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SOME FIXED POINT CONVERGENCE THEOREMS FOR NONEXPANSIVE TYPE MAPPINGS IN GEODESIC SPACES

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Abstract. This paper aims to approximate fixed points for nonexpansive type mappings in geodesic spaces, particularly when the set of fixed points is not empty. We delve into the general Picard-Mann (GPM) algorithm, deriving convergence theorems both in terms of Δ and strong convergence under different conditions.

Keywords: nonexpansive mapping, condition E, geodesic space

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1. INTRODUCTION

Let $(\mathscr{Y},\Theta)$ denote a metric space. A mapping $\Upsilon:\mathscr{Y}\to\mathscr{Y}$ is termed nonexpansive if it satisfies the inequality:

$$\Theta(\Upsilon(\vartheta),\Upsilon(\eta))\leq\Theta(\vartheta,\eta) \text{ for all } \vartheta,\eta\in\mathscr{Y}. \tag{1.1}$$

A point $\vartheta\in\mathscr{Y}$ is considered a fixed point of Υ if $\Upsilon(\vartheta)=\vartheta$. We denote $Fix(\Upsilon):=\{\upsilon\in\mathscr{Y}:\Upsilon(\upsilon)=\upsilon\}$. The quest for fixed points of nonlinear mappings holds significant importance in the study of transition operators for initial value problems, differential inclusions, monotone operators, accretive operators, variational inequality problems and equilibrium problems.

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This motivates numerous researchers to explore various algorithms. The Picard algorithm, introduced by Picard [19], stands out as a commonly employed approach due to its simplicity and popularity, particularly for finding fixed points of contractive mappings. However, it is worth noting that the Picard algorithm may fail to converge to a fixed point for nonexpansive mappings. In light of this, Krasnosel'skiĭ [13], Schaefer [21], and Mann [16] proposed more generalized algorithms aimed at discovering fixed points of nonexpansive mappings.

Researchers have been developing iterative methods in the last 20 years to solve problems faster. These methods involve repeating steps (1 to 4 times) to get closer to the answer. Inspired by this progress, Shukla et al. [28] introduced a new method called the General Picard-Mann (GPM) for solving a specific mathematical problem in Banach spaces. This GPM method appears to be very efficient and outperforms many existing methods. We can also use this approach to develop new algorithms for various other problems, including finding roots of equations, solving optimization problems, and analyzing equilibrium points. Shukla and Panicker [26] extended GPM method for the class of mapping satisfying condition (E). They also compared GPM method with other algorithms in the literature and observed that GPM method converges faster to a fixed point of mapping satisfying condition (E) than other algorithms.

On the other hand, Takahashi [31] introduced a convex structure to the metric space and established theorems regarding the existence of fixed points for nonexpansive mappings. Goebel and Kirk [7] employed Krasnosel'skiĭ-Mann iterative method to approximate fixed points of nonexpansive mappings in nonlinear spaces. In recent years, numerous studies have investigated significant fixed point results within the framework of geodesic spaces ([20, 15, 11, 1, 14, 17, 4, 25, 27, 29, 24, 10, 9]). Ariza-Ruiz et al. [1] extended well-known theorems on firmly nonexpansive mappings, including the asymptotic behavior of the Picard iterative method, from linear spaces to geodesic spaces. Leuştean [15] expanded celebrated fixed-point theory results in geodesic spaces, such as the monotone modulus of uniform convexity and asymptotic regularity for the Ishikawa iterative method. Shukla [23] extended the general Picard-Mann iterative method from Banach spaces to geodesic spaces, establishing Δ and strong convergence theorems under specific assumptions. Additionally, he obtained solutions for constrained minimization problems and the zero of monotone operators.

Building upon the aforementioned advancements, we expand the scope of the GPM method to encompass a broader class of mappings. Through this extension, we establish both Δ and strong convergence theorems, subject to specific assumptions. Our findings serve to generalize, extend, and complement numerous results outlined in prior works such as [28, 23, 26].

2. PRELIMINARIES

Consider a metric space $(\mathscr{Y},\Theta)$ and let $[0,l]\subset\mathbb{R}$. Given a pair of points $\upsilon,\zeta\in\mathscr{Y}$, a path $\chi:[0,1]\to\mathscr{Y}$ is said to connect υ and ζ if $\chi(0)=\upsilon$ and $\chi(1)=\zeta$. Such a path χ is termed a geodesic if it satisfies the condition:

$$\Theta(\chi(s),\chi(t))=\Theta(\chi(0),\chi(1))|s-t|,\ \text{for all } s,t\in[0,1].$$

The metric space $(\mathscr{Y},\Theta)$ earns the designation of a geodesic space if every pair of points $\upsilon,\zeta\in\mathscr{Y}$ is connected by at least one geodesic. It's important to note that the geodesic segment joining υ and ζ may not be unique. The precise formulation of hyperbolic spaces, as introduced by Kohlenbach [12], adheres to these principles.

Definition 2.1. [12]. A triple $(\mathscr{Y},\Theta,W)$ earns the designation of a hyperbolic metric space, or simply a W-hyperbolic space, if $(\mathscr{Y},\Theta)$ forms a metric space and the function $W:\mathscr{Y}\times\mathscr{Y}\times[0,1]\to\mathscr{Y}$ adheres to the following conditions for all $\upsilon,\zeta,z,w\in\mathscr{Y}$ and $\lambda,\theta\in[0,1]$:

(W1) $\Theta(z,W(\upsilon,\zeta,\lambda))\le(1-\lambda)\Theta(z,\upsilon)+\lambda\Theta(z,\zeta)$;

(W2) $\Theta(W(\upsilon,\zeta,\lambda),W(\upsilon,\zeta,\theta))=|\lambda-\theta|\Theta(\upsilon,\zeta)$;

(W3) $W(\upsilon,\zeta,\lambda)=W(\zeta,\upsilon,1-\lambda)$;

(W4) $\Theta(W(\upsilon,z,\lambda),W(\zeta,w,\lambda))\le(1-\lambda)\Theta(\upsilon,\zeta)+\lambda\Theta(z,w)$.

Furthermore, any Busemann space possesses the property of being uniquely geodesic, meaning that for any pair of points $\upsilon,\zeta\in\mathscr{Y}$, there exists a single geodesic segment joining υ and ζ (as demonstrated in [5]).

Several well-known examples of W-hyperbolic spaces include normed spaces, Hadamard manifolds, CAT(0)-spaces, and the Hilbert open unit ball equipped with the hyperbolic metric (as outlined in [1, 12]).

Remark 2.2. If $W(\upsilon,\zeta,\lambda)=(1-\lambda)\upsilon+\lambda\zeta$ for all $\upsilon,\zeta\in\mathscr{Y},\lambda\in[0,1]$, then it follows that all normed linear spaces are W-hyperbolic spaces.

We shall write

$$W(\upsilon,\zeta,\lambda):=(1-\lambda)\upsilon\oplus\lambda\zeta$$

to denote a point $W(\upsilon,\zeta,\lambda)$ in a W-hyperbolic space. For $\upsilon,\zeta\in\mathscr{Y}$, we denote

$$[\upsilon,\zeta]=\{(1-\lambda)\upsilon\oplus\lambda\zeta:\lambda\in[0,1]\}$$

as a geodesic segment. A nonempty subset $\mathscr{D}$ of W-hyperbolic space $(\mathscr{Y},\Theta,W)$ is said to be convex if $[\upsilon,\zeta]\subset\mathscr{D}$ for all $\upsilon,\zeta\in\mathscr{D}$. Please refer to the definition of uniformly convex W hyperbolic space (UCW-hyperbolic space) provided in [14].

Remark 2.3. Leuştean [15] proved that complete CAT(0) spaces are complete uniformly convex hyperbolic spaces (or UCW-hyperbolic spaces).

Let $\{\upsilon_n\}$ be a bounded sequence in a hyperbolic space $(\mathscr{Y},\Theta,W)$ and $\mathscr{D}$ a nonempty subset of $\mathscr{Y}$. A functional $r(\ .\ ,\{\upsilon_n\}):\mathscr{Y}\to[0,+\infty)$ can be defined as follows:

$$r(\zeta,\{\upsilon_n\})=\limsup_{n\to+\infty}\Theta(\zeta,\upsilon_n).$$

The asymptotic radius of $\{\upsilon_n\}$ with respect to (in short, wrt) $\mathscr{D}$ is described as

$$r(\mathscr{D},\{\upsilon_n\})=\inf\{r(\zeta,\{\upsilon_n\}):\zeta\in\mathscr{D}\}.$$

A point υ in $\mathscr{D}$ is called as an asymptotic center of $\{\upsilon_n\}$ wrt $\mathscr{D}$ if

$$r(\upsilon,\{\upsilon_n\})=r(\mathscr{D},\{\upsilon_n\}).$$

$A(\mathscr{D},\{\upsilon_n\})$ is denoted as set of all asymptotic centers of $\{\upsilon_n\}$ wrt $\mathscr{D}$.

Definition 2.4. [30]. Let $\{\upsilon_n\}$ be a bounded sequence in a W hyperbolic space $(\mathscr{Y},\Theta)$. The sequence $\{\upsilon_n\}$ Δ-converges to υ if υ is the unique asymptotic center for every subsequence $\{\rho_n\}$ of $\{\upsilon_n\}$.

Let $(\mathscr{Y},\Theta)$ be a W hyperbolic space and $\mathscr{D}\subset\mathscr{Y}$ such that $\mathscr{D}\neq\emptyset$. A sequence $\{\upsilon_n\}$ in $\mathscr{Y}$ is said to be Fejér monotone wrt $\mathscr{D}$ if

$$\Theta(\upsilon^\dagger,\upsilon_{n+1})\leq\Theta(\upsilon^\dagger,\upsilon_n),\ \text{for all } n\geq 0,\ \text{for all } \upsilon^\dagger\in\mathscr{D}.$$

Definition 2.5. [22]. A mapping $\Upsilon:\mathscr{D}\to\mathscr{D}$ with $Fix(\Upsilon)\neq\emptyset$ satisfies Condition (I) if there exists a function $g:[0,+\infty)\to[0,+\infty)$ with the following assumptions:

(1) $g(s)>0$ for $s\in(0,+\infty)$ and $g(0)=0$.

(2) $\Theta(\upsilon,\Upsilon(\upsilon))\geq g(\Theta(\upsilon,Fix(\Upsilon)))\ \forall\ \upsilon\in\mathscr{D}$,

where $\Theta(\upsilon,Fix(\Upsilon))=\inf\{\Theta(\upsilon,\zeta):\zeta\in Fix(\Upsilon)\}$.

Definition 2.6. Let $(\mathscr{Y},\Theta)$ be a metric space and $\mathscr{D}\subset\mathscr{Y}$ such that $\mathscr{D}\neq\emptyset$. A mapping $\Upsilon:\mathscr{D}\to\mathscr{D}$ is compact if $\Upsilon(\mathscr{D})$ has a compact closure.

Lemma 2.7. [15]. *Let $\{\upsilon_n\}$ be a bounded sequence in $(\mathscr{Y},\Theta,W)$ and $A(\mathscr{D},\{\upsilon_n\})=\{z\}$. Let $\{r_n\}$ and $\{s_n\}$ be two sequences in $\mathbb{R}$ such that $r_n\in[0,+\infty)$ for all $n\in\mathbb{N}$, $\limsup r_n\leq 1$ and $\limsup s_n\leq 0$. Suppose that $\zeta\in\mathscr{D}$ and there exist $m,N\in\mathbb{N}$ such that*

$$\Theta(\zeta,\upsilon_{n+m})\leq r_n\Theta(z,\upsilon_n)+s_n,\ \text{for all } n\geq N.$$

Then $\zeta=z$.

Lemma 2.8. [1]. *Let $(\mathscr{Y},\Theta,W)$ be a W-hyperbolic space, $\mathscr{D}\subset\mathscr{Y}$ such that $\mathscr{D}\neq\emptyset$. If $\{\upsilon_n\}$ is Fejér monotone wrt $\mathscr{D}$, $A(\mathscr{D},\{\upsilon_n\})=\{\upsilon\}$ and $A(\mathscr{Y},\{\rho_n\})\subseteq\mathscr{D}$ for every subsequence $\{\rho_n\}$ of $\{\upsilon_n\}$. Then the sequence $\{\upsilon_n\}$ Δ-converges to $\upsilon\in\mathscr{D}$.*

Definition 2.9. [6]. Let $\mathscr{Y}$ be a metric space. A mapping $\Upsilon:\mathscr{Y}\to\mathscr{Y}$ is said to fulfill condition (E) if there exists $\mu\geq 1$ such that

$$\Theta(\upsilon,\Upsilon(\eta))\leq\mu\Theta(\upsilon,\Upsilon(\upsilon))+\Theta(\upsilon,\eta)\ \forall\ \upsilon,\eta\in\mathscr{Y}.$$

3. MAIN RESULTS

In their work [28], Shukla *et al.* introduced an iterative approach known as the GPM method. Let $\mathscr{B}$ be a Banach space and $\mathscr{D}\subset\mathscr{B}$ such that $\mathscr{D}\neq\emptyset$, and $\mathscr{D}$ be convex. Let $\Upsilon:\mathscr{D}\to\mathscr{D}$ be a mapping. Consider a sequence as follows:

$$\begin{cases} \upsilon_1 = \upsilon \in \mathscr{D} \\ \upsilon_{n+1} = \Upsilon^k\{(1-\alpha_n)\upsilon_n + \alpha_n \Upsilon(\upsilon_n)\}, \ n \in \mathbb{N}, \end{cases}$$

where k is a fixed natural number and $\{\alpha_n\}$ is a sequence in $[0,1]$.

In the context of a geodesic space, the method described above can be outlined as follows, as detailed in [23]: Consider a W-hyperbolic space $(\mathscr{Y}, \Theta, W)$, where $\mathscr{D} \subset \mathscr{Y}$ is a non-empty, convex subset. Let $\Upsilon : \mathscr{D} \to \mathscr{D}$ be a mapping. Consider a sequence as follows:

$$\begin{cases} \upsilon_1 = \upsilon \in \mathscr{D} \\ \upsilon_{n+1} = \Upsilon^k\{(1-\alpha_n)\upsilon_n \oplus \alpha_n \Upsilon(\upsilon_n)\}, \ n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where k is a fixed natural number and $\{\alpha_n\}$ is a sequence in $[0,1]$.

3.1. Stability results. We consider the stability analysis of the GPM method (3.1) now. A fixed-point iteration method is deemed numerically stable if minor perturbations arising from rounding errors or approximations during computations result in only slight alterations to the approximate value of the fixed point obtained through this method, as elaborated in [2]. The stability of an iterative technique holds significant importance across various fields such as fractal geometry, computational analysis, game theory, and beyond.

The concept of a Υ-stable fixed-point iterative method was initially introduced by Harder and Hicks [8]. Osilike [18] later expanded on this notion by introducing the concept of almost Υ-stable methods. Berinde [3] further refined this idea by introducing a weaker form of stability termed summably almost Υ-stable. In the context of W-hyperbolic space, this concept can be defined as follows:

Definition 3.1. Consider a nonempty convex subset $\mathscr{D}$ of a W-hyperbolic space $\mathscr{Y}$. Let $\Upsilon : \mathscr{D} \to \mathscr{D}$ be a mapping with non-empty fixed point set $Fix(\Upsilon)$. Starting with a given point $\upsilon_1 \in \mathscr{D}$, the fixed-point iteration method produces a sequence $\{\upsilon_n\}$ within $\mathscr{D}$ according to the following process:

$$\upsilon_{n+1} = f(\Upsilon, \upsilon_n) \tag{3.2}$$

where f is some function.

Assume that the method (3.2) exhibits strong convergence to a fixed point $\upsilon^\dagger$ of Υ. Consider an arbitrary sequence $\{\eta_n\}$ in $\mathscr{D}$, and define:

$$\varepsilon_n = \Theta(\eta_{n+1}, f(\Upsilon, \eta_n)). \tag{3.3}$$

Then iterative method (3.2) is said to be summably almost Υ-stable (or summably almost stable with respect to Υ) if and only if

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty \text{ implies that } \sum_{n=0}^{\infty} \Theta(\eta_n, \upsilon^\dagger) < \infty.$$

Every almost stable iterative approach also qualifies as summably almost stable, yet the converse is not true.

Now, we demonstrate that the iterative method (3.1) exhibits summably almost stable with respect to Υ behavior when dealing with mappings of contractive nature.

Theorem 3.2. *Consider $\mathscr{D}$, a nonempty closed convex subset of a complete W-hyperbolic space $\mathscr{Y}$. Let $\Upsilon : \mathscr{D} \to \mathscr{D}$ be a contraction mapping with a fixed point $\upsilon^\dagger$ and contraction constant $\theta \in (0,1)$. Assume $\{\alpha_n\}$ is a sequence within $[0,1]$. Given $\upsilon_1 \in \mathscr{D}$ and a fixed $k \in \mathbb{N}$, define the sequence $\{\upsilon_n\}$ according to* (3.1). *Let $\{\eta_n\}$ be any arbitrary sequence in $\mathscr{D}$, and define:*

$$\varepsilon_n = \Theta(\eta_{n+1}, \Upsilon^k\{(1-\alpha_n)\upsilon_n \oplus \alpha_n \Upsilon(\upsilon_n)\}).$$

Then we have the followings:

(1) *The sequence $\{\upsilon_n\}$ strongly converges to the fixed point $\upsilon^\dagger$.*
(2) $\sum\limits_{n=0}^{\infty} \varepsilon_n < \infty$ *implies that* $\sum\limits_{n=0}^{\infty} \Theta(\eta_n, \upsilon^\dagger) < \infty$, *so that $\{\upsilon_n\}$ is summably almost Υ-stable*
(3) $\lim\limits_{n\to\infty} \eta_n = \upsilon^\dagger$ *implies* $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

Proof. By (3.1) and let $\upsilon^\dagger \in Fix(\Upsilon)$, we have

$$\begin{aligned} \Theta(\upsilon_{n+1}, \upsilon^\dagger) &= \Theta(\Upsilon^k\{(1-\alpha_n)\upsilon_n \oplus \alpha_n\Upsilon(\upsilon_n)\}, \upsilon^\dagger) \\ &\leq \theta^k \Theta((1-\alpha_n)\upsilon_n \oplus \alpha_n \Upsilon(\upsilon_n), \upsilon^\dagger) \\ &\leq \theta^k((1-\alpha_n)\Theta(\upsilon_n, \upsilon^\dagger) + \alpha_n \Theta(\Upsilon(\upsilon_n), \upsilon^\dagger)) \\ &\leq \theta^k \Theta(\upsilon_n, \upsilon^\dagger) \end{aligned} \tag{3.4}$$

by successively induction, we get

$$\Theta(\upsilon_{n+1}, \upsilon^{\dagger}) \leq \left(\theta^k\right)^n \Theta(\upsilon_1, \upsilon^{\dagger}).$$

Since $\theta^k < 1$, $\{\upsilon_n\}$ strongly converges to $\upsilon^{\dagger}$.

We show (2), for each $\upsilon^{\dagger} \in Fix(\Upsilon)$, we get

$$\Theta(\Upsilon(\eta_n), \upsilon^{\dagger}) \leq \theta\Theta(\eta_n, \upsilon^{\dagger}).$$

For each $k \in \mathbb{N}$

$$\Theta(\Upsilon^k(\eta_n), \upsilon^{\dagger}) \leq \theta^k\Theta(\eta_n, \upsilon^{\dagger}).$$

In view triangle inequality

$$\begin{aligned}
\Theta(\eta_{n+1}, \upsilon^{\dagger}) &\leq \Theta(\eta_{n+1}, \Upsilon^k\{(1-\alpha_n)\eta_n \oplus \alpha_n\Upsilon(\eta_n)\}) + \Theta(\Upsilon^k\{(1-\alpha_n)\eta_n \oplus \alpha_n\Upsilon(\eta_n)\}, \upsilon^{\dagger}) \\
&\leq \theta^k\Theta((1-\alpha_n)\eta_n \oplus \alpha_n\Upsilon(\eta_n), \upsilon^{\dagger}) + \varepsilon_n \\
&\leq \theta^k\{(1-\alpha_n)\Theta(\eta_n, \upsilon^{\dagger}) + \alpha_n\theta\Theta(\eta_n, \upsilon^{\dagger})\} + \varepsilon_n \qquad (3.5) \\
&\leq \theta^k\Theta(\eta_n, \upsilon^{\dagger}) + \varepsilon_n.
\end{aligned}$$

In view of assumption $\sum\limits_{n=0}^{\infty} \varepsilon_n < \infty$ and [3, Lemma 1 (ii)], it implies that $\sum\limits_{n=0}^{\infty} \Theta(\eta_n, \upsilon^{\dagger}) < \infty$. Finally, we prove (3), suppose $\lim\limits_{n\to\infty} \eta_n = \upsilon^{\dagger}$. Now

$$\begin{aligned}
\varepsilon_n &= \Theta(\eta_{n+1}, \Upsilon^k\{(1-\alpha_n)\eta_n \oplus \alpha_n\Upsilon(\eta_n)\}) \\
&\leq \Theta(\eta_{n+1}, \upsilon^{\dagger}) + \Theta\left(\Upsilon^k\{(1-\alpha_n)\eta_n \oplus \alpha_n\Upsilon(\eta_n)\}, \upsilon^{\dagger}\right) \\
&\leq \Theta(\eta_{n+1}, \upsilon^{\dagger}) + \Theta\left((1-\alpha_n)\eta_n \oplus \alpha_n\Upsilon(\eta_n), \upsilon^{\dagger}\right) \\
&\leq \Theta(\eta_{n+1}, \upsilon^{\dagger}) + (1-\alpha_n)\Theta(\eta_n, \upsilon^{\dagger}) + \alpha_n\Theta\left(\Upsilon(\eta_n), \upsilon^{\dagger}\right) \\
&\leq \Theta(\eta_{n+1}, \upsilon^{\dagger}) + \Theta(\eta_n, \upsilon^{\dagger}) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

This completes the proof. □

Lemma 3.3. *Consider $(\mathscr{Y}, \Theta, W)$, a complete UCW-hyperbolic space, $\mathscr{D} \subset \mathscr{Y}$ satisfies $\mathscr{D} \neq \emptyset$ and is closed and convex. Let $\Upsilon : \mathscr{D} \to \mathscr{D}$ be a QNE mapping with $Fix(\Upsilon) \neq \emptyset$. Given $\upsilon_1 \in \mathscr{D}$ and $\alpha_n \in [\alpha, \beta]$ with $\alpha, \beta \in (0,1)$, the sequence $\{\upsilon_n\}$ defined by (3.1) yields the following results.*

(1) $\lim_{n\to+\infty}\Theta(\upsilon_n,\upsilon^\dagger)$ *exists for all* $\upsilon^\dagger\in Fix(\Upsilon)$.

(2) $\lim_{n\to+\infty}\Theta(\upsilon_n,\Upsilon(\upsilon_n))=0$.

Proof. From (3.1), we obtain

$$\Theta(\upsilon_{n+1},\upsilon^\dagger)=\Theta(\Upsilon^k\{(1-\alpha_n)\upsilon_n\oplus\alpha_n\Upsilon(\upsilon_n)\},\upsilon^\dagger).$$

Using the fact that Υ is QNE,

$$\Theta(\upsilon_{n+1},\upsilon^\dagger)\le\Theta((1-\alpha_n)\upsilon_n\oplus\alpha_n\Upsilon(\upsilon_n),\upsilon^\dagger).$$

From (W1), we get

$$\Theta(\upsilon_{n+1},\upsilon^\dagger)\le(1-\alpha_n)\Theta(\upsilon_n,\upsilon^\dagger)+\alpha_n\Theta(\Upsilon(\upsilon_n),\upsilon^\dagger).$$

Again Υ is QNE,

$$\begin{aligned}\Theta(\upsilon_{n+1},\upsilon^\dagger)&\le(1-\alpha_n)\Theta(\upsilon_n,\upsilon^\dagger)+\alpha_n\Theta(\upsilon_n,\upsilon^\dagger)\\&=\Theta(\upsilon_n,\upsilon^\dagger).\end{aligned}$$

Thus, the sequence $\{\Theta(\upsilon_n,\upsilon^\dagger)\}$ is monotone nonincreasing. Because of that $\lim_{n\to+\infty}\Theta(\upsilon_n,\upsilon^\dagger)$ exists. Let

$$\lim_{n\to+\infty}\Theta(\upsilon_n,\upsilon^\dagger)=h>0. \tag{3.6}$$

Since Υ is QNE,

$$\lim_{n\to+\infty}\Theta(\Upsilon(\upsilon_n),\upsilon^\dagger)\le h. \tag{3.7}$$

From (3.6)

$$\begin{aligned}h&=\lim_{n\to+\infty}\Theta(\upsilon_{n+1},\upsilon^\dagger)=\limsup_{n\to+\infty}\Theta(\Upsilon^k\{(1-\alpha_n)\upsilon_n\oplus\alpha_n\Upsilon(\upsilon_n)\},\upsilon^\dagger)\\&\le\limsup_{n\to+\infty}\Theta((1-\alpha_n)\upsilon_n\oplus\alpha_n\Upsilon(\upsilon_n),\upsilon^\dagger)\\&\le\lim_{n\to+\infty}\Theta(\upsilon_n,\upsilon^\dagger)=h.\end{aligned}$$

Therefore,

$$\lim_{n\to+\infty}\Theta((1-\alpha_n)\upsilon_n\oplus\alpha_n\Upsilon(\upsilon_n),\upsilon^\dagger)=h. \tag{3.8}$$

From (3.6), (3.7), (3.8) and [11, Lemma 2.5],

$$\lim_{n\to+\infty}\Theta(\upsilon_n,\Upsilon(\upsilon_n))=0. \tag{3.9}$$

□

Lemma 3.4. *Let $(\mathscr{Y},\Theta)$ be a uniquely geodesic space, $\mathscr{D}$ a nonempty closed convex subset of $\mathscr{Y}$ and $\Upsilon:\mathscr{D}\to\mathscr{D}$ a QNE mapping with $Fix(\Upsilon)\neq\emptyset$. Then $Fix(\Upsilon)$ is closed and convex.*

Proof. First we show that $Fix(\Upsilon)$ is closed. Let $\{\upsilon_n\}$ be a sequence in $Fix(\Upsilon)$ such that $\{\upsilon_n\}$ strongly converges to $\upsilon\in\mathscr{D}$. From the definition of mapping Υ, we have

$$\Theta(\Upsilon(\upsilon),\upsilon_n)\leq\Theta(\upsilon,\upsilon_n)\to 0 \text{ as } n\to+\infty.$$

Thus $\Upsilon(\upsilon)=\upsilon\in Fix(\Upsilon)$. Now, we show that $Fix(\Upsilon)$ is convex. Let $\upsilon\neq\eta\in Fix(\Upsilon)$ and $\gamma\in[\upsilon,\eta]$. It can be seen that

$$\Theta(\upsilon,\Upsilon(\gamma))\leq\Theta(\upsilon,\eta) \tag{3.10}$$

and

$$\Theta(\eta,\Upsilon(\gamma))\leq\Theta(\eta,\gamma) \tag{3.11}$$

Then from (3.10) and (3.11)

$$\Theta(\upsilon,\eta)\leq\Theta(\upsilon,\Upsilon(\gamma))+\Theta(\Upsilon(\gamma),\eta)\leq\Theta(\upsilon,\gamma)+\Theta(\gamma,\eta)=\Theta(\upsilon,\eta).$$

Therefore,

$$\Theta(\upsilon,\Upsilon(\gamma))+\Theta(\Upsilon(\gamma),\eta)=\Theta(\upsilon,\eta)$$

and $\Upsilon(\gamma)\in[\upsilon,\eta]$. From [1, Lemma 2.3 (ii)] we get the followings:

$$\Theta(\upsilon,\gamma)+\Theta(\gamma,\Upsilon(\gamma))=\Theta(\upsilon,\Upsilon(\gamma))\leq\Theta(\upsilon,\gamma),$$

or

$$\Theta(\eta,\gamma)+\Theta(\gamma,\Upsilon(\gamma))=\Theta(\eta,\Upsilon(\gamma))\leq\Theta(\eta,\gamma).$$

In both occasions, we get $\gamma=\Upsilon(\gamma)$. □

Theorem 3.5. *Let $\mathscr{Y}$, $\mathscr{D}$ and $\{\upsilon_n\}$ be same as in Lemma 3.3. Let $\Upsilon : \mathscr{D} \to \mathscr{D}$ be a mapping satisfying condition (E) with $Fix(\Upsilon) \neq \emptyset$. Then the sequence $\{\upsilon_n\}$ Δ-converges to a point in $Fix(\Upsilon)$.*

Proof. Considering Lemma 3.3, the sequence $\Theta(\upsilon_n, z^\dagger)$ exhibits monotonic non-increasing behavior for all $z^\dagger \in Fix(\Upsilon)$. Furthermore, the sequence $\{\upsilon_n\}$ demonstrates Fejér monotonicity with respect to $Fix(\Upsilon)$, given the closed and convex nature of $Fix(\Upsilon)$ as mentioned in Lemma 3.4. In view of [1, Proposition 2.12] establishes that the sequence $\{\upsilon_n\}$ possesses a unique asymptotic center $w^\dagger$ in relation to $Fix(\Upsilon)$. If $\{\rho_n\}$ is a subsequence of $\{\upsilon_n\}$, then from [1, Proposition 2.12], $\{\rho_n\}$ also possesses a unique asymptotic center $\rho^\dagger$ with respect to $Fix(\Upsilon)$.

$$\Theta(\rho_n, \Upsilon(\rho^\dagger)) \quad \leq \quad \Theta(\rho_n, \rho^\dagger) + \mu\Theta(\Upsilon(\rho_n), \rho_n).$$

From (3.9)

$$\Theta(\rho_n, \Upsilon(\rho^\dagger)) \leq \Theta(\rho_n, \rho^\dagger).$$

Considering Lemma 2.7, we conclude that $\Upsilon(\rho^\dagger) = \rho^\dagger$. Moreover, from Lemma 2.8, it can be inferred that the sequence $\{\upsilon_n\}$ Δ-converges to a point within $Fix(\Upsilon)$. □

Theorem 3.6. *Consider $\mathscr{Y}$, $\mathscr{D}$, and $\{\upsilon_n\}$ as defined in Lemma 3.3. Let $\Upsilon : \mathscr{D} \to \mathscr{D}$ be a mapping satisfying condition (E) with $Fix(\Upsilon) \neq \emptyset$. If the mapping Υ satisfies condition (I), then the sequence $\{\upsilon_n\}$ strongly converges to a point within $Fix(\Upsilon)$.*

Proof. According to Lemma 3.3, the sequences $\Theta(\upsilon_n, w^\dagger)$ are monotonic and non-increasing for all $w^\dagger \in Fix(\Upsilon)$. Consequently, the sequence $\Theta(\upsilon_n, Fix(\Upsilon))$ is also monotonic and non-increasing. This ensures the existence of $\lim_{n\to+\infty} \Theta(\upsilon_n, Fix(\Upsilon))$. In view of Lemma 3.3

$$\lim_{n\to+\infty} \Theta(\upsilon_n, \Upsilon(\upsilon_n)) = 0. \tag{3.12}$$

Since Υ satisfies condition (I),

$$\Theta(\upsilon_n, \Upsilon(\upsilon_n)) \geq g(\Theta(\upsilon_n, Fix(\Upsilon))).$$

From (3.12), $\lim_{n\to+\infty} g(\Theta(\upsilon_n, Fix(\Upsilon))) = 0$ and

$$\lim_{n\to+\infty} \Theta(\upsilon_n, Fix(\Upsilon)) = 0. \tag{3.13}$$

Now, it can be confirmed that $\{\upsilon_n\}$ is Cauchy sequence. Given any $\varepsilon > 0$, as per (3.13), there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\Theta(\upsilon_n, Fix(\Upsilon)) < \frac{\varepsilon}{4}$$

and

$$\inf\{\Theta(\upsilon_{n_0}, w^\dagger) : w^\dagger \in Fix(\Upsilon)\} < \frac{\varepsilon}{4},$$

and there is $w^\dagger \in Fix(\Upsilon)$ in such a way that

$$\Theta(\upsilon_{n_0}, w^\dagger) < \frac{\varepsilon}{2}.$$

Thus

$$\Theta(\upsilon_{n+m}, \upsilon_n) \leq \Theta(\upsilon_{n+m}, w^\dagger) + \Theta(w^\dagger, \upsilon_n) \leq 2\Theta(\upsilon_{n_0}, w^\dagger) < 2\frac{\varepsilon}{2} = \varepsilon$$

for all $m, n \geq n_0$. Given the sequence $\{\upsilon_n\}$ is Cauchy, and considering the fact that $\mathscr{D}$ is closed within $\mathscr{Y}$, it follows that $\{\upsilon_n\}$ converges to a point $\upsilon^\dagger \in \mathscr{D}$. Now,

$$\begin{aligned} \Theta(\upsilon^\dagger, \Upsilon(\upsilon^\dagger)) &\leq \Theta(\upsilon^\dagger, \upsilon_n) + \Theta(\upsilon_n, \Upsilon(\upsilon^\dagger)) \\ &\leq 2\Theta(\upsilon^\dagger, \upsilon_n) + \mu\Theta(\upsilon_n, \Upsilon(\upsilon_n)) \end{aligned}$$

from (3.12), $\upsilon^\dagger = \Upsilon(\upsilon^\dagger)$. Therefore, the sequence $\{\upsilon_n\}$ strongly converges to a point within $Fix(\Upsilon)$. □

Theorem 3.7. *Consider* $(\mathscr{Y}, \Theta, W)$*, a complete UCW-hyperbolic space. Let* $\mathscr{D}$ *and* $\{\upsilon_n\}$ *be as defined in Lemma 3.3. Let* $\Upsilon : \mathscr{D} \to \mathscr{D}$ *be a mapping satisfying condition (E) with* $Fix(\Upsilon) \neq \emptyset$*. If* Υ *is a compact mapping, then the sequence* $\{\upsilon_n\}$ *strongly converges to a point within* $Fix(\Upsilon)$*.*

Proof. As per Lemma 3.3, $\{\upsilon_n\}$ is a bounded sequence. Additionally, referring to Lemma 3.3,

$$\lim_{n\to+\infty} \Theta(\upsilon_n, \Upsilon(\upsilon_n)) = 0. \tag{3.14}$$

Considering the definition of a compact mapping, the range of $\mathscr{D}$ under Υ is confined within a compact set. Consequently, there exists a subsequence $\{\Upsilon(\upsilon_{n_j})\}$ of $\{\Upsilon(\upsilon_n)\}$ that strongly converges to $\upsilon^\dagger \in \mathscr{D}$. Referring to (3.14), it follows that the subsequence $\{\upsilon_{n_j}\}$ also strongly converges to $\upsilon^\dagger$. Mapping Υ satisfying condition (E), we have

$$\Theta(\upsilon_{n_j}, \Upsilon(\upsilon^\dagger)) \leq \mu\Theta(\upsilon_{n_j}, \Upsilon(\upsilon_{n_j})) + \Theta(\upsilon_{n_j}, \upsilon^\dagger)$$

From (3.9)

$$\Theta(\upsilon_{n_j}, \Upsilon(\upsilon^\dagger)) \leq \Theta(\upsilon_{n_j}, \upsilon^\dagger) = 0.$$

Therefore, subsequence $\{\upsilon_{n_j}\}$ strongly converges to $\Upsilon(\upsilon^\dagger)$, it implies that $\Upsilon(\upsilon^\dagger) = \upsilon^\dagger$. Since $\lim_{n\to+\infty} \Theta(\upsilon_n, \upsilon^\dagger)$ exists, the sequence $\{\upsilon_n\}$ strongly converges to a point in $Fix(\Upsilon)$. □

Corollary 3.8. *Let $(\mathscr{Y}, \Theta, W)$ be a complete UCW-hyperbolic space and $\mathscr{D} \subset \mathscr{Y}$ such that $\mathscr{D} \neq \emptyset$, $\mathscr{D}$ be a compact convex. Let $\Upsilon : \mathscr{D} \to \mathscr{D}$ be a mapping satisfying condition (E) with $Fix(\Upsilon) \neq \emptyset$. Define the sequence $\{\upsilon_n\}$ by the successive iteration*

$$\begin{cases} \upsilon_1 = \upsilon \in \mathscr{D} \\ \upsilon_{n+1} = \Upsilon^k\{(1-\alpha_n)\upsilon_n \oplus \alpha_n \Upsilon(\upsilon_n)\}, \ n \in \mathbb{N}, \end{cases}$$

where k is a fixed, $k \in \mathbb{N}$ and $\alpha_n \in [\alpha, \beta]$ with $\alpha, \beta \in (0,1)$, Then the sequence $\{\upsilon_n\}$ strongly converges to a point in $Fix(\Upsilon)$.

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CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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