0,s_0),\mathfrak{H}_b(\ell_1,s_1)) = \alpha(\ell_1,\ell_1,\ell_2) \succeq 1_{\mathfrak{A}}$ and $\beta(\ell_0,\ell_0,\ell_1) \succeq 1_{\mathfrak{A}}$ implies $\beta(\mathfrak{H}_b(\ell_0,s_0),\mathfrak{H}_b(\ell_0,s_0),\mathfrak{H}_b(\ell_1,s_1)) = \beta(\ell_1,\ell_1,\ell_2) \succeq 1_{\mathfrak{A}}$. By repeating similar process, we obtain $\alpha(\ell_p,\ell_p,\ell_{p+1}) \succeq 1_{\mathfrak{A}}$ and $\beta(\ell_p,\ell_p,\ell_{p+1}) \succeq 1_{\mathfrak{A}}$ for every $p \in N$. So, we assume that

$$\eta_b\left(\begin{array}{c} 2\kappa^2\alpha(\ell_p,\ell_p,\ell_{p+1})\beta(\ell_{p+1},\ell_{p+1},\ell_{p+2})S_b(\mathfrak{H}_b(\ell_p,s_p),\mathfrak{H}_b(\ell_p,s_p),\mathfrak{H}_b(\ell_{p+1},s_p)), \\ \varphi(S_b(\ell_p,\ell_p,\ell_{p+1}))a^*S_b(\ell_p,\ell_p,\ell_{p+1})a \end{array}\right) \succeq 0_{\mathfrak{A}}.$$

From $(\eta_{b_1})$, we have

$$\begin{aligned} & 2\kappa^2\alpha(\ell_p,\ell_p,\mathfrak{H}_b(\ell_p,s_p))\beta(\ell_{p+1},\ell_{p+1},\mathfrak{H}_b(\ell_{p+1},s_{p+1}))S_b(\mathfrak{H}_b(\ell_p,s_p),\mathfrak{H}_b(\ell_p,s_p),\mathfrak{H}_b(\ell_{p+1},s_p)) \\ \prec\ & \varphi(S_b(\ell_p,\ell_p,\ell_{p+1}))a^*S_b(\ell_p,\ell_p,\ell_{p+1})a. \end{aligned}$$

Now from above inequality, we see that

$$\begin{aligned} & S_b(\ell_{p+1},\ell_{p+1},\ell_{p+2}) = S_b(\mathfrak{H}_b(\ell_p,s_p),\mathfrak{H}_b(\ell_p,s_p),\mathfrak{H}_b(\ell_{p+1},s_{p+1})) \\ \preceq\ & 2\kappa S_b(\mathfrak{H}_b(\ell_p,s_p),\mathfrak{H}_b(\ell_p,s_p),\mathfrak{H}_b(\ell_{p+1},s_p)) + \kappa S_b(\mathfrak{H}_b(\ell_{p+1},s_{p+1}),\mathfrak{H}_b(\ell_{p+1},s_{p+1}),\mathfrak{H}_b(\ell_{p+1},s_p)) \\ \preceq\ & 2\kappa^2\alpha(\ell_p,\ell_p,\mathfrak{H}_b(\ell_p,s_p))\beta(\ell_{p+1},\ell_{p+1},\mathfrak{H}_b(\ell_{p+1},s_{p+1}))S_b(\mathfrak{H}_b(\ell_p,s_p),\mathfrak{H}_b(\ell_p,s_p),\mathfrak{H}_b(\ell_{p+1},s_p)) \\ & +\kappa M_b||s_{p+1}-s_p|| \\ \prec\ & \varphi(S_b(\ell_p,\ell_p,\ell_{p+1}))a^*S_b(\ell_p,\ell_p,\ell_{p+1})a + \kappa||M_b||||s_{p+1}-s_p|| \\ \preceq\ & a^*S_b(\ell_p,\ell_p,\ell_{p+1})a + \kappa||M_b||||s_{p+1}-s_p||. \end{aligned}$$

Letting $p \to \infty$, we obtain

$$\begin{aligned} \lim_{p\to\infty} S_b(\ell_{p+1},\ell_{p+1},\ell_{p+2}) & \preceq \lim_{p\to\infty} a^*S_b(\ell_p,\ell_p,\ell_{p+1})a \\ & \preceq \lim_{p\to\infty} (a^*)^2 S_b(\varkappa_{p-1},\varkappa_{p-1},\varkappa_p)(a)^2 \\ & \vdots \end{aligned}$$

$$\preceq \lim_{p\to\infty}(a^*)^{p+1}S_b(\varkappa_0,\varkappa_0,\varkappa_1)(a)^{p+1}$$

which, together with the property, if $a,b\in\mathfrak{A}_h$ and $a\preceq b$ implies $u^*au\preceq u^*bu$, yields that for each $p\in N\cup\{0\}$, put $\delta_p=S_b(\ell_{p+1},\ell_{p+1},\ell_{p+2})$, we have

$$0_{\mathfrak{A}}\preceq\lim_{p\to\infty}\delta_p=\lim_{p\to\infty}S_b(\ell_{p+1},\ell_{p+1},\ell_{p+2})\preceq\lim_{p\to\infty}(a^*)^{p+1}\delta_0(a)^{p+1}.$$

Since $||a||<1$, it follows that $\lim\limits_{p\to\infty}S_b(\ell_{p+1},\ell_{p+1},\ell_{p+2})=0_{\mathfrak{A}}$ It is now time to demonstrate the $C^*$-AV-$S_b$-CS $\{\ell_p\}$ in $(\mathscr{G},\mathfrak{A},S_b)$. On the other hand, suppose $\{\ell_p\}$ is not a $C^*$-AV-$S_b$-CS. Natural numbers $\{q_k\}$ and $\{p_k\}$ can be arranged in a monotone increasing sequence with $\varepsilon>0$ such that $p_k>q_k$,

$$S_b\left(\ell_{q_k},\ell_{q_k},\ell_{p_k}\right)\succeq\varepsilon \tag{5.1}$$

and

$$S_b\left(\ell_{q_k},\ell_{q_k},\ell_{p_{k-1}}\right)\prec\varepsilon. \tag{5.2}$$

From (5.1) and (5.2), we have

$$\begin{aligned}
\varepsilon &\preceq S_b\left(\ell_{q_k},\ell_{q_k},\ell_{p_k}\right)\\
&\preceq 2\kappa S_b\left(\ell_{q_k},\ell_{q_k},\ell_{q_{k+1}}\right)+\kappa S_b\left(\ell_{q_{k+1}},\ell_{q_{k+1}},\ell_{p_k}\right)\\
&\leq 2\kappa S_b\left(\ell_{q_k},\ell_{q_k},\ell_{q_{k+1}}\right)+\kappa S_b\left(\mathfrak{H}_b(\ell_{q_k},s_{q_k}),\mathfrak{H}_b(\ell_{q_k},s_{q_k}),\mathfrak{H}_b(\ell_{p_{k-1}},s_{p_{k-1}})\right)\\
&\leq 2\kappa S_b\left(\ell_{q_k},\ell_{q_k},\ell_{q_{k+1}}\right)+2\kappa^2 S_b\left(\mathfrak{H}_b(\ell_{q_k},s_{q_k}),\mathfrak{H}_b(\ell_{q_k},s_{q_k}),\mathfrak{H}_b(\ell_{p_{k-1}},s_{q_k})\right)\\
&\qquad+\kappa^2 S_b\left(\mathfrak{H}_b(\ell_{p_{k-1}},s_{q_k}),\mathfrak{H}_b(\ell_{p_{k-1}},s_{q_k}),\mathfrak{H}_b(\ell_{p_{k-1}},s_{p_{k-1}})\right).\\
&\leq 2\kappa S_b\left(\ell_{q_k},\ell_{q_k},\ell_{q_{k+1}}\right)+\kappa^2||M_b|||s_{q_k}-s_{p_{k-1}}|\\
&\quad+\ 2\kappa^2\begin{pmatrix}\alpha\left(\ell_{q_k},\ell_{q_k},\mathfrak{H}_b(\ell_{q_k},s_{q_k})\right)\beta\left(\ell_{p_{k-1}},\ell_{p_{k-1}},\mathfrak{H}_b(\ell_{p_{k-1}},s_{p_{k-1}})\right)\\ S_b\left(\mathfrak{H}_b(\ell_{q_k},s_{q_k}),\mathfrak{H}_b(\ell_{q_k},s_{q_k}),\mathfrak{H}_b(\ell_{p_{k-1}},s_{q_k})\right)\end{pmatrix}\\
&\prec \varphi(S_b(\ell_{q_k},\ell_{q_k},\ell_{p_{k-1}}))a^*S_b(\ell_{q_k},\ell_{q_k},\ell_{p_{k-1}})a+2\kappa S_b\left(\ell_{q_k},\ell_{q_k},\ell_{q_{k+1}}\right)+\kappa^2||M_b|||s_{q_k}-s_{p_{k-1}}|\\
&\preceq a^*S_b(\ell_{q_k},\ell_{q_k},\ell_{p_{k-1}})a+2\kappa S_b\left(\ell_{q_k},\ell_{q_k},\ell_{q_{k+1}}\right)+\kappa^2||M_b|||s_{q_k}-s_{p_{k-1}}|.
\end{aligned}$$

We obtain that by setting $k\to\infty$ on both sides.

$$\begin{aligned} 0<\varepsilon &\leq \lim_{k\to\infty}||a^*S_b(\ell_{q_k},\ell_{q_k},\ell_{p_{k-1}})a|| \\ &\leq ||a||\varepsilon<\varepsilon. \end{aligned}$$

This leads to the conclusion that $||a||<1$, it contradicts itself. In the $C^*$-AV-$S_b$MS $(\mathscr{G},\mathfrak{A},S_b)$, the sequence $\{\ell_p\}$ is a $C^*$-AV-$S_b$-CS. The sequence $\{\ell_p\}\to\nu \in (\mathscr{G},\mathfrak{A},S_b)$ comes from the completeness of $(\mathscr{G},\mathfrak{A},S_b)$.

$$\lim_{p\to\infty}\ell_{p+1}=\varkappa=\lim_{p\to\infty}\ell_p.$$

We can prove $\varkappa=\mathfrak{H}_b(\varkappa,s)$.

Now, consider that $\mathscr{G}$ is $(\alpha,\beta)$-regular space then there exists a subsequence $\{\ell_{p_k}\}$ of $\{\ell_p\}$ such that $\alpha\left(\ell_{p_k},\ell_{p_k},\ell_{p_{k+1}}\right)\succeq 1_{\mathfrak{A}}$ and $\beta\left(\ell_{p_k},\ell_{p_k},\ell_{p_{k+1}}\right)\succeq 1_{\mathfrak{A}}$ for each $k\in N$ and $\alpha\left(\varkappa,\varkappa,\mathfrak{H}_b(\varkappa,s)\right)\succeq 1_{\mathfrak{A}}$, $\beta\left(\varkappa,\varkappa,\mathfrak{H}_b(\varkappa,s)\right)\succeq 1_{\mathfrak{A}}$.
From $(\tau_1)$, we have

$$\eta_b\left(\begin{array}{c} 2\kappa^2\alpha(\ell_{p_k},\ell_{p_k},\mathfrak{H}_b(\ell_{p_k},s))\beta(\varkappa,\varkappa,\mathfrak{H}_b(\varkappa,s))S_b(\mathfrak{H}_b(\ell_{p_k},s),\mathfrak{H}_b(\ell_{p_k},s),\mathfrak{H}_b(\varkappa,s)),\\ \varphi(S_b(\ell_{p_k},\ell_{p_k},\varkappa))a^*S_b(\ell_{p_k},\ell_{p_k},\varkappa)a \end{array}\right)\succeq 0_{\mathfrak{A}}.$$

From $(\eta_{b_1})$, and $(\eta_{b_2})$, we have

$$\begin{aligned} & 2\kappa^2\alpha(\ell_{p_k},\ell_{p_k},\mathfrak{H}_b(\ell_{p_k},s))\beta(\varkappa,\varkappa,\mathfrak{H}_b(\varkappa,s))S_b(\mathfrak{H}_b(\ell_{p_k},s),\mathfrak{H}_b(\ell_{p_k},s),\mathfrak{H}_b(\varkappa,s)) \\ \prec\ & \varphi(S_b(\ell_{p_k},\ell_{p_k},\varkappa))a^*S_b(\ell_{p_k},\ell_{p_k},\varkappa)a \\ \preceq\ & a^*S_b(\ell_{p_k},\ell_{p_k},\varkappa)a. \end{aligned}$$

Now from above inequality, we see that

$$\begin{aligned} 0 &< ||S_b(\varkappa,\varkappa,\mathfrak{H}_b(\varkappa,s))||=\lim_{n\to\infty}||S_b(\mathfrak{H}_b(\ell_{p_k},s),\mathfrak{H}_b(\ell_{p_k},s),\mathfrak{H}_b(\varkappa,s))|| \\ &\leq \lim_{n\to\infty}||2\kappa^2\alpha(\ell_{p_k},\ell_{p_k},\mathfrak{H}_b(\ell_{p_k},s))\beta(\varkappa,\varkappa,\mathfrak{H}_b(\varkappa,s))S_b(\mathfrak{H}_b(\ell_{p_k},s),\mathfrak{H}_b(\ell_{p_k},s),\mathfrak{H}_b(\varkappa,s))|| \\ &\leq \lim_{n\to\infty}\inf||a^*S_b(\ell_{p_k},\ell_{p_k},\varkappa)a|| \\ &\leq \lim_{n\to\infty}\sup||a^*S_b(\ell_{p_k},\ell_{p_k},\varkappa)a||\leq||a||\lim_{n\to\infty}\sup||S_b(\ell_{p_k},\ell_{p_k},\varkappa)||=0. \end{aligned}$$

Accordingly, $\varkappa=\mathfrak{H}_b(\varkappa,s)$ indicates that $S_b(\varkappa,\varkappa,\mathfrak{H}_b(\varkappa,s))=0$. So, $s$ in $\mathfrak{B}$. It is obvious that $\mathfrak{B}$ is closed in $[0,1]$. Let $\mathfrak{B}$ be $s_0$. Then, $\ell_0$ exists in $\mathfrak{U}$ such that $\ell_0=\mathfrak{H}_b(\ell_0,s_0)$. Because $\mathfrak{U}$ is open,

$\delta > 0$ must exist for $B_{S_b}(\ell_0, \delta) \subseteq \mathfrak{U}$. Select the value of $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$ such that $|s - s_0| \leq \frac{1}{||M_b||^p} < \varepsilon$. Consequently, for $\overline{B_b(\ell_0, \delta)} = \{\ell \in \mathscr{G} : ||S_b(\ell, \ell, \ell_0)|| \leq \delta + \kappa^2 ||S_b(\ell_0, \ell_0, \ell_0)||\}$. Now

$$\eta_b \begin{pmatrix} 2\kappa^2 \alpha(\ell, \ell, \mathfrak{H}_b(\ell, s_0))\beta(\ell_0, \ell_0, \mathfrak{H}_b(\ell_0, s_0)) S_b\left(\mathfrak{H}_b(\ell, s_0), \mathfrak{H}_b(\ell, s_0), \mathfrak{H}_b(\ell_0, s_0)\right), \\ \varphi(S_b(\ell, \ell, \ell_0)) a^* S_b(\ell, \ell, \ell_0) a \end{pmatrix} \succeq 0_{\mathfrak{A}}.$$

From $(\eta_{b_1})$, we have

$$\begin{aligned} & 2\kappa^2 \alpha(\ell, \ell, \mathfrak{H}_b(\ell, s_0))\beta(\ell_0, \ell_0, \mathfrak{H}_b(\ell_0, s_0)) S_b\left(\mathfrak{H}_b(\ell, s_0), \mathfrak{H}_b(\ell, s_0), \mathfrak{H}_b(\ell_0, s_0)\right) \\ \prec \; & \varphi(S_b(\ell, \ell, \ell_0)) a^* S_b(\ell, \ell, \ell_0) a \\ \preceq \; & a^* S_b(\ell, \ell, \ell_0) a. \end{aligned}$$

Now from above inequality, we see that

$$\begin{aligned} & S_b\left(\mathfrak{H}_b(\ell, s), \mathfrak{H}_b(\ell, s), \ell_0)\right) \\ = \; & S_b\left(\mathfrak{H}_b(\ell, s), \mathfrak{H}_b(\ell, s), \mathfrak{H}_b(\ell_0, s_0)\right) \\ \preceq \; & 2\kappa S_b\left(\mathfrak{H}_b(\ell, s), \mathfrak{H}_b(\ell, s), \mathfrak{H}_b(\ell, s_0)\right) + \kappa S_b\left(\mathfrak{H}_b(\ell, s_0), \mathfrak{H}_b(\ell, s_0), \mathfrak{H}_b(\ell_0, s_0)\right) \\ \preceq \; & 2\kappa M_b |s - s_0| \\ + \; & 2\kappa^2 \alpha(\ell, \ell, \mathfrak{H}_b(\ell, s_0))\beta(\ell_0, \ell_0, \mathfrak{H}_b(\ell_0, s_0)) S_b\left(\mathfrak{H}_b(\ell, s_0), \mathfrak{H}_b(\ell, s_0), \mathfrak{H}_b(\ell_0, s_0)\right) \\ \preceq \; & 2\kappa M_b |s - s_0| + a^* S_b(\ell, \ell, \ell_0) a \\ \preceq \; & \frac{2\kappa}{||M_b||^{p-1}} + a^* S_b(\ell, \ell, \ell_0) a. \end{aligned} \tag{5.3}$$

Letting $p \to \infty$, we get that

$$||S_b\left(\mathfrak{H}_b(\ell, s), \mathfrak{H}_b(\ell, s), \ell_0)\right)|| \leq ||a||^2 ||S_b(\ell, \ell, \ell_0)||.$$

Since $||a|| < 1$ implies that

$$||S_b\left(\mathfrak{H}_b(\ell, s), \mathfrak{H}_b(\ell, s), \ell_0)\right)|| < ||S_b(\ell, \ell, \ell_0)|| \leq \delta + \kappa^2 ||S_b(\ell_0, \ell_0, \ell_0)||.$$

Thus for each fixed $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$, $\mathfrak{H}_b(.,0) : \overline{B_b(\ell_0, \delta)} \to \overline{B_b(\ell_0, \delta)}$. Then, all the conditions of Theorem (5.1) holds.Thus, we conclude that $\mathfrak{H}_b(.,0)$ has a FP in $\overline{\mathfrak{U}}$. But this must be in $\mathfrak{U}$.

Therefore, $s \in \mathfrak{B}$ for $s \in (s_0-\varepsilon, s_0+\varepsilon)$. Hence $(s_0-\varepsilon, s_0+\varepsilon) \subseteq \mathfrak{B}$. Clearly $\mathfrak{B}$ is open in $[0,1]$. A similar procedure can be used to demonstrate the opposite. □

## 6. Conclusion

This paper uses contractive mappings of the $(\alpha,\beta)$-$Z_b$-Geraghty type via $b$-simulation functions to demonstrate certain FPT in the context of complete $C^*$-algebra valued $S_b$-metric spaces, along with appropriate examples that highlight the main findings. Applications for integral equations and homotopy are also given.

## Conflict of Interests

The authors declare that there is no conflict of interests.

## References


[1] S. Janković, Z. Golubović, S. Radenović, Compatible and weakly compatible mappings in cone metric spaces, Math. Computer Model. 52 (2010), 1728–1738. https://doi.org/10.1016/j.mcm.2010.06.043.

[2] M. Bina Devi, N. Priyobarta, Y. Rohen, Fixed point theorems for $(\alpha,\beta)-(\phi,\psi)$-rational contractive type mappings, J. Math. Comput. Sci. 11 (2021), 955–969. https://doi.org/10.28919/jmcs/5244.

[3] B. Khomdram, M. Bina Devi, Y. Rohen, Fixed point of theorems of generalised $\alpha$-ratonal contractive mappings on rectangular $b$-metric spaces, J. Math. Comput. Sci. 11 (2021), 991–1010. https://doi.org/10.28919/jmcs/5255.

[4] Z. Mustafa, M.M.M. Jaradat, H.M. Jaradat, Some common fixed point results of graphs on b-metric space, J. Nonlinear Sci. Appl. 9 (2016), 4838–4851. https://doi.org/10.22436/jnsa.009.06.119.

[5] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha-\psi$-contractive type mappings, Nonlinear Anal.: Theory Methods Appl. 75 (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014.

[6] Z. Ma, L. Jiang, H. Sun, $C^*$-algebra-valued metric spaces and related fixed point theorems, Fixed Point Theory Appl. 2014 (2014), 206. https://doi.org/10.1186/1687-1812-2014-206.

[7] Z. Ma, L. Jiang, $C^*$-Algebra-valued b-metric spaces and related fixed point theorems, Fixed Point Theory Appl. 2015 (2015), 222. https://doi.org/10.1186/s13663-015-0471-6.

[8] S.S. Razavi, H.P. Masiha, Common fixed point theorems in $C^*$-algebra valued $b$-metric spaces with applications to integral equations, Fixed Point Theory, 20 (2019), 649–662. https://doi.org/10.24193/fpt-ro.2019.2.43.

[9] S. Sedghi, A. Gholidahneh, T. Dosenovic, et al. Common fixed point of four maps in $S_b$-metric spaces, J. Linear Topol. Algebra, 5 (2016), 93–104.

[10] N. Souayah, N. Mlaiki A fixed point theorems in $S_b$-metric spaces, J. Math. Computer Sci. 16 (2016), 131–139.

[11] N. Souayah, A fixed point in partial $S_b$-metric spaces, An. Univ. "Ovidius" Constanța, Ser. Mat. 24 (2016), 351–362. https://doi.org/10.1515/auom-2016-0062.

[12] N. Mlaiki, A. Mukheimer, Y. Rohen, et al. Fixed point theorems for $\alpha$-contractive mapping in $S_b$-metric spaces, J. Math. Anal. 8 (2017), 40–46.

[13] Y. Rohen, T. Dosenović, S. Radenović, A note on the paper "A fixed point theorems in $S_b$-metric spaces", Filomat, 31 (2017), 3335–3346. https://doi.org/10.2298/fil1711335r.

[14] G. Kishore, K. Rao, D. Panthi, et al. Some applications via fixed point results in partially ordered $S_b$-metric spaces, Fixed Point Theory Appl. 2017 (2016), 10. https://doi.org/10.1186/s13663-017-0603-2.

[15] S.S. Razavi, H.P. Masiha, $C^*$-algebra-valued $S_b$-metric spaces and applications to integral equations, AUT J. Math. Comput. (2023). https://doi.org/10.22060/ajmc.2023.22211.1140.

[16] S.S. Razavi, H.P. Masiha, M. De La Sen, Applications in integral equations through common results in $C^*$-algebra-valued $S_b$-metric spaces, Axioms, 12 (2023), 413. https://doi.org/10.3390/axioms12050413.

[17] F. Khojasteh, S. Shukla, S. Radenovic, A new approach to the study of fixed point theory for simulation functions, Filomat, 29 (2015), 1189–1194. https://doi.org/10.2298/fil1506189k.

[18] M. Demmaa, R. Saadatib, P. Vetroa, Fixed point results on $b$-metric space via Picard sequences and $b$-simulation functions, Iran. J. Math. Sci. Inform. 11 (2016), 123–136. https://doi.org/10.7508/ijmsi.2016.01.011.

[19] M. Imdad, A.Perveen, R. Gubran, Some common fixed point results via $b$-simulation function, TWMS J. Appl. Eng. Math. 10 (2020), 606–615. https://hdl.handle.net/11729/2846.

[20] B. Rodjanadid, P. Cherachapridi, J. Tanthaunch, Some fixed point theorems in $b$-metric spaces with $b$-simulation functions, Thai J. Math. 19 (2021), 1625–1636.

[21] G.S. Rad, D. Dolicanin-Dekic, A shorter and simple approach to study fixed point results via $b$-simulation functions, Iran. J. Math. Sci. Inform. 13(2018), 97–102. https://doi.org/10.7508/ijmsi.2018.1.009.

[22] S. Chandok, Some fixed point theorems for $(\alpha,\beta)$-admissible Geraghty type contractive mappings and related results, Math. Sci. 9 (2015), 127–135. https://doi.org/10.1007/s40096-015-0159-4.

[23] H. Afshari, H. Alsulami, E. Karapinar, On the extended multivalued Geraghty type contractions, J. Nonlinear Sci. Appl. 9 (2016), 4695–4706. https://doi.org/10.22436/jnsa.009.06.108.

[24] H. Afshari, H. Aydi, E. Karapinar, On generalized $\alpha-\psi$-Geraghty contractions on $b$-metric spaces, Georgian Math. J. 27 (2018), 9–21. https://doi.org/10.1515/gmj-2017-0063.

[25] H. Aydi, A. Felhi, H. Afshari, New Geraghty type contractions on metric-like spaces, J. Nonlinear Sci. Appl. 10 (2017), 780–788. https://doi.org/10.22436/jnsa.010.02.38.

[26] S. Negi, S. Antal, U.C. Gairola, $(\alpha,\beta)-\mathscr{Z}_b$-Geraghty type contraction in $b$-metric-like spaces via $b$-simulation function, TWMS J. Appl. Eng. Math. 14 (2024), 268–283. https://hdl.handle.net/11729/5869.

[27] G.J. Murphy, $C^*$-algebras and operator theory, Academic Press, London, (1990).

[28] Q. Xin, L. Jiang, Z. Ma, Common fixed point theorems in $C^*$-algebra-valued metric spaces, J. Nonlinear Sci. Appl. 9 (2016), 4617–4627.

[29] M. Arshad, M. Mohammed, E. Ameer, et al. Common fixed point theorems for generalized $F$-Geraghty type contraction in partial $b$-metric-like Spaces, Adv. Fixed Point Theory, 10 (2020), 7. https://doi.org/10.28919/afpt/4155.