Available online at http://scik.org
Adv. Fixed Point Theory, 2025, 15:29
https://doi.org/10.28919/afpt/8863
ISSN: 1927-6303

EXISTENCE OF FIXED POINT IN INTUITIONISTIC FUZZY B-METRIC SPACE WITH APPLICATION

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Abstract. In this paper, extended intuitionistic fuzzy b-metric space is defined and showing the properties crucial to its structure. Subsequently, the fixed point theorem within the framework of these extended intuitionistic fuzzy b-metric spaces has been established. Our findings not only extend but also generalize the existing results in the current literature, demonstrating symmetrical patterns and properties within these mathematical frameworks. To underscore the effectiveness of the proposed methodologies, a nontrivial example is employed. Additionally, application related to the existence and uniqueness of solutions for a specific class of Fredholm integral equations is explored to highlight the practical implications of our main results.

Keywords: fixed point; intuitionistic fuzzy *b*-metric space; extended intuitionistic fuzzy *b*-metric space.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The Banach contraction principle, also known as the BFP theorem, is a significant mathematical result that has been widely extended and explored by numerous authors. This principle

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Received August 23, 2024

was first established by Banach and has since served as a fundamental tool in the field of functional analysis. It provides conditions under which a mapping on a complete metric space has a unique fixed point. Numerous researchers have expanded upon the remarkable achievements of the BCP, exploring various avenues of development. One notable contribution was made by Frechet [1], who introduced the concept of metric space.

Another significant advancement came from Czerwik [21], who provided a formal definition of b-metric space while investigating Banach theorems. The initial development of b-metric spaces can be attributed to Bourbaki and Bakhtin. Building upon this foundation, Kamran and Samreen [23] focused on the triangle inequality within b-metric space and devised a novel distance function known as extended b-metric space that relaxed certain constraints imposed by the inequality. In 1965, Zadeh [13] introduced fuzzy set theory as a mathematical framework to address situations in everyday life that involve imprecise or ambiguous information. Kaleva et al. [11] proposed the notion of FMS, where the imprecision in distance measurements between elements is incorporated into the metric structure.

The groundwork for FMS was laid by Kramosil et al. [10], followed by the contributions of George et al. [3] in further developing the concept. In the same year, Lopez et al. [16] introduced a Hausdorff fuzzy metric, designed specifically for a set of non-empty compact subsets that are within a defined FMS. Arshad et al. [1] established necessary and sufficient conditions for the existence of fixed points in multivalued maps within FMS. In 2014, Shojaei [7] introduced the idea of HIFMS, see also [4, 5].

Additionally, Atanassaov [2] extended this theory in 1986, introducing the concept of IFS, which considers both the degree of membership (belongs) and non-membership (not-belongs) of elements within a set. In contrast to classical logic, where an element's membership in a set is represented by a number within the interval [0, 1], intuitionistic fuzzy logic expresses an element's membership as a number also within the interval [0, 1]. In 2004, Park [8] introduced IFMS by utilizing continuous t-norm and t-conorm as a generalization of FMS originally proposed by George and Veeramani [3].

Apart from FMS, there exist various extensions and modifications to the concepts of metrics and metric spaces. The relationship between b-metric spaces and FMS has been explored in several referenced papers [6, 15, 17, 13, 19, 22]. A novel concept known as FbMS was introduced, where a less stringent form of triangle inequality is employed. Furthermore, Kamran et al. [23] proposed the concept of extended b-metric spaces and revised FP theorems that are specifically designed for these types of spaces. FbMS has emerged as a recent area of investigation. The current study serves as a motivation to further explore the adaptable applicability of these concepts.

2. Preliminary

Definition 2.1. [8] *If a binary operation, denoted as* $\diamond : [0,1]^2 \to [0,1]$*, satisfies the following conditions, it is called a continuous triangular norm:*
(i) $\mathfrak{e} \diamond 1 = 1$ *for all* $\mathfrak{e} \in [0,1]$*;*
(ii)The operation $\diamond$ *is associative, commutative and continuous;*
(iii) $\mathfrak{e} \diamond \mathfrak{d} \leq \mathfrak{p} \diamond \mathfrak{q}$ *whenever* $\mathfrak{e} \leq \mathfrak{p}$ *and* $\mathfrak{d} \leq \mathfrak{q}$.
For example $\mathfrak{e} \diamond \mathfrak{d} = \min(\mathfrak{e}, \mathfrak{d}), \mathfrak{e} \diamond \mathfrak{d} = \mathfrak{e}.\mathfrak{d}, \mathfrak{e} \diamond \mathfrak{d} = \max(\mathfrak{e} + \mathfrak{d} - 1, 0)$.

Definition 2.2. [8] *A binary operation, denoted as* $o : [0,1]^2 \to [0,1]$*, is categorized as a continuous triangular co-norm if it meets the following requirements:*
(i) The operation o exhibits associativity, commutativity and continuity;
(ii) $\mathfrak{e}\ o\ 0 = \mathfrak{e}$*, for all* $\mathfrak{e} \in [0,1]$;
(iii) $\mathfrak{e}\ o\ \mathfrak{d} \leq \mathfrak{p}\ o\ \mathfrak{q}$*, whenever* $\mathfrak{e} \leq \mathfrak{p}$ *and* $\mathfrak{d} \leq \mathfrak{q}\ \forall \mathfrak{e}, \mathfrak{d}, \mathfrak{p}, \mathfrak{q} \in [0,1]$.
For example $\mathfrak{e}\ o\ \mathfrak{d} = \ min(\mathfrak{e} + \mathfrak{d}, 1), \mathfrak{e}\ o\ \mathfrak{d} = \mathfrak{e} + \mathfrak{d} - \mathfrak{e}\mathfrak{d}, \mathfrak{e}\ o\ \mathfrak{d} = \max(\mathfrak{e}, \mathfrak{d})$.

Definition 2.3. [3] *Let E denote a non-empty set. Consider a continuous triangular norm represented by the symbol* $\diamond$ *and a fuzzy set* $\mathbb{H}$ *defined on* $E \times E \times (0, \infty)$*. The combination of set E, fuzzy set* $\mathbb{H}$*, and the operation* $\diamond$ *is referred to as a fuzzy metric space if it fulfills the following conditions. For any* $s, y \geq 0$*, and for all* $\rho_1, \rho_2, \rho_3 \in E$*:*
(FMS1) $\mathbb{H}(\rho_1, \rho_2, y) > 0$;
(FMS2) $\mathbb{H}(\rho_1, \rho_2, y) = 1$ *if and only if* $\rho_1 = \rho_2$;
(FMS3) $\mathbb{H}(\rho_1, \rho_2, y) = \mathbb{H}(\rho_2, \rho_1, y)$*;*
(FMS4) $\mathbb{H}(\rho_1, \rho_2, y) \diamond \mathbb{H}(\rho_2, \rho_3, s) \leq \mathbb{H}(\rho_1, \rho_3, y + s)$*;*
(FMS5) $\mathbb{H}(\rho_1, \rho_2, .) : (0, \infty) \to (0, 1]$ *is continuous.*

Example 2.1. [3] *Consider* $(E,\ddot{d})$ *is a metric space and* $\mathfrak{e}\diamond\mathfrak{d}=\mathfrak{e}\mathfrak{d}$ $(Also\ take\ \mathfrak{e}\diamond\mathfrak{d}=\min(\mathfrak{e},\mathfrak{d}))$, *for all* $\mathfrak{e},\mathfrak{d}\in[0,1]$ *and* $\mathbb{H}$ *is fuzzy set on* $E\times E\times(0,\infty)$, *then*

$\mathbb{H}(\rho_1,\rho_2,y)=\frac{y}{y+\ddot{d}(\rho_1,\rho_2)}$ *become a fuzzy metric space where* $\rho_1,\rho_2\in E$ *and* $y>0$.

Definition 2.4. [20] *Let* E *be a non-empty set and a binary operation* $\diamond$ *be a continuous triangular-norm and let* $b\geq 1$ *must be a real number. A fuzzy set* $\mathbb{H}$ *in* $E\times E\times(0,\infty)$ *is called fuzzy b-metric space if, for all* $\rho_1,\rho_2,\rho_3\in E$ *the following conditions hold:*

(FbMS1) $\mathbb{H}(\rho_1,\rho_2,0)=0$;

(FbMS2) $\mathbb{H}(\rho_1,\rho_2,y)=1$ *for all* $y>0$ *if and only if* $\rho_1=\rho_2$;

(FbMS3) $\mathbb{H}(\rho_1,\rho_2,y)=\mathbb{H}(\rho_2,\rho_1,y)$ $y\geq 0$;

(FbMS4) $\mathbb{H}(\rho_1,\rho_2,y)\diamond\mathbb{H}(\rho_2,\rho_3,s)\leq\mathbb{H}(\rho_1,\rho_3,b(y+s))$;

(FbMS5) $\mathbb{H}(\rho_1,\rho_2,.):[0,\infty)\to[0,1]$ *is left continuous mapping and* $\lim_{y\to\infty}\mathbb{H}(\rho_1,\rho_2,y)=1$.

The quadruple $(E,\mathbb{H},\diamond,b)$ *is called a fuzzy b-metric space, when* $b=1$, *a fuzzy b-metric space is reduced to a fuzzy metric space.*

Definition 2.5. [2] *Let* E *is an arbitrary set,* $\diamond$ *and* o *are continuous t-norm and continuous t-conorm respectively. Let* $\mathbb{H}$ *and* $\mathbb{O}$ *be fuzzy sets on* $E\times E\times(0,\infty)$ *satisfying the following conditions for all* $\rho_1,\rho_2,\rho_3\in E$,

(a) $\mathbb{H}(\rho_1,\rho_2,y)+\mathbb{O}(\rho_1,\rho_2,y)\leq 1$;

(b) $\mathbb{H}(\rho_1,\rho_2,y)>0$;

(c) $\mathbb{H}(\rho_1,\rho_2,y)=1$, *iff* $\rho_1=\rho_2$ *for all* $y>0$;

(d) $\mathbb{H}(\rho_1,\rho_2,y)=\mathbb{H}(\rho_2,\rho_1,y)$, *for all* $y>0$;

(e) $\mathbb{H}(\rho_1,\rho_3,(y+s))\geq\mathbb{H}(\rho_1,\rho_2,y)\diamond\mathbb{H}(\rho_2,\rho_3,s)$, *for all* $y,s>0$;

(f) $\mathbb{H}(\rho_1,\rho_2,.):[0,\infty)\to[0,1]$ *are continuous;*

(g) $\mathbb{O}(\rho_1,\rho_2,y)>0$;

(h) $\mathbb{O}(\rho_1,\rho_2,y)=0$, *for all* $y>0$ *iff* $\rho_1=\rho_2$;

(i) $\mathbb{O}(\rho_1,\rho_2,y)=\mathbb{O}(\rho_2,\rho_1,y)$, *for all* $y>0$;

(j) $\mathbb{O}(\rho_1,\rho_3,(y+s))\leq\mathbb{O}(\rho_1,\rho_2,y)\ o\ \mathbb{O}(\rho_2,\rho_3,s)$, *for all* $y,s>0$;

(k) $\mathbb{O}(\rho_1,\rho_2,.):[0,\infty)\to[0,1]$ *are continuous map, where* $\mathbb{O}(\rho_1,\rho_2,y)$ *and* $\mathbb{H}(\rho_1,\rho_2,y)$ *represent the degree of non-nearness and the degree of nearness between* ρ_1 *and* ρ_2 *with respect*

to y, respectively. Also, $\lim_{y\to\infty}\mathbb{H}(\rho_1,\rho_2,y)=1$ *and* $\lim_{y\to\infty}\mathbb{O}(\rho_1,\rho_2,y)=0$. *Then a 5-tuple* $(E,\mathbb{H},\mathbb{O},\diamond,o)$ *is called an IFMS.*

Example 2.2. [2] *Let* $(E,\ddot{d})$ *be a metric space and* $\mathfrak{e}\diamond\mathfrak{d}=min(\mathfrak{e},\mathfrak{d})$, $\mathfrak{e}\ o\ \mathfrak{d}=max(\mathfrak{e},\mathfrak{d})$ *for all* $\mathfrak{e},\mathfrak{d}\in[0,1]$*and let* $\mathbb{H},\mathbb{O}$ *be fuzzy sets on* $E\times E\times(0,\infty)$, *which are defined as follows:*

$$\mathbb{H}(\rho_1,\rho_2,y)=\begin{cases}\frac{y}{y+\ddot{d}(\rho_1,\rho_2)} & if\ y>0\\ 0 \quad if\ y=0\end{cases}$$

and $$\mathbb{O}(\rho_1,\rho_2,y)=\begin{cases}\frac{\ddot{d}(\rho_1,\rho_2)}{y+\ddot{d}(\rho_1,\rho_2)} & if\ y>0.\\ 1 \quad if\ y=0\end{cases}$$

Then 5-tuple $(E,\mathbb{H},\mathbb{O},\diamond,o)$ *is called an IFMS.*

Definition 2.6. [9] *Let E is an arbitrary set,* $\diamond$ *and o are continuous t-norm and continuous t-conorm respectively. Let* $\mathbb{H}$ *and* $\mathbb{O}$ *be fuzzy sets on* $E\times E\times(0,\infty)$ *satisfying the following state, for all* $\rho_1,\rho_2,\rho_3\in E, b\geq 1$, $y,s>0$.

(a) $\mathbb{H}(\rho_1,\rho_2,y)+\mathbb{O}(\rho_1,\rho_2,y)\leq 1$;

(b) $\mathbb{H}(\rho_1,\rho_2,y)>0$;

(c) $\mathbb{H}(\rho_1,\rho_2,y)=1$, *for all* $y>0$ *iff* $\rho_1=\rho_2$;

(d) $\mathbb{H}(\rho_1,\rho_2,y)=\mathbb{H}(\rho_2,\rho_1,y)$, *for all* $y>0$;

(e) $\mathbb{H}(\rho_1,\rho_3,(y+s))\geq\mathbb{H}(\rho_1,\rho_2,\frac{y}{b})\diamond\mathbb{H}(\rho_2,\rho_3,\frac{s}{b})$, *for all* $y,s>0$;

(f) $\mathbb{H}(\rho_1,\rho_2,.)$ *is non decreasing function of* R^+ *and* $\lim_{y\to\infty}\mathbb{H}(\rho_1,\rho_2,y)=1$;

(g) $\mathbb{O}(\rho_1,\rho_2,y)<1$;

(h) $\mathbb{O}(\rho_1,\rho_2,y)=0$, *for all* $y>0$ *iff* $\rho_1=\rho_2$;

(i) $\mathbb{O}(\rho_1,\rho_2,y)=\mathbb{O}(\rho_2,\rho_1,y)$, *for all* $y>0$;

(j) $\mathbb{O}(\rho_1,\rho_3,(y+s))\leq\mathbb{O}(\rho_1,\rho_2,\frac{y}{b})\ o\ \mathbb{O}(\rho_2,\rho_3,\frac{s}{b})$, *for all* $y,s>0$;

(k) $\mathbb{O}(\rho_1,\rho_2,.)$ *is non increasing function of* R^+ *and* $\lim_{y\to\infty}\mathbb{O}(\rho_1,\rho_2,y)=0$.

Also, $\lim_{y\to\infty}\mathbb{H}(\rho_1,\rho_2,y)=1$ *and* $\lim_{y\to\infty}\mathbb{O}(\rho_1,\rho_2,y)=0$. *Then a 5-tuple* $(E,\mathbb{H},\mathbb{O},\diamond,o)$ *is called an IFbMS.*

3. Main Results

Definition 3.1. *A 5-tuple set* $(E,\mathbb{H}_\zeta,\mathbb{O}_\zeta,\diamond,o)$ *is said to be an EIFbMS, if E is a non-empty set,* $\diamond$ *and o is continuous t-norm and continuous t-conorm respectively and* $\zeta: E\times E\to[1,\infty)$ *be* *a function, $\mathbb{H}_\zeta$ and $\mathbb{O}_\zeta$ are fuzzy sets on $E \times E \times (0,\infty)$ satisfying the following conditions with $b \geq 1$, $s, y > 0$ and for all $\rho_1, \rho_2, \rho_3 \in E$.*

(a) $\mathbb{H}_\zeta(\rho_1,\rho_2,y) + \mathbb{O}_\zeta(\rho_1,\rho_2,y) \leq 1$*;*

(b) $\mathbb{H}_\zeta(\rho_1,\rho_2,y) > 0$;

(c) $\mathbb{H}_\zeta(\rho_1,\rho_2,y) = 1$*, for all* $y > 0$ *iff* $\rho_1 = \rho_2$*;*

(d) $\mathbb{H}_\zeta(\rho_1,\rho_2,y) = \mathbb{H}_\zeta(\rho_2,\rho_1,y)$*, for all* $y > 0$*;*

(e) $\mathbb{H}_\zeta(\rho_1,\rho_3,\zeta(\rho_1,\rho_3)(y+s)) \geq \mathbb{H}_\zeta(\rho_1,\rho_2,\frac{y}{b}) \diamond \mathbb{H}_\zeta(\rho_2,\rho_3,\frac{s}{b})$*, for all* $y, s > 0$*;*

(f) $\mathbb{H}_\zeta(\rho_1,\rho_2,.)$ *is non decreasing function of* $\mathbb{R}^+$ *and* $\lim_{y\to\infty} \mathbb{H}_\zeta(\rho_1,\rho_2,y) = 1$;

(g) $\mathbb{O}_\zeta(\rho_1,\rho_2,y) < 1$*;*

(h) $\mathbb{O}_\zeta(\rho_1,\rho_2,y) = 0$*, for all* $y > 0$ *iff* $\rho_1 = \rho_2$*;*

(i) $\mathbb{O}_\zeta(\rho_1,\rho_2,y) = \mathbb{O}_\zeta(\rho_2,\rho_1,y)$*, for all* $y > 0$*;*

(j) $\mathbb{O}_\zeta(\rho_1,\rho_3,\zeta(\rho_1,\rho_3)(y+s)) \leq \mathbb{O}_\zeta(\rho_1,\rho_2,\frac{y}{b})\ o\ \mathbb{O}_\zeta(\rho_2,\rho_3,\frac{s}{b})$*, for all* $y, s > 0$*;*

(k) $\mathbb{O}_\zeta(\rho_1,\rho_2,.)$ *is non increasing function of* $\mathbb{R}^+$ *and* $\lim_{y\to\infty} \mathbb{O}_\zeta(\rho_1,\rho_2,y) = 0$.

Also, the value $\lim_{y\to\infty} \mathbb{H}_\zeta(\rho_1,\rho_2,y) = 1$ *and* $\lim_{y\to\infty} \mathbb{O}_\zeta(\rho_1,\rho_2,y) = 0$.

Example 3.1. *Let $\mathbb{H}_\zeta, \mathbb{O}_\zeta$ be fuzzy sets on $E \times E \times (0,\infty)$ and $E = \{1,2,3\}$, $\ddot{d}_b : E \times E \to R$ with $\ddot{d}_b(\mathfrak{e},\mathfrak{d}) = (\mathfrak{e}-\mathfrak{d})^2$ and $\mathfrak{e} \diamond \mathfrak{d} = min(\mathfrak{e},\mathfrak{d})$, $\mathfrak{e}\ o\ \mathfrak{d} = max(\mathfrak{e},\mathfrak{d})$ for all $\mathfrak{e},\mathfrak{d} \in [0,1]$, $\zeta(\rho_1,\rho_3) = 1+\rho_1+\rho_3$ defined as follows:*

$$\mathbb{H}(\rho_1,\rho_2,y) = \begin{cases} \frac{y}{y+\ddot{d}_b(\rho_1,\rho_2)} & if\ y > 0 \\ 0 \quad if\ y = 0 \end{cases}$$

and $$\mathbb{O}(\rho_1,\rho_2,y) = \begin{cases} \frac{\ddot{d}_b(\rho_1,\rho_2)}{y+\ddot{d}_b(\rho_1,\rho_2)} & if\ y > 0. \\ 1 \quad if\ y = 0 \end{cases}$$

$\ddot{d}_b(1,1) = 0$, $\ddot{d}_b(2,2) = 0$, $\ddot{d}_b(3,3) = 0$, $\ddot{d}_b(1,2) = 1$, $\ddot{d}_b(2,1) = 1$, $\ddot{d}_b(2,3) = 1$, $\ddot{d}_b(3,2) = 1$, $\ddot{d}_b(3,1) = 4$, $\ddot{d}_b(1,3) = 4$, $\zeta(1,1) = 3$, $\zeta(2,2) = 5$, $\zeta(3,3) = 7$, $\zeta(1,2) = 4$, $\zeta(2,1) = 4$, $\zeta(2,3) = 6$, $\zeta(3,2) = 6$, $\zeta(1,3) = 5$, $\zeta(3,1) = 5$.

Properties (a)- (d) of Definition (3.1) are satisfied we need to check only property (e),

$$\mathbb{H}_\zeta(\rho_1,\rho_3,\zeta(\rho_1,\rho_3)(y+s)) \geq \mathbb{H}_\zeta(\rho_1,\rho_2,\frac{y}{b}) \diamond \mathbb{H}_\zeta(\rho_2,\rho_3,\frac{s}{b}),\ for\ all\ y,s > 0. \tag{1}$$

Now, take L.H.S of Equation (1),

$$\mathbb{H}_{\zeta}(\rho_1,\rho_3,\zeta(\rho_1,\rho_3)(y+s)) = \frac{\zeta(\rho_1,\rho_3)(y+s)}{\zeta(\rho_1,\rho_3)(y+s)+\ddot{d}_b(\rho_1,\rho_3)}. \tag{2}$$

Put $\rho_1 = 1, \rho_3 = 2$ *in Equation (2),*

$$\mathbb{H}_{\zeta}(1,2,\zeta(1,2)(y+s)) = \frac{\zeta(1,2)(y+s)}{\zeta(1,2)(y+s)+\ddot{d}_b(1,2)}$$

$$= \frac{4(y+s)}{4(y+s)+1} = 1-\frac{1}{4(y+s)+1} \tag{3}$$

Take, R.H.S of Equation (1), put b=1, $\rho_1 = 1, \rho_3 = 2, \rho_2 = 3,$

$$\mathbb{H}_{\zeta}(1,3,y) = \frac{y}{y+\ddot{d}_b(1,3)} = \frac{y}{y+4} = 1-\frac{4}{y+4} \tag{4}$$

and

$$\mathbb{H}_{\zeta}(3,2,s) = \frac{s}{s+\ddot{d}_b(3,2)} = \frac{s}{s+1} = 1-\frac{1}{s+1}. \tag{5}$$

So, that $\mathbb{H}_{\zeta}(1,2,\zeta(1,2)(y+s)) = 1-\frac{1}{4(y+s)+1}$

$$= 1-\frac{4}{16y+16s+4} > 1-\frac{4}{16y+4} > 1-\frac{4}{y+4} \tag{6}$$

This show that, $\mathbb{H}_{\zeta}(1,2,\zeta(1,2)(y+s)) > \mathbb{H}_{\zeta}(1,3,y)$.

Similarly , $\mathbb{H}_{\zeta}(1,2,\zeta(1,2)(y+s)) > \mathbb{H}_{\zeta}(3,2,s)$. *So that,*

$$\mathbb{H}_{\zeta}(1,2,\zeta(1,2)(y+s)) \geq min\{\mathbb{H}_{\zeta}(1,3,y),\mathbb{H}_{\zeta}(3,2,s)\}$$

$$= \mathbb{H}_{\zeta}(1,3,y)\diamond\mathbb{H}_{\zeta}(3,2,s).$$

Analogously, it can be demonstrated that,

$$\mathbb{H}_{\zeta}(1,3,\zeta(1,3)(y+s)) \geq \mathbb{H}_{\zeta}(1,2,y)\diamond\mathbb{H}_{\zeta}(2,3,s)$$

and

$$\mathbb{H}_{\zeta}(2,3,\zeta(2,3)(y+s)) \geq \mathbb{H}_{\zeta}(2,1,y)\diamond\mathbb{H}_{\zeta}(1,3,s).$$

Hence for all $\rho_1,\rho_2,\rho_3 \in E$,

$$\mathbb{H}_{\zeta}(\rho_1,\rho_3,\zeta(\rho_1,\rho_3)(y+s)) \geq \mathbb{H}_{\zeta}(\rho_1,\rho_2,\frac{y}{b})\diamond\mathbb{H}_{\zeta}(\rho_2,\rho_3,\frac{s}{b}).$$

Properties from (f)-(i) of Definition 3.1 are clearly satified we will check only property (j),

$$\mathbb{O}_\zeta(\rho_1,\rho_3,\zeta(\rho_1,\rho_3)(y+s)) \leq \mathbb{O}_\zeta(\rho_1,\rho_2,\frac{y}{b}) \diamond \mathbb{O}_\zeta(\rho_2,\rho_3,\frac{s}{b}) \tag{7}$$

for all $y,s>0$

Take L.H.S of Equation (7),

$$\mathbb{O}_\zeta(\rho_1,\rho_3,\zeta(\rho_1,\rho_3)(y+s)) = \frac{\zeta(\rho_1,\rho_3)(y+s)}{\zeta(\rho_1,\rho_3)(y+s)+\ddot{d}_b(\rho_1,\rho_3)} \tag{8}$$

Put $\rho_1=1,\rho_3=2$ *in Equation (8),*

$$\mathbb{O}_\zeta(1,2,\zeta(1,2)(y+s)) = \frac{\zeta(1,2)(y+s)}{\zeta(1,2)(y+s)+\ddot{d}_b(1,2)}$$

$$= \frac{4(y+s)}{4(y+s)+1} = 1-\frac{1}{4(y+s)+1} \tag{9}$$

Take R.H.S of Equation (7) with b=1,

$$\mathbb{O}_\zeta(1,3,y) = \frac{y}{y+\ddot{d}_b(1,3)} = \frac{y}{y+4} = 1-\frac{4}{y+4}$$

and

$$\mathbb{O}_\zeta(3,2,s) = \frac{s}{s+\ddot{d}_b(3,2)} = \frac{s}{s+1} = 1-\frac{1}{s+1}.$$

So, $\mathbb{O}_\zeta(1,2,\zeta(1,2)(y+s))= 1-\frac{1}{4(y+s)+1} = 1-\frac{4}{16y+16s+4}$

$$< 1-\frac{4}{16y+4} < 1-\frac{4}{y+4}. \tag{10}$$

This show that, $\mathbb{O}_\zeta(1,2,\zeta(1,2)(y+s)) < \mathbb{O}_\zeta(1,3,y)$.

Similarly , $\mathbb{O}_\zeta(1,2,\zeta(1,2)(y+s)) < \mathbb{O}_\zeta(3,2,s)$ *so that*

$$\mathbb{O}_\zeta(1,2,\zeta(1,2)(y+s)) \leq min\{\mathbb{O}_\zeta(1,3,y),\mathbb{O}_\zeta(3,2,s)\}$$

$$= \mathbb{O}_\zeta(1,3,y) \diamond \mathbb{O}_\zeta(3,2,s).$$

Analogously, it can be demonstrated that,

$$\mathbb{O}_\zeta(1,3,\zeta(1,3)(y+s)) \leq \mathbb{O}_\zeta(1,2,y) \diamond \mathbb{O}_\zeta(2,3,s)$$

and

$$\mathbb{O}_\zeta(2,3,\zeta(2,3)(y+s)) \leq \mathbb{O}_\zeta(2,1,y) \diamond \mathbb{O}_\zeta(1,3,s)$$

Hence, all value of $\rho_1, \rho_2, \rho_3 \in E$,

$$\mathbb{O}_\zeta(\rho_1, \rho_3, \zeta(\rho_1, \rho_3)(y+s)) \leq \mathbb{O}_\zeta(\rho_1, \rho_2, \frac{y}{b}) \diamond \mathbb{O}_\zeta(\rho_2, \rho_3, \frac{s}{b})$$

Then, $(E, \mathbb{H}_\zeta, \mathbb{O}_\zeta, \diamond, o)$ *is an EIFbMS.*

Lemma 3.1. *Let* $\{v_n\}$ *be a sequence in EIFbMS* $(E, \mathbb{H}_\zeta, \mathbb{O}_\zeta, \diamond, o)$. *Suppose that there exist* $\lambda \in (0, \frac{1}{b})$ *such that*

$$\mathbb{H}_\zeta(v_{n-1}, v_n, y) \geq \mathbb{H}_\zeta(v_{n-1}, v_n, \frac{y}{\lambda}), n \in \mathbb{N} \tag{11}$$

and

$$\mathbb{O}_\zeta(v_{n-1}, v_n, y) \leq \mathbb{O}_\zeta(v_{n-1}, v_n, \frac{y}{\lambda}), n \in \mathbb{N}. \tag{12}$$

Also, there exist $v_0, v_1 \in E, y > 0$ *and* $l \in (0,1)$ *such that*

$$\lim_{n\to\infty} \sum_{i=n}^{\infty} \mathbb{H}_\zeta(v_0, v_1, \frac{y}{l^i}) = 1 \text{ and } \lim_{n\to\infty} \sum_{i=n}^{\infty} \mathbb{O}_\zeta(v_0, v_1, \frac{y}{l^i}) = 0. \tag{13}$$

Then $\{v_n\}$ *is a Cauchy sequence.*

Proof. For every $y > 0$, considering $n > m > n_0$, since $\mathbb{H}_\zeta$ is b-non decreasing and $\mathbb{G}_\zeta$ is b-non increasing.

The series $\sum_{i=1}^{\infty} c^i$ is convergent. Hence, there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} c^i < 1$ for every $n > n_0$, $c \in (b\lambda, 1)$.

Now,

$$\begin{aligned}
\mathbb{H}_\zeta(v_n, v_{n+m}, y) &\geq \mathbb{H}_\zeta(v_n, v_{n+m}, \frac{y\sum_{i=n}^{n+m-1} c^i}{b}) \\
&\geq \mathbb{H}_\zeta(v_n, v_{n+1}, \frac{yc^n}{b}) \diamond \mathbb{H}_\zeta(v_{n+1}, v_{n+m}, \frac{y\sum_{i=n+1}^{n+m-1} c^i}{b^2}) \\
&\geq \mathbb{H}_\zeta(v_n, v_{n+1}, \frac{yc^n}{b}) \diamond \mathbb{H}_\zeta(v_{n+1}, v_{n+2}, \frac{yc^{n+1}}{b^2}) \ldots \diamond \mathbb{H}_\zeta(v_{n+m-1}, v_{n+m}, \frac{yc^{n+m-1}}{b^m})
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{O}_\zeta(v_n, v_{n+m}, y) &\leq \mathbb{O}_\zeta(v_n, v_{n+m}, \frac{y\sum_{i=n}^{n+m-1} c^i}{b}) \\
&\leq \mathbb{O}_\zeta(v_n, v_{n+1}, \frac{yc^n}{b}) \; o \; \mathbb{O}_\zeta(v_{n+1}, v_{n+m}, \frac{y\sum_{i=n+1}^{n+m-1} c^i}{b^2})
\end{aligned}$$

$$\leq \mathbb{O}_\zeta(v_n, v_{n+1}, \frac{yc^n}{b}) \ o \ \mathbb{O}_\zeta(v_{n+1}, v_{n+2}, \frac{yc^{n+1}}{b^2})... \ o \ \mathbb{O}_\zeta(v_{n+m-1}, v_{n+m}, \frac{yc^{n+m-1}}{b^m}).$$

From Equation (11) and (12) we have

$$\mathbb{H}_\zeta(v_{n-1}, v_n, y) \geq \mathbb{H}_\zeta(v_{n-1}, v_n, \frac{y}{\lambda}), n \in \mathbb{N}$$

and

$$\mathbb{O}_\zeta(v_{n-1}, v_n, y) \leq \mathbb{O}_\zeta(v_{n-1}, v_n, \frac{y}{\lambda}), n \in \mathbb{N}.$$

if $n > m$ and $b > 1$, we have

$$\begin{aligned}
\mathbb{H}_\zeta(v_n, v_{n+m}, y) &\geq \mathbb{H}_\zeta(v_0, v_1, \frac{yc^n}{b^2\lambda^n}) \diamond \mathbb{H}_\zeta(v_0, v_1, \frac{yc^{n+1}}{b^3\lambda^{n+1}}) \diamond ... \mathbb{H}_\zeta(v_0, v_1, \frac{yc^{n+m-1}}{b^m\lambda^{n+m-1}}) \\
&\geq \sum_{i=n}^{n+m-1} \mathbb{H}_\zeta(v_0, v_1, \frac{yc^i}{b^{i-n+2}\lambda^i}) \\
&\geq \sum_{i=n}^{n+m-1} \mathbb{H}_\zeta(v_0, v_1, \frac{yc^i}{b^i\lambda^i}) \\
&\geq \sum_{i=n}^{\infty} \mathbb{H}_\zeta(v_0, v_1, \frac{y}{l^i})
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{O}_\zeta(v_n, v_{n+m}, y) &\leq \mathbb{O}_\zeta(v_0, v_1, \frac{yc^n}{b^2\lambda^n}) \ o \ \mathbb{O}_\zeta(v_0, v_1, \frac{yc^{n+1}}{b^3\lambda^{n+1}}) \ o \ ... \mathbb{H}_\zeta(v_0, v_1, \frac{yc^{n+m-1}}{b^m\lambda^{n+m-1}}) \\
&\leq \sum_{i=n}^{n+m-1} \mathbb{O}_\zeta(v_0, v_1, \frac{yc^i}{b^{i-n+2}\lambda^i}) \\
&\leq \sum_{i=n}^{n+m-1} \mathbb{O}_\zeta(v_0, v_1, \frac{yc^i}{b^i\lambda^i}) \\
&\leq \sum_{i=n}^{\infty} \mathbb{O}_\zeta(v_0, v_1, \frac{y}{l^i}),
\end{aligned}$$

where $l = \frac{b\lambda}{c}$. As $l \in (0,1)$, from (13) we get $\{v_n\}$ is a Cauchy sequence. □

Theorem 3.1. *Let* $(E, \mathbb{H}_\zeta, \mathbb{O}_\zeta, \diamond, o)$ *is a complete EIFBMS with the mapping* $\zeta : E \times E \to [1, \infty)$ *and* $for\ all$ $\rho_1, \rho_2 \in E$, *such that*

$$\lim_{n\to\infty} \mathbb{H}_\zeta(\rho_1, \rho_2, y) = 1 \tag{14}$$

and

$$\lim_{n\to\infty} \mathbb{O}_\zeta(\rho_1,\rho_2,y)=0 \tag{15}$$

and a mapping $g:E\to E$ which satisfies the following Equation

$$\mathbb{H}_\zeta(g\rho_1,g\rho_2,ky)\geq \mathbb{H}_\zeta(\rho_1,\rho_2,y) \tag{16}$$

and

$$\mathbb{O}_\zeta(g\rho_1,g\rho_2,ky)\leq \mathbb{O}_\zeta(\rho_1,\rho_2,y), \tag{17}$$

where $k\in(0,1)$. Let us take an arbitrary $a_0\in E,n,q\in\mathbb{N}$ we have $\zeta(a_n,a_{n+q})<\frac{1}{k}$ where $a_n=g^n a_0$, then a mapping g has a unique FP.

Proof. Let's start from an arbitrary point $a_0\in E$ and make a sequence $\{a_n\}$ with the help of iterative process $a_n=g^n a_0, n\in\mathbb{N}$. For all $n,y>0$, by an iterative process of the contractive condition from Equation (16), we get that

$$\begin{aligned}\mathbb{H}_\zeta(a_n,a_{n+1},ky)&=\mathbb{H}_\zeta(ga_{n-1},ga_n,ky)\\&\geq \mathbb{H}_\zeta(a_{n-1},a_n,y)\\&\geq \mathbb{H}_\zeta(a_{n-2},a_{n-1},\frac{y}{k})\\&\geq \mathbb{H}_\zeta(a_{n-3},a_{n-2},\frac{y}{k^2})\ldots\\&\geq \mathbb{H}_\zeta(a_0,a_1,\frac{y}{k^{n-1}}).\end{aligned}$$

So, we have

$$\mathbb{H}_\zeta(a_n,a_{n+1},ky)\geq \mathbb{H}_\zeta(a_0,a_1,\frac{y}{k^{n-1}}). \tag{18}$$

For any $q\in\mathbb{N}$, write down $y=\frac{qy}{q}=\frac{y}{q}+....+\frac{y}{q}$ and make use of Definition 3.1(e) repeatedly,

$$\begin{aligned}\mathbb{H}_\zeta(a_n,a_{n+q},y)\geq{}& \mathbb{H}_\zeta(a_n,a_{n+1},\frac{y}{bq\zeta(a_n,a_{n+q})})\diamond\\&\mathbb{H}_\zeta(a_{n+1},a_{n+2},\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})})\diamond\\&\mathbb{H}_\zeta(a_{n+2},a_{n+3},\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})(a_{n+2},a_{n+q})})\diamond....\diamond\end{aligned} \tag{19}$$

$$\mathbb{H}_\zeta(a_{n+q-1},a_{n+q},\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})(a_{n+2},a_{n+q})...(a_{n+q-1},a_{n+q})}).$$

Using Equation (18) and Definition 3.1(e) we obtain,

$$\mathbb{H}_\zeta(a_n,a_{n+q},y) \geq \mathbb{H}_\zeta(a_0,a_1,\frac{y}{bq\zeta(a_n,a_{n+q})k^n}) \diamond \mathbb{H}_\zeta(a_0,a_1,\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})k^{n+1}}) \diamond$$

$$\mathbb{H}_\zeta(a_0,a_1,\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})(a_{n+2},a_{n+q})k^{n+3}}) \diamond ... \diamond$$

$$\mathbb{H}_\zeta(a_0,a_1,\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})(a_{n+2},a_{n+q})...(a_{n+q-1},a_{n+q})k^{n+q-1}}).$$

And from contractive condition (17),

$$\begin{aligned}\mathbb{O}_\zeta(a_n,a_{n+1},ky) = \mathbb{O}_\zeta(ga_{n-1},ga_n,ky) &\leq \mathbb{O}_\zeta(a_{n-1},a_n,y)\\ &\leq \mathbb{O}_\zeta(a_{n-2},a_{n-1},\frac{y}{k})\\ &\leq \mathbb{O}_\zeta(a_{n-3},a_{n-2},\frac{y}{k^2})...\\ &\leq \mathbb{O}_\zeta(a_0,a_1,\frac{y}{k^{n-1}}).\end{aligned}$$

So, we have

$$\mathbb{O}_\zeta(a_n,a_{n+1},ky) \leq \mathbb{O}_\zeta(a_0,a_1,\frac{y}{k^{n-1}}). \tag{20}$$

For any $q \in \mathbb{N}$, write down $y = \frac{qt}{q} = \frac{y}{q} + ... + \frac{y}{q}$ and make use of Definition 3.1(j) repeatedly,

$$\mathbb{O}_\zeta(a_n,a_{n+q},y) \leq \mathbb{O}_\zeta(a_n,a_{n+1},\frac{y}{bq\zeta(a_n,a_{n+q})})\ o\ \mathbb{O}_\zeta(a_{n+1},a_{n+2},\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})})$$

$$o\ \mathbb{O}_\zeta(a_{n+2},a_{n+3},\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})(a_{n+2},a_{n+q})})\ o\ ...$$

$$o\ \mathbb{O}_\zeta(a_{n+q-1},a_{n+q},\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})(a_{n+2},a_{n+q})...(a_{n+q-1},a_{n+q})}).$$

Using Equation (20) and Definition 3.1(j) we obtain,

$$\mathbb{O}_\zeta(a_n,a_{n+q},y) \leq \mathbb{O}_\zeta(a_0,a_1,\frac{y}{bq\zeta(a_n,a_{n+q})k^n})\ o\ \mathbb{O}_\zeta(a_0,a_1,\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})k^{n+1}})$$

$$o\ \mathbb{O}_\zeta(a_0,a_1,\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})(a_{n+2},a_{n+q})k^{n+3}})\ o\ .....$$

$$o\ \mathbb{O}_\zeta(a_0,a_1,\frac{y}{bq\zeta(a_n,a_{n+q})\zeta(a_{n+1},a_{n+q})(a_{n+2},a_{n+q})...(a_{n+q-1},a_{n+q})k^{n+q-1}}).$$

For all $n,q \in \mathbb{N}$ we know that $\zeta(a_n,a_{n+q})k < 1$ with $k \in (0,1)$.

Now using Equation (18) and (20) and $n \to \infty$, we get

$$\lim_{n\to\infty} \mathbb{H}_\zeta(a_n, a_{n+q}, y) = 1 \diamond 1 \diamond 1 \diamond ...1 = 1.$$

and

$$\lim_{n\to\infty} \mathbb{O}_\zeta(a_n, a_{n+q}, y) = 0 \ o \ 0 \ o \ 0 \ o \ ...0 = 0.$$

Therefore $\{a_n\}$ is a Cauchy sequence. After $(E, \mathbb{H}_\zeta, \mathbb{O}_\zeta, \diamond, o)$ is a complete EIFBMS there exist $a \in E$ such that $\lim_{n\to\infty}\{a_n\} = a$. We have to prove a is the FP of g.

Using 3.1(e) and 3.1(j), we get

$$\begin{aligned}
\mathbb{H}_\zeta(ga, g, y) &\geq \mathbb{H}_\zeta(ga, ga_n, \frac{y}{2\zeta(ga,a)}) \diamond \mathbb{H}_\zeta(ga_n, a, \frac{y}{2\zeta(ga,a)}) \\
&\geq \mathbb{H}_\zeta(a, a_n, \frac{y}{2\zeta(ga,a)k}) \diamond \mathbb{H}_\zeta(a_{n+1}, a_n, \frac{y}{2\zeta(ga,a)}) \\
&\longrightarrow 1 \diamond 1 = 1, \ as \ n \to \infty \text{ and} \\
\mathbb{O}_\zeta(ga, g, y) &\leq \mathbb{O}_\zeta(ga, ga_n, \frac{y}{2\zeta(ga,a)}) \ o \ \mathbb{O}_\zeta(ga_n, a, \frac{y}{2\zeta(ga,a)}) \\
&\leq \mathbb{O}_\zeta(a, a_n, \frac{y}{2\zeta(ga,a)k}) \ o \ \mathbb{O}_\zeta(a_{n+1}, a_n, \frac{y}{2\zeta(ga,a)}) \\
&\longrightarrow 0 \ o \ 0 = 0, \ \text{ as } n \longrightarrow \infty.
\end{aligned}$$

ga=g so, a is the FP of g. To prove uniqueness, consider l is the other FP such that $gl = l$ for arbitrary $l \in E$ then

$$\begin{aligned}
\mathbb{H}_\zeta(l, a, y) &= \mathbb{H}_\zeta(gl, ga, y) \\
&\geq \mathbb{H}_\zeta(l, a, \frac{y}{k}) \\
&= \mathbb{H}_\zeta(gl, ga, \frac{y}{k}) \\
&\geq \mathbb{H}_\zeta(l, a, \frac{y}{k^2}) \\
&\geq ...\mathbb{H}_\zeta(l, a, \frac{y}{k^n}) \\
&\longrightarrow 1 \ as \ n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{O}_\zeta(l, a, y) &= \mathbb{O}_\zeta(gl, ga, y) \\
&\leq \mathbb{O}_\zeta(l, a, \frac{y}{k})
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{O}_\zeta(gl, ga, \frac{y}{k}) \\
&\leq \mathbb{O}_\zeta(l, a, \frac{y}{k^2}) \\
&\leq ...\mathbb{O}_\zeta(l, a, \frac{y}{k^n}) \\
&\longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

Therefore $a = l$.

This completes the proof. □

Example 3.2. *Let* $E = [0,1]$ *and* $\mathbb{H}_\zeta(\rho_1, \rho_2, y) = (\frac{1}{y})^{(\rho_1-\rho_2)^2}$ *and* $\mathbb{O}_\zeta(\rho_1, \rho_2, y) = 1-(\frac{1}{y})^{(\rho_1-\rho_2)^2}$ *and sequence* $\{a_n\} = \frac{1}{2n}$ *and* $\zeta = 1+\rho_1+\rho_2$ *. It is simple to confirm that* $(E, \mathbb{H}_\zeta, \mathbb{O}_\zeta, \diamond, o)$ *is a complete EIFBMS. Let* $g : E \to E$ *be a mapping defined as* $g(\rho_1) = 1-\rho_1$ *for all* $y > 0$, $k \in (0,1)$ *we have,*

$$
\begin{aligned}
\mathbb{H}_\zeta(g\rho_1, g\rho_2, y) &= \mathbb{H}_\zeta(1-\rho_1, 1-\rho_2, ky) \\
&= (\frac{1}{ky})^{(1-\rho_1-1+\rho_2)^2} \\
&= (\frac{1}{ky})^{(\rho_2-\rho_1)^2} \\
&= (\frac{1}{ky})^{(\rho_1-\rho_2)^2}.
\end{aligned}
$$

Now, $k \in (0,1)$ *i.e* $k < 1$ *implies* $ky < y$ *because* $y > 0$ *implies* $\frac{1}{ky} > \frac{1}{y}$.

So,

$$
\mathbb{H}_\zeta(1-\rho_1, 1-\rho_2, ky) > (\frac{1}{y})^{(\rho_1-\rho_2)^2} = \mathbb{H}_\zeta(\rho_1, \rho_2, y) \text{ and}
$$

$$
\begin{aligned}
\mathbb{O}_\zeta(g\rho_1, g\rho_2, y) &= \mathbb{O}_\zeta(1-\rho_1, 1-\kappa_2, ky) \\
&= 1-(\frac{1}{ky})^{(1-\rho_1-1+\rho_2)^2} \\
&= 1-(\frac{1}{ky})^{(\rho_2-\rho_1)^2} \\
&= 1-(\frac{1}{ky})^{(\rho_1-\rho_2)^2}.
\end{aligned}
$$

Now, $k \in (0,1)$ *i.e* $k < 1$ *implies* $ky < y$ *because* $y > 0$

$$
\frac{1}{ky} > \frac{1}{y}
$$

$$(\frac{1}{ky})^{(\rho_1-\rho_2)^2} > (\frac{1}{y})^{(\rho_1-\rho_2)^2}$$

$$-(\frac{1}{ky})^{(\rho_1-\rho_2)^2} < -(\frac{1}{y})^{(\rho_1-\rho_2)^2}$$

$$1-(\frac{1}{ky})^{(\rho_1-\rho_2)^2} < 1-(\frac{1}{y})^{(\rho_1-\rho_2)^2}.$$

So, from above Equation $\mathbb{O}_\zeta(1-\rho_1, 1-\rho_2, ky) < 1-(\frac{1}{t})^{(\rho_1-\rho_2)^2} = \mathbb{O}_\zeta(\rho_1,\rho_2,y)$ $\zeta(a_n.a_{n+1}) < \frac{1}{k}$ *satisfy so* g *has a unique fixed point and that point is* $\frac{1}{2}$.

4. Application to Fredholm Integral Equations

Let $E = C([e,s]^2,\mathbb{R})$ be the set of all continuous real-valued functions defined on the interval $[e,s]\times[e,s]$. Now, we let the fuzzy integral Equation

$$\rho_1(v) = s(j) + \beta\int_e^s F(v,j)\rho_1(v)\,dj \quad \text{for } v, j \in [e,s] \tag{21}$$

where $\beta > 0$ is a tringular shaped fuzzy number, $s(j)$ is a fuzzy function of $j \in [e,s]$, and $F \in E$. Define $\mathbb{H}_\zeta$ and $\mathbb{G}_\zeta$ by

$$\mathbb{H}_\zeta(\rho_1(v),\rho_2(v),y) = \sup_{v\in[e,s]} \frac{y}{y+\max\{\rho_1(v),\rho_2(v)\}^2}$$

$$\mathbb{G}_\zeta(\rho_1(v),\rho_2(v),y) = 1-\sup_{v\in[e,s]} \frac{y}{y+\max\{\rho_1(v),\rho_2(v)\}^2}$$

for all $\rho_1,\rho_2 \in E$ and $y > 0$, with the CTN and CTCN defined by $\rho_1 \diamond \rho_2 = \rho_1.\rho_2$ and $\rho_1 \circ \rho_2 = \max(\rho_1,\rho_2)$. Define $\zeta : E\times E \to [1,\infty)$ by

$$\zeta(\rho_1,\rho_2) = \begin{cases} 1 & \text{if } \rho_1 = \rho_2; \\ 1+\max\{\rho_1,\rho_2\} & \text{if otherwise.} \end{cases}$$

Then $(E,\mathbb{H}_\zeta,\mathbb{G}_\zeta,\diamond,\circ)$ is a complete EIEBMS.

Assume that $\max\{F(v,j)\rho_1(v), F(v,j)\rho_2(v)\} \le \max\{\rho_1(v),\rho_2(v)\}$ for $\rho_1,\rho_2 \in E$, $k \in (0,1)$, and for all $v, j \in [e,s]$. Also consider

$$\beta\int_e^s \frac{dj}{\sqrt{2}} \le k < 1.$$

Then the fuzzy integral equation in Eq. (21) has a unique solution.

Proof. Define $g : E \to E$ by

$$g\rho_1(\nu) = s(j) + \beta \int_e^s F(\nu, j)\rho_1(\nu)\, dj \quad \text{for all } \nu, j \in [e,s].$$

Now for all $\rho_1, \rho_2 \in E$, we obtain

$$\begin{aligned}
&\mathbb{H}_\zeta(g\rho_1(\nu), g\rho_2(\nu), ky)\\
&= \sup_{\nu\in[e,s]} \frac{ky}{ky + \max\{g\rho_1(\nu), g\rho_2(\nu)\}^2}\\
&= \sup_{\nu\in[e,s]} \frac{ky}{ky + \max\left(s(j) + \beta \int_e^s F(\nu, j)\rho_1(\nu)\, dj, s(j) + \beta \int_e^s F(\nu, j)\rho_1(\nu)\, dj\right)^2}\\
&= \sup_{\nu\in[e,s]} \frac{ky}{ky + \max\left(\beta \int_e^s F(\nu, j)\rho_1(\nu)\, dj, \beta \int_e^s F(\nu, j)\rho_1(\nu)\, dj\right)^2}\\
&= \sup_{\nu\in[e,s]} \frac{ky}{ky + \max\{F(\nu, j)\rho_1(\nu), F(\nu, j)\rho_2(\nu)\}^2}\\
&\geq \sup_{\nu\in[e,s]} \frac{y}{y + \max\{\rho_1(\nu), \rho_2(\nu)\}^2}\\
&\geq \mathbb{H}_\zeta(\rho_1(\nu), \rho_2(\nu), y).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{G}_\zeta(g\rho_1(\nu), g\rho_2(\nu), ky)\\
&= 1 - \sup_{\nu\in[e,s]} \frac{ky}{ky + \max\{g\rho_1(\nu), g\rho_2(\nu)\}^2}\\
&= 1 - \sup_{\nu\in[e,s]} \frac{ky}{ky + \max\left(s(j) + \beta \int_e^s F(\nu, j)\rho_1(\nu)\, dj, s(j) + \beta \int_e^s F(\nu, j)\rho_1(\nu)\, dj\right)^2}\\
&= 1 - \sup_{\nu\in[e,s]} \frac{ky}{ky + \max\left(\beta \int_e^s F(\nu, j)\rho_1(\nu)\, dj, \beta \int_e^s F(\nu, j)\rho_1(\nu)\, dj\right)^2}\\
&= 1 - \sup_{\nu\in[e,s]} \frac{ky}{ky + \max\{F(\nu, j)\rho_1(\nu), F(\nu, j)\rho_2(\nu)\}^2}\\
&\leq \sup_{\nu\in[e,s]} \frac{y}{y + \max\{\rho_1(\nu), \rho_2(\nu)\}^2}\\
&\leq \mathbb{G}_\zeta(\rho_1(\nu), \rho_2(\nu), y).
\end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are fulfilled. Hence, operator g has a unique fixed point (FP). This implies that the fuzzy integral Eq. (21) has a unique solution. $\square$

Corollary 4.1. *Let $(E, \mathbb{H}_\zeta, \mathbb{G}_\zeta, \diamond, \circ)$ be a complete EIFBMS. Define $g : E \to E$ as*

$$g\rho_1(\nu) = s(j) + \beta \int_e^s F(\nu, j)\rho_1(\nu)\, dj \quad \textit{for all } \nu, j \in [e, s].$$

Suppose the below conditions are met:

I. $\max\{F(\nu, j)\rho_1(\nu), F(\nu, j)\rho_2(\nu)\} \leq \max\{\rho_1(\nu), \rho_2(\nu)\}$ *for* $\rho_1, \rho_2 \in E$, $k \in (0,1)$, *and for all* $\nu, j \in [e, s]$.

II. $\beta \int_e^s \frac{dj}{\sqrt{2}} \leq k < 1$.

Then the integral Equation (21) has a solution. We can easily prove this by following the above proof.

Conclusion

The aims of this paper is introduced and explored different concepts in EIFbMS, focusing on a satisfactory notion for the EIFbMS in a specific scenario. In the context of EIFbMS, the researchers established a reasonable condition for agreement, allowing for the consideration of Cauchy sequences. To show that our ideas work, we created some examples. Using our findings, we ensured that there is only one unique solution to Fredholm integral equation. This means we can now confidently solve more types of these equations under these conditions.

Abbreviation

FP - Fixed point.

MS - Metric space.

FMS - Fuzzy metric space.

FbMS - Fuzzy b-metric space.

CFP - Common fixed point.

BCP - Banach contraction principle.

IFMS - Intuitionistic fuzzy metric space.

IFbMS - Intuitionistic fuzzy b- metric space.

EIFbMS - Extended Intuitionistic fuzzy b- metric space.

FUNDING INFORMATION

This work was supported by Directorate of Research and Innovation, Walter Sisulu University, South Africa.

AUTHOR CONTRIBUTIONS

All authors contributed equally to the writing of this article. All authors read and approved the final manuscript.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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