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COMMON FIXED POINT THEOREMS IN METRIC DOMAINS

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Abstract. The aim of this contribution is to prove two unique common fixed point theorems for two pairs of occasionally weakly biased mappings of type $(\mathbb{A})$ on complete metric domains. These theorems improve some results on metric, partial metric and metric domains. Also, two suitable examples are given in order to support our results. Again, an application is furnished in order to convince the reader about our useful results.

Keywords: metric domains; occasionally weakly biased mappings of type $(\mathbb{A})$; common fixed points.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction and Needed Definitions

Fixed point theory is a very important axis in mathematics which has a huge number of applications in different subjects [1, 2, 3, 4, 5]. It presents prominent tools for solving many issues. The strong starting of fixed point theory was in 1922 with Stefan Banach, who gave the contractive mapping theorem which considered as an important tool in the theory of metric spaces, and can be understood as an abstract formulation of Émile Picard's method of successive

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approximations. Several mathematicians made a lot of generalizations of the notion of metric space by weakening the defining axioms for the notion of metric. For instance, in 1985, in his thesis [6], Stephen G. Matthews suggested the class of metric domains. According to him, these domains allow a natural distinction to be made between ”complete” and ”partial” (non complete) objects. Always according to Matthews, metric domain has been introduced in order to promote the notion of completeness in domain theory and, he pointed out that there is a one to one correspondence between the class of metric domains and the class of metric spaces. In 1992, in his paper [7], the same author provided another generalisation of metric spaces under the name of partial metric spaces in which he keeps the symmetry axiom. In 2001, in his dissertation [8], Pascal Hitzler used metric domains under the name of dislocated metrics and he investigated the topological structure underlying the notion of dislocated metric, which leads to a proof of the Matthews theorem which is in the spirit of the proof of Banach contraction mapping theorem. In 2012, in his paper [9], Alireza Amini-Harandi introduced a new generalization of a partial metric space which is called a metric-like space. Then, he gave some fixed point theorems in such spaces which generalize and improve some well-known results in both metric-like and partial metric spaces. However, to get a more description about these topics, the reader my refer to the references [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

Remark 1. *The notions of metric domains, metric-like spaces and dislocated metric spaces are exactly the same, and they also named d-metric spaces.*

On the other hand, in 2022, we put in a new concept called occasionally weakly biased mappings of type $(\mathbb{A})$ and we asserted that our notion has an edge over weak and occasionally weak compatibility; that is, weakly compatible and occasionally weakly compatible mappings are subclasses of occasionally weakly biased mappings of type $(\mathbb{A})$.

Definition 1. *(*[6]*) A metric domain is a pair* $<\mathscr{M}, m>$ *where* $\mathscr{M}$ *is a non-empty set, and* m *is a function from* $\mathscr{M}\times\mathscr{M}$ *to* $\mathbb{R}_+$ *such that*

(1) $\forall\ \varsigma_1,\ \varsigma_2 \in \mathscr{M},\ m(\varsigma_1,\varsigma_2) = 0 \Rightarrow \varsigma_1 = \varsigma_2$

(2) $\forall\ \varsigma_1,\ \varsigma_2 \in \mathscr{M},\ m(\varsigma_1,\varsigma_2) = m(\varsigma_2,\varsigma_1)$

(3) $\forall\ \varsigma_1,\ \varsigma_2,\ \varsigma_3 \in \mathscr{M},\ m(\varsigma_1,\varsigma_2) \leq m(\varsigma_1,\varsigma_3) + m(\varsigma_3,\varsigma_2)$.

Definition 2. *([7]) A partial metric is a function $p : \mathscr{P} \times \mathscr{P} \to \mathbb{R}$, such that*

(1) $\forall\, \varsigma_1,\, \varsigma_2 \in \mathscr{P}$, $\varsigma_1 = \varsigma_2 \Leftrightarrow p(\varsigma_1,\varsigma_1) = p(\varsigma_1,\varsigma_2) = p(\varsigma_2,\varsigma_2)$

(2) $\forall\, \varsigma_1,\, \varsigma_2 \in \mathscr{P}$, $p(\varsigma_1,\varsigma_1) \leq p(\varsigma_1,\varsigma_2)$

(3) $\forall\, \varsigma_1,\, \varsigma_2 \in \mathscr{P}$, $p(\varsigma_1,\varsigma_2) = p(\varsigma_2,\varsigma_1)$

(4) $\forall\, \varsigma_1,\, \varsigma_2,\, \varsigma_3 \in \mathscr{P}$, $p(\varsigma_1,\varsigma_3) \leq p(\varsigma_1,\varsigma_2) + p(\varsigma_2,\varsigma_3) - p(\varsigma_2,\varsigma_2)$.

Definition 3. *([9]) A mapping $l : \mathscr{L} \times \mathscr{L} \to \mathbb{R}_+$, where $\mathscr{L}$ is a nonempty set, is said to be metric-like on $\mathscr{L}$ if for any $\varsigma_1,\, \varsigma_2,\, \varsigma_3 \in \mathscr{L}$, the following three conditions hold true:*

(1) $l(\varsigma_1,\varsigma_2) = 0 \Rightarrow \varsigma_1 = \varsigma_2$

(2) $l(\varsigma_1,\varsigma_2) = l(\varsigma_2,\varsigma_1)$

(3) $l(\varsigma_1,\varsigma_2) \leq l(\varsigma_1,\varsigma_3) + l(\varsigma_3,\varsigma_2)$.

The pair $(\mathscr{L}, l)$ is then called a metric-like space.

Then a metric-like on $\mathscr{L}$ satisfies all of the conditions of a metric except that $l(\varsigma_1,\varsigma_1)$ may be positive for $\varsigma_1 \in \mathscr{L}$.

Definition 4. *([21]) Two self-mappings $\mathscr{P}$ and $\mathscr{Q}$ of a metric space $(\mathscr{M}, m)$ are called weakly compatible if and only if $\mathscr{P}$ and $\mathscr{Q}$ commute on the set of coincidence points.*

Definition 5. *([22]) Two self-mappings $\mathscr{P}$ and $\mathscr{Q}$ of a set $\mathscr{M}$ are occasionally weakly compatible if and only if, there is a point ν in $\mathscr{M}$ which is a coincidence point of $\mathscr{P}$ and $\mathscr{Q}$ at which $\mathscr{P}$ and $\mathscr{Q}$ commute.*

Definition 6. *([23]) Let $\mathscr{S}$ and $\mathscr{T}$ be self-mappings on a metric domain $(\mathscr{M}, m)$. The pair $(\mathscr{S}, \mathscr{T})$ is said to be occasionally weakly $\mathscr{S}$-biased of type $(\mathbb{A})$ and occasionally weakly $\mathscr{T}$-biased of type $(\mathbb{A})$, respectively, if and only if, there exists a point ς in $\mathscr{M}$ such that $\mathscr{S}\varsigma = \mathscr{T}\varsigma$ implies*

$$m(\mathscr{S}\mathscr{S}\varsigma, \mathscr{T}\varsigma) \leq m(\mathscr{T}\mathscr{S}\varsigma, \mathscr{S}\varsigma),$$

$$m(\mathscr{T}\mathscr{T}\varsigma, \mathscr{S}\varsigma) \leq m(\mathscr{S}\mathscr{T}\varsigma, \mathscr{T}\varsigma),$$

respectively.

Example 1. *Consider the metric domain* $(\mathscr{D},\mathfrak{D})$ *where* $\mathscr{D}=(-\infty,+\infty)$ *and* $\mathfrak{D}(\varsigma_1,\varsigma_2)=|\varsigma_1|+|\varsigma_2|$. *Let* $\mathscr{M}$ *and* $\mathscr{N}$ *be two mappings from* $\mathscr{D}$ *into itself defined as follows:*

$$\mathscr{M}\varsigma_1=\begin{cases}2\varsigma_1 & \text{when } -\infty<\varsigma_1\le 1\\ -\dfrac{25}{\varsigma_1} & \text{when } 1<\varsigma_1<+\infty,\end{cases}\quad \mathscr{N}\varsigma_1=\begin{cases}\varsigma_1+1 & \text{when } -\infty<\varsigma_1\le 1\\ \varsigma_1-10 & \text{when } 1<\varsigma_1<+\infty.\end{cases}$$

It is clear to see that $\mathscr{M}1=\mathscr{N}1=2$ *and* $\mathscr{M}5=\mathscr{N}5=-5$ *and*

$$10=\mathfrak{D}(\mathscr{N}\mathscr{N}1,\mathscr{M}1)\le\mathfrak{D}(\mathscr{M}\mathscr{N}1,\mathscr{N}1)=\frac{29}{2},$$
$$9=\mathfrak{D}(\mathscr{N}\mathscr{N}5,\mathscr{M}5)\le\mathfrak{D}(\mathscr{M}\mathscr{N}5,\mathscr{N}5)=15,$$

that is mappings $\mathscr{M}$ *and* $\mathscr{N}$ *are occasionally weakly* $\mathscr{N}$*-biased of type* $(\mathbb{A})$.

However,

$$\mathscr{M}\mathscr{N}1=-\frac{25}{2}\neq-8=\mathscr{N}\mathscr{M}1,$$
$$\mathscr{M}\mathscr{N}5=-10\neq-4=\mathscr{N}\mathscr{M}5,$$

that is, $\mathscr{M}$ *and* $\mathscr{N}$ *are neither weakly compatible nor occasionally weakly compatible.*

In this investigation, we will use our new definition to prove the existence and uniqueness of common fixed points for quadruple mappings in complete metric domains. These theorems improve and/or extend some results in metric and partial metric spaces as well as in metric domains.

2. Existence and Uniqueness of Common Fixed Points with Examples

Theorem 1. *Let* $(\mathscr{M},m)$ *be a complete metric domain. Let* $\mathscr{A}$, $\mathscr{B}$, $\mathscr{S}$, $\mathscr{T}:\mathscr{M}\to\mathscr{M}$ *be mappings satisfying*

(1) the pairs $(\mathscr{A},\mathscr{T})$ *and* $(\mathscr{B},\mathscr{S})$ *are occasionally weakly* $\mathscr{T}$*-biased (respectively* $\mathscr{S}$*-biased) of type* $(\mathbb{A})$ *and*

(2) $m(\mathscr{A}x,\mathscr{B}y)\le\kappa(m(\mathscr{S}y,\mathscr{A}x)+m(\mathscr{T}x,\mathscr{S}y)+m(\mathscr{T}x,\mathscr{A}x)+m(\mathscr{B}y,\mathscr{S}y)+m(\mathscr{T}x,\mathscr{B}y))$

for all $x,y\in\mathscr{M}$, *where* $\kappa\in\left(0,\frac{1}{7}\right)$. *Then* $\mathscr{A}$, $\mathscr{B}$, $\mathscr{S}$ *and* $\mathscr{T}$ *have a unique common fixed point.*

Proof. By hypotheses, there are two points μ and ν in $\mathscr{M}$ such that $\mathscr{A}\mu = \mathscr{T}\mu$ implies $m(\mathscr{T}\mathscr{T}\mu, \mathscr{A}\mu) \leq m(\mathscr{A}\mathscr{T}\mu, \mathscr{T}\mu)$ and $\mathscr{B}\nu = \mathscr{S}\nu$ implies $m(\mathscr{S}\mathscr{S}\nu, \mathscr{B}\nu) \leq m(\mathscr{B}\mathscr{S}\nu, \mathscr{S}\nu)$.

First, we are going to prove that $\mathscr{A}\mu = \mathscr{B}\nu$. Suppose that $\mathscr{A}\mu \neq \mathscr{B}\nu$, from inequality (2) we have

$$\begin{aligned}
m(\mathscr{A}\mu,\mathscr{B}\nu) &\leq \kappa(m(\mathscr{S}\nu,\mathscr{A}\mu)+m(\mathscr{T}\mu,\mathscr{S}\nu)+m(\mathscr{T}\mu,\mathscr{A}\mu)+m(\mathscr{B}\nu,\mathscr{S}\nu)+m(\mathscr{T}\mu,\mathscr{B}\nu))\\
&= \kappa(m(\mathscr{B}\nu,\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{B}\nu)+m(\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{B}\nu,\mathscr{B}\nu)+m(\mathscr{A}\mu,\mathscr{B}\nu))\\
&\leq \kappa(m(\mathscr{B}\nu,\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{B}\nu)+2m(\mathscr{A}\mu,\mathscr{B}\nu)+2m(\mathscr{B}\nu,\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{B}\nu))\\
&= 7\kappa m(\mathscr{A}\mu,\mathscr{B}\nu)\\
&< m(\mathscr{A}\mu,\mathscr{B}\nu),
\end{aligned}$$

which is a contradiction, thus $\mathscr{A}\mu = \mathscr{B}\nu$. Now, we assert that $\mathscr{A}\mathscr{A}\mu = \mathscr{A}\mu$. If not, then the use of condition (2) gives

$$\begin{aligned}
m(\mathscr{A}\mathscr{A}\mu,\mathscr{B}\nu) \quad \leq \quad & \kappa(m(\mathscr{S}\nu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{S}\nu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{B}\nu,\mathscr{S}\nu)\\
&+m(\mathscr{T}\mathscr{A}\mu,\mathscr{B}\nu));
\end{aligned}$$

i.e.,

$$\begin{aligned}
m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu) &\leq \kappa(m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{A}\mu)\\
&\quad +m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu))\\
&\leq \kappa(m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{A}\mu)\\
&\quad +m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu))\\
&= \kappa(m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+2m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{A}\mu))\\
&= \kappa(m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+2m(\mathscr{T}\mathscr{T}\mu,\mathscr{A}\mu)+m(\mathscr{T}\mathscr{T}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{A}\mu))\\
&\leq \kappa(m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+2m(\mathscr{A}\mathscr{T}\mu,\mathscr{T}\mu)+m(\mathscr{T}\mathscr{T}\mu,\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)\\
&\quad +m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu)),
\end{aligned}$$

or

$$\begin{aligned}
m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu) &\leq \kappa(m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+2m(\mathscr{A}\mathscr{T}\mu,\mathscr{T}\mu)+m(\mathscr{A}\mathscr{T}\mu,\mathscr{T}\mu)+m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)\\
&\quad +m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu))
\end{aligned}$$

$$= \kappa(m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+2m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)$$
$$+m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu))$$
$$= 7\kappa m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu)$$
$$< m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu),$$

which is a contradiction, therefore $\mathscr{A}\mathscr{A}\mu = \mathscr{A}\mu$, consequently, $\mathscr{T}\mathscr{A}\mu = \mathscr{A}\mu$. Now, suppose that $\mathscr{B}\mathscr{B}\nu \neq \mathscr{B}\nu$. Using inequality (2) we obtain

$$\begin{aligned} m(\mathscr{A}\mu,\mathscr{B}\mathscr{B}\nu) \quad &\leq \quad \kappa(m(\mathscr{S}\mathscr{B}\nu,\mathscr{A}\mu)+m(\mathscr{T}\mu,\mathscr{S}\mathscr{B}\nu)+m(\mathscr{T}\mu,\mathscr{A}\mu)+m(\mathscr{B}\mathscr{B}\nu,\mathscr{S}\mathscr{B}\nu) \\ &\quad +m(\mathscr{T}\mu,\mathscr{B}\mathscr{B}\nu)); \end{aligned}$$

i.e.,

$$\begin{aligned} m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu) \quad &\leq \quad \kappa(m(\mathscr{S}\mathscr{B}\nu,\mathscr{B}\nu)+m(\mathscr{B}\nu,\mathscr{S}\mathscr{B}\nu)+m(\mathscr{B}\nu,\mathscr{B}\nu)+m(\mathscr{B}\mathscr{B}\nu,\mathscr{S}\mathscr{B}\nu) \\ &\quad +m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)) \\ &= \quad \kappa(2m(\mathscr{S}\mathscr{S}\nu,\mathscr{B}\nu)+m(\mathscr{B}\nu,\mathscr{B}\nu)+m(\mathscr{B}\mathscr{B}\nu,\mathscr{S}\mathscr{S}\nu)+m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)) \\ &\leq \quad \kappa(2m(\mathscr{S}\mathscr{S}\nu,\mathscr{B}\nu)+m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)+m(\mathscr{B}\mathscr{B}\nu,\mathscr{B}\nu)+m(\mathscr{B}\mathscr{B}\nu,\mathscr{B}\nu) \\ &\quad +m(\mathscr{B}\nu,\mathscr{S}\mathscr{S}\nu)+m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)) \\ &= \quad \kappa(3m(\mathscr{S}\mathscr{S}\nu,\mathscr{B}\nu)+4m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)) \\ &\leq \quad \kappa(3m(\mathscr{B}\mathscr{S}\nu,\mathscr{S}\nu)+4m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)) \\ &= \quad 7\kappa m(\mathscr{B}\mathscr{B}\nu,\mathscr{B}\nu) \\ &< \quad m(\mathscr{B}\mathscr{B}\nu,\mathscr{B}\nu) \end{aligned}$$

this contradiction implies that $\mathscr{B}\mathscr{B}\nu = \mathscr{B}\nu$ and so $\mathscr{S}\mathscr{B}\nu = \mathscr{B}\nu$; i.e., $\mathscr{B}\mathscr{A}\mu = \mathscr{A}\mu$ and $\mathscr{S}\mathscr{A}\mu = \mathscr{A}\mu$. Put $\mathscr{A}\mu = \mathscr{S}\mu = \mathscr{B}\nu = \mathscr{T}\nu = \rho$, therefore ρ is a common fixed point of mappings $\mathscr{A}$, $\mathscr{B}$, $\mathscr{S}$ and $\mathscr{T}$.

Finally, let ρ and ρ be two distinct common fixed points of mappings $\mathscr{A}$, $\mathscr{B}$, $\mathscr{S}$ and $\mathscr{T}$. Then, $\rho = \mathscr{A}\rho = \mathscr{B}\rho = \mathscr{S}\rho = \mathscr{T}\rho$ and $\rho = \mathscr{A}\rho = \mathscr{B}\rho = \mathscr{S}\rho = \mathscr{T}\rho$. From (2) we have

$$m(\mathscr{A}\rho,\mathscr{B}\rho) \quad \leq \quad \kappa(m(\mathscr{S}\rho,\mathscr{A}\rho)+m(\mathscr{T}\rho,\mathscr{S}\rho)+m(\mathscr{T}\rho,\mathscr{A}\rho)+m(\mathscr{B}\rho,\mathscr{S}\rho)+m(\mathscr{T}\rho,\mathscr{B}\rho));$$

i.e.,

$$\begin{aligned} m(\rho,\rho) &\leq \kappa(m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho)) \\ &\leq \kappa(m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho) \\ &\quad +m(\rho,\rho)+m(\rho,\rho)) \\ &= 7\kappa m(\rho,\rho) \\ &< m(\rho,\rho) \end{aligned}$$

which is a contradiction, thus $\rho = \rho$. □

Now, we produce an illustrative example which highlights our first result.

Example 2. *Let* $(\mathscr{M} = [-1,100], m)$ *be a complete metric domain, with* $m(x,y) = \max\{|x|,|y|\}$. *Define*

$$\mathscr{A}x = \begin{cases} 0 \text{ if } -1 \leq x \leq 0 \\ -\dfrac{1}{9} \text{ if } 0 < x \leq 100, \end{cases} \quad \mathscr{B}x = \begin{cases} 0 \text{ if } -1 \leq x \leq 0 \\ -\dfrac{1}{8} \text{ if } 0 < x \leq 100, \end{cases}$$

$$\mathscr{T}x = \begin{cases} -60x \text{ if } -1 \leq x \leq 0 \\ 70 \text{ if } 0 < x \leq 100, \end{cases} \quad \mathscr{S}x = \begin{cases} -80x \text{ if } -1 \leq x \leq 0 \\ 90 \text{ if } 0 < x \leq 100. \end{cases}$$

First, it is clear to see that $\mathscr{A}$ *and* $\mathscr{T}$ *are occasionally weakly* $\mathscr{T}$*-biased of type* $(\mathbb{A})$ *and* $\mathscr{B}$ *and* $\mathscr{S}$ *are occasionally weakly* $\mathscr{S}$*-biased of type* $(\mathbb{A})$. *Take* $\kappa = \frac{1}{8}$, *we get*

(1) for $-1 \leq x,y \leq 0$, *we have* $\mathscr{A}x = 0$, $\mathscr{B}y = 0$, $\mathscr{T}x = -60x$, $\mathscr{S}y = -80y$ *and*

$$\begin{aligned} m(\mathscr{A}x,\mathscr{B}y) &= 0 \\ &\leq \frac{1}{8}(-120x-160y+\max\{-60x,-80y\}) \\ &= \kappa(m(\mathscr{S}y,\mathscr{A}x)+m(\mathscr{T}x,\mathscr{S}y)+m(\mathscr{T}x,\mathscr{A}x)+m(\mathscr{B}y,\mathscr{S}y)+m(\mathscr{T}x,\mathscr{B}y)), \end{aligned}$$

(2) for $0 < x,y \leq 100$, *we have* $\mathscr{A}x = -\frac{1}{9}$, $\mathscr{B}y = -\frac{1}{8}$, $\mathscr{T}x = 70$, $\mathscr{S}y = 90$ *and*

$$\begin{aligned} m(\mathscr{A}x,\mathscr{B}y) &= \frac{1}{8} \\ &\leq \frac{205}{4} \\ &= \kappa(m(\mathscr{S}y,\mathscr{A}x)+m(\mathscr{T}x,\mathscr{S}y)+m(\mathscr{T}x,\mathscr{A}x)+m(\mathscr{B}y,\mathscr{S}y)+m(\mathscr{T}x,\mathscr{B}y)), \end{aligned}$$

(3) for $-1 \leq x \leq 0 < y \leq 100$*, we have* $\mathscr{A}x = 0$*,* $\mathscr{B}y = -\frac{1}{8}$*,* $\mathscr{T}x = -60x$*,* $\mathscr{S}y = 90$ *and*

$$\begin{aligned} m(\mathscr{A}x, \mathscr{B}y) &= \frac{1}{8} \\ &\leq \frac{1}{8}\left(270 - 60x + \max\left\{-60x, \frac{1}{8}\right\}\right) \\ &= \kappa(m(\mathscr{S}y, \mathscr{A}x) + m(\mathscr{T}x, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{A}x) + m(\mathscr{B}y, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{B}y)), \end{aligned}$$

(4) for $-1 \leq y \leq 0 < x \leq 100$*, we have* $\mathscr{A}x = -\frac{1}{9}$*,* $\mathscr{B}y = 0$*,* $\mathscr{T}x = 70$*,* $\mathscr{S}y = -80y$ *and*

$$\begin{aligned} m(\mathscr{A}x, \mathscr{B}y) &= \frac{1}{9} \\ &\leq \frac{1}{8}\left(140 - 80y + \max\left\{-80y, \frac{1}{9}\right\} + \max\{70, -80y\}\right) \\ &= \kappa(m(\mathscr{S}y, \mathscr{A}x) + m(\mathscr{T}x, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{A}x) + m(\mathscr{B}y, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{B}y)), \end{aligned}$$

so, all hypotheses of Theorem 1 are satisfied and 0 *is the unique common fixed point of mappings* $\mathscr{A}$*,* $\mathscr{B}$*,* $\mathscr{T}$ *and* $\mathscr{S}$*.*

Remark 2. *Note that* $\mathscr{A}\mathscr{M} = \left\{0, -\frac{1}{9}\right\} \nsubseteq \mathscr{S}\mathscr{M} = [0, 80] \cup \{90\}$ *and* $\mathscr{B}\mathscr{M} = \left\{0, -\frac{1}{8}\right\} \nsubseteq \mathscr{T}\mathscr{M} = [0, 60] \cup \{70\}$.

In the next, we will extend constant κ of Theorem 1 to a function.

Theorem 2. *Let* $(\mathscr{M}, m)$ *be a complete metric domain. Let* $\mathscr{A}$*,* $\mathscr{B}$*,* $\mathscr{S}$*,* $\mathscr{T} : \mathscr{M} \to \mathscr{M}$ *be mappings satisfying*

(1) the pairs $(\mathscr{A}, \mathscr{T})$ *and* $(\mathscr{B}, \mathscr{S})$ *are occasionally weakly* $\mathscr{T}$*-biased (respectively* $\mathscr{S}$*-biased) of type* $(\mathbb{A})$ *and*

(2) $m(\mathscr{A}x, \mathscr{B}y) \leq \kappa(m(\mathscr{T}x, \mathscr{S}y))[m(\mathscr{S}y, \mathscr{A}x) + m(\mathscr{T}x, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{A}x) + m(\mathscr{B}y, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{B}y)]$

for all x*,* $y \in \mathscr{M}$*, where* $\kappa : [0, +\infty) \to [0, 1)$ *is a non-decreasing function such that* $7\kappa(t) < 1$ *for* $t > 0$*. Then* $\mathscr{A}$*,* $\mathscr{B}$*,* $\mathscr{S}$ *and* $\mathscr{T}$ *have a unique common fixed point.*

Proof. Again, by hypotheses, there are two points μ and ν in $\mathscr{M}$ such that $\mathscr{A}\mu = \mathscr{T}\mu$ implies $m(\mathscr{T}\mathscr{T}\mu, \mathscr{A}\mu) \leq m(\mathscr{A}\mathscr{T}\mu, \mathscr{T}\mu)$ and $\mathscr{B}\nu = \mathscr{S}\nu$ implies $m(\mathscr{S}\mathscr{S}\nu, \mathscr{B}\nu) \leq m(\mathscr{B}\mathscr{S}\nu, \mathscr{S}\nu)$.

First, we are going to prove that $\mathscr{A}\mu = \mathscr{B}\nu$. Suppose that $\mathscr{A}\mu \neq \mathscr{B}\nu$, from inequality (2) we have

$$\begin{aligned}
m(\mathscr{A}\mu,\mathscr{B}\nu) &\leq \kappa(m(\mathscr{T}\mu,\mathscr{S}\nu))[m(\mathscr{S}\nu,\mathscr{A}\mu)+m(\mathscr{T}\mu,\mathscr{S}\nu)+m(\mathscr{T}\mu,\mathscr{A}\mu)+m(\mathscr{B}\nu,\mathscr{S}\nu)\\
&\quad +m(\mathscr{T}\mu,\mathscr{B}\nu)]\\
&= \kappa(m(\mathscr{A}\mu,\mathscr{B}\nu))[m(\mathscr{B}\nu,\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{B}\nu)+m(\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{B}\nu,\mathscr{B}\nu)\\
&\quad +m(\mathscr{A}\mu,\mathscr{B}\nu)]\\
&\leq \kappa(m(\mathscr{A}\mu,\mathscr{B}\nu))[m(\mathscr{B}\nu,\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{B}\nu)+2m(\mathscr{A}\mu,\mathscr{B}\nu)+2m(\mathscr{B}\nu,\mathscr{A}\mu)\\
&\quad +m(\mathscr{A}\mu,\mathscr{B}\nu)]\\
&= 7\kappa(m(\mathscr{A}\mu,\mathscr{B}\nu))m(\mathscr{A}\mu,\mathscr{B}\nu)\\
&< m(\mathscr{A}\mu,\mathscr{B}\nu)
\end{aligned}$$

which is a contradiction, thus $\mathscr{A}\mu = \mathscr{B}\nu$. Now, we assert that $\mathscr{A}\mathscr{A}\mu = \mathscr{A}\mu$. If not, then the use of condition (2) gives

$$\begin{aligned}
m(\mathscr{A}\mathscr{A}\mu,\mathscr{B}\nu) &\leq \kappa(m(\mathscr{T}\mathscr{A}\mu,\mathscr{S}\nu))[m(\mathscr{S}\nu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{S}\nu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)\\
&\quad +m(\mathscr{B}\nu,\mathscr{S}\nu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{B}\nu)];
\end{aligned}$$

i.e.,

$$\begin{aligned}
m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu) &\leq \kappa(m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu))[m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)\\
&\quad +m(\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu)]\\
&= \kappa(m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu))[m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+2m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)\\
&\quad +m(\mathscr{A}\mu,\mathscr{A}\mu)]\\
&\leq \kappa(m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu))[m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+2m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu)+m(\mathscr{T}\mathscr{A}\mu,\mathscr{A}\mu)\\
&\quad +m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu)]\\
&= \kappa(m(\mathscr{T}\mathscr{T}\mu,\mathscr{A}\mu))[m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+2m(\mathscr{T}\mathscr{T}\mu,\mathscr{A}\mu)+m(\mathscr{T}\mathscr{T}\mu,\mathscr{A}\mu)\\
&\quad +m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu)]\\
&= \kappa(m(\mathscr{T}\mathscr{T}\mu,\mathscr{A}\mu))[4m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+3m(\mathscr{T}\mathscr{T}\mu,\mathscr{A}\mu)]
\end{aligned}$$

$$
\begin{aligned}
&\leq \kappa(m(\mathscr{A}\mathscr{T}\mu,\mathscr{T}\mu))[4m(\mathscr{A}\mu,\mathscr{A}\mathscr{A}\mu)+3m(\mathscr{A}\mathscr{T}\mu,\mathscr{T}\mu)]\\
&=7\kappa(m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu))m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu)\\
&< m(\mathscr{A}\mathscr{A}\mu,\mathscr{A}\mu),
\end{aligned}
$$

which is a contradiction, therefore $\mathscr{A}\mathscr{A}\mu=\mathscr{A}\mu$, consequently, $\mathscr{T}\mathscr{A}\mu=\mathscr{A}\mu$. Now, suppose that $\mathscr{B}\mathscr{B}\nu\neq\mathscr{B}\nu$. Using inequality (2) we obtain

$$
\begin{aligned}
m(\mathscr{A}\mu,\mathscr{B}\mathscr{B}\nu) \;\leq\; & \kappa(m(\mathscr{B}\nu,\mathscr{S}\mathscr{B}\nu))[m(\mathscr{S}\mathscr{B}\nu,\mathscr{A}\mu)+m(\mathscr{T}\mu,\mathscr{S}\mathscr{B}\nu)+m(\mathscr{T}\mu,\mathscr{A}\mu)\\
&+m(\mathscr{B}\mathscr{B}\nu,\mathscr{S}\mathscr{B}\nu)+m(\mathscr{T}\mu,\mathscr{B}\mathscr{B}\nu)];
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu) &\leq \kappa(m(\mathscr{B}\nu,\mathscr{S}\mathscr{B}\nu))[m(\mathscr{S}\mathscr{B}\nu,\mathscr{B}\nu)+m(\mathscr{B}\nu,\mathscr{S}\mathscr{B}\nu)+m(\mathscr{B}\nu,\mathscr{B}\nu)\\
&+m(\mathscr{B}\mathscr{B}\nu,\mathscr{S}\mathscr{B}\nu)+m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)]\\
&=\kappa(m(\mathscr{B}\nu,\mathscr{S}\mathscr{S}\nu))[2m(\mathscr{S}\mathscr{S}\nu,\mathscr{B}\nu)+m(\mathscr{B}\nu,\mathscr{B}\nu)+m(\mathscr{B}\mathscr{B}\nu,\mathscr{S}\mathscr{S}\nu)\\
&+m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)],
\end{aligned}
$$

or

$$
\begin{aligned}
m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu) &\leq \kappa(m(\mathscr{B}\nu,\mathscr{S}\mathscr{S}\nu))[m(\mathscr{B}\nu,\mathscr{S}\mathscr{B}\nu))[2m(\mathscr{S}\mathscr{S}\nu,\mathscr{B}\nu)+m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)\\
&+m(\mathscr{B}\mathscr{B}\nu,\mathscr{B}\nu)+m(\mathscr{B}\mathscr{B}\nu,\mathscr{B}\nu)+m(\mathscr{B}\nu,\mathscr{S}\mathscr{S}\nu)+m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)]\\
&=\kappa(m(\mathscr{B}\nu,\mathscr{S}\mathscr{S}\nu))[3m(\mathscr{S}\mathscr{S}\nu,\mathscr{B}\nu)+4m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)]\\
&\leq \kappa(m(\mathscr{S}\nu,\mathscr{B}\mathscr{S}\nu))[3m(\mathscr{B}\mathscr{S}\nu,\mathscr{S}\nu)+4m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu)]\\
&=7\kappa(m(\mathscr{B}\nu,\mathscr{B}\mathscr{B}\nu))m(\mathscr{B}\mathscr{B}\nu,\mathscr{B}\nu)\\
&< m(\mathscr{B}\mathscr{B}\nu,\mathscr{B}\nu).
\end{aligned}
$$

This contradiction implies that $\mathscr{B}\mathscr{B}\nu=\mathscr{B}\nu$ and so $\mathscr{S}\mathscr{B}\nu=\mathscr{B}\nu$; i.e., $\mathscr{B}\mathscr{A}\mu=\mathscr{A}\mu$ and $\mathscr{S}\mathscr{A}\mu=\mathscr{A}\mu$. Put $\mathscr{A}\mu=\mathscr{S}\mu=\mathscr{B}\nu=\mathscr{T}\nu=\rho$, therefore ρ is a common fixed point of mappings $\mathscr{A}$, $\mathscr{B}$, $\mathscr{S}$ and $\mathscr{T}$.

Finally, let ρ and ρ be two distinct common fixed points of mappings $\mathscr{A}$, $\mathscr{B}$, $\mathscr{S}$ and $\mathscr{T}$. Then, $\rho=\mathscr{A}\rho=\mathscr{B}\rho=\mathscr{S}\rho=\mathscr{T}\rho$ and $\rho=\mathscr{A}\rho=\mathscr{B}\rho=\mathscr{S}\rho=\mathscr{T}\rho$. From (2) we have

$$\begin{aligned} m(\mathscr{A}\rho,\mathscr{B}\rho) &\leq \kappa(m(\mathscr{T}\rho,\mathscr{S}\rho))[m(\mathscr{S}\rho,\mathscr{A}\rho)+m(\mathscr{T}\rho,\mathscr{S}\rho)+m(\mathscr{T}\rho,\mathscr{A}\rho) \\ &\quad +m(\mathscr{B}\rho,\mathscr{S}\rho)]+m(\mathscr{T}\rho,\mathscr{B}\rho)]; \end{aligned}$$

i.e.,

$$\begin{aligned} m(\rho,\rho) &\leq \kappa(m(\rho,\rho))[m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho)] \\ &\leq \kappa(m(\rho,\rho))[m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho)+m(\rho,\rho) \\ &\quad +m(\rho,\rho)+m(\rho,\rho)] \\ &= 7\kappa(m(\rho,\rho))m(\rho,\rho) \\ &< m(\rho,\rho) \end{aligned}$$

which is a contradiction, thus $\rho = \rho$. □

Again, we furnish an example which illustrates our second theorem.

Example 3. *Let* $\left(\mathscr{M} = \left[0, \frac{\pi}{2}\right], d\right)$ *be a complete metric domain, with* $m(x,y) = x+y$. *Define*

$$\mathscr{A}x = \begin{cases} 0 \text{ if } 0 \leq x \leq \frac{\pi}{4} \\ \frac{\pi}{1000} \text{ if } \frac{\pi}{4} < x \leq \frac{\pi}{2}, \end{cases} \quad \mathscr{B}x = \begin{cases} 0 \text{ if } 0 \leq x \leq \frac{\pi}{4} \\ \frac{\pi}{2000} \text{ if } \frac{\pi}{4} < x \leq \frac{\pi}{2}, \end{cases}$$

$$\mathscr{T}x = \begin{cases} x \text{ if } 0 \leq x \leq \frac{\pi}{4} \\ \frac{499\pi}{1000} \text{ if } \frac{\pi}{4} < x \leq \frac{\pi}{2}, \end{cases} \quad \mathscr{S}x = \begin{cases} x \text{ if } 0 \leq x \leq \frac{\pi}{4} \\ \frac{249\pi}{500} \text{ if } \frac{\pi}{4} < x \leq \frac{\pi}{2}. \end{cases}$$

First, it is clear to see that $\mathscr{A}$ *and* $\mathscr{T}$ *are occasionally weakly* $\mathscr{T}$*-biased of type* $(\mathbb{A})$ *and* $\mathscr{B}$ *and* $\mathscr{S}$ *are occasionally weakly* $\mathscr{S}$*-biased of type* $(\mathbb{A})$. *Take* $\kappa(t) = \frac{\sin t}{8}$, *we get*

(1) for $0 \leq x,y \leq \frac{\pi}{4}$, *we have* $\mathscr{A}x = 0$, $\mathscr{B}y = 0$, $\mathscr{T}x = x$, $\mathscr{S}y = y$ *and*

$$\begin{aligned} m(\mathscr{A}x,\mathscr{B}y) &= 0 \\ &\leq \frac{3}{8}(x+y)\sin(x+y) \\ &= \kappa(m(\mathscr{T}x,\mathscr{S}y))[m(\mathscr{S}y,\mathscr{A}x)+m(\mathscr{T}x,\mathscr{S}y)+m(\mathscr{T}x,\mathscr{A}x) \\ &\quad +m(\mathscr{B}y,\mathscr{S}y)+m(\mathscr{T}x,\mathscr{B}y)], \end{aligned}$$

(2) for $\frac{\pi}{4} < x, y \leq \frac{\pi}{2}$, *we have* $\mathscr{A}x = \frac{\pi}{1000}$, $\mathscr{B}y = \frac{\pi}{2000}$, $\mathscr{T}x = \frac{499\pi}{1000}$, $\mathscr{S}y = \frac{249\pi}{500}$ *and*

$$\begin{aligned} m(\mathscr{A}x, \mathscr{B}y) &= \frac{3\pi}{2000} \\ &\leq \frac{1497\pi}{4000} \sin\left(\frac{997\pi}{1000}\right) \\ &= \kappa(m(\mathscr{T}x, \mathscr{S}y))[m(\mathscr{S}y, \mathscr{A}x) + m(\mathscr{T}x, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{A}x) \\ &\quad + m(\mathscr{B}y, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{B}y)], \end{aligned}$$

(3) for $0 \leq x \leq \frac{\pi}{4} < y \leq \frac{\pi}{2}$ *we have* $\mathscr{A}x = 0$, $\mathscr{B}y = \frac{\pi}{2000}$, $\mathscr{T}x = x$, $\mathscr{S}y = \frac{249\pi}{500}$ *and*

$$\begin{aligned} m(\mathscr{A}x, \mathscr{B}y) &= \frac{\pi}{2000} \\ &\leq \frac{1}{8} \sin\left(x + \frac{249\pi}{500}\right)\left(3x + \frac{299\pi}{200}\right) \\ &= \kappa(m(\mathscr{T}x, \mathscr{S}y))[m(\mathscr{S}y, \mathscr{A}x) + m(\mathscr{T}x, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{A}x) \\ &\quad + m(\mathscr{B}y, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{B}y)], \end{aligned}$$

(4) for $0 \leq y \leq \frac{\pi}{4} < x \leq \frac{\pi}{2}$, *we have* $\mathscr{A}x = \frac{\pi}{1000}$, $\mathscr{B}y = 0$, $\mathscr{T}x = \frac{499\pi}{1000}$, $\mathscr{S}y = y$ *and*

$$\begin{aligned} m(\mathscr{A}x, \mathscr{B}y) &= \frac{\pi}{1000} \\ &\leq \frac{1}{8} \sin\left(y + \frac{499\pi}{1000}\right)\left(3y + \frac{1499\pi}{1000}\right) \\ &= \kappa(m(\mathscr{T}x, \mathscr{S}y))[m(\mathscr{S}y, \mathscr{A}x) + m(\mathscr{T}x, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{A}x) \\ &\quad + m(\mathscr{B}y, \mathscr{S}y) + m(\mathscr{T}x, \mathscr{B}y)], \end{aligned}$$

so, all hypotheses of Theorem 2 are satisfied and 0 *is the unique common fixed point of mappings* $\mathscr{A}$, $\mathscr{B}$, $\mathscr{T}$ *and* $\mathscr{S}$.

3. Application to Integral Equation

Take the next equation:

(1) $$S(x) = \int_0^x \mathscr{B}(x, y).\mathscr{D}(y, S(y))dy$$

for $x \in [0,\sigma]$, where $\sigma > 0$, $\mathscr{B} : [0,\sigma] \times [0,\sigma] \to \mathbb{R}$, $\mathscr{D} : [0,\sigma] \times \mathbb{R} \to \mathbb{R}$ are continuous mappings. Let $\mathscr{M} = C([0,\sigma],\mathbb{R})$ be endowed with the metric domain

$$\begin{aligned} m(S,S') &= \|S\|_\infty + \|S'\|_\infty \\ &= \max_{x\in[0,\sigma]} |S(x)| + \max_{x\in[0,\sigma]} |S'(x)|, \end{aligned}$$

for all S, S' in $\mathscr{M}$. Define on $\mathscr{M}$ the self-operator $\mathscr{A}$ by

$$\mathscr{A}S(x) = \int_0^x \mathscr{B}(x,y).\mathscr{D}(y,S(y))dy.$$

Theorem 3. *Suppose that the next requirements hold true:*

(1) $\max_{0\leq x\leq\sigma} \int_0^x \mathscr{B}(x,y)dy \leq 1$,

(2) $|\mathscr{D}(y,S(y))| + |\mathscr{D}(y,S'(y))| \leq \kappa[|S| + |S'|]$, *for each* $y \in [0,\sigma]$, S, $S' \in \mathscr{M}$, *where* $\kappa \in \left(0,\frac{1}{7}\right)$.

Then, integral equation (1) *has a unique solution.*

Proof. For all S, S' in $\mathscr{M}$, we have

$$\begin{aligned} m(\mathscr{A}S,\mathscr{A}S') &= \|\mathscr{A}S\|_\infty + \|\mathscr{A}S'\|_\infty \\ &= \max_{x\in[0,\sigma]} |\mathscr{A}S(x)| + \max_{x\in[0,\sigma]} |\mathscr{A}S'(x)| \\ &= \max_{x\in[0,\sigma]} \left|\int_0^x \mathscr{B}(x,y).\mathscr{D}(y,S(y))dy\right| + \max_{x\in[0,\sigma]} \left|\int_0^x \mathscr{B}(x,y).\mathscr{D}(y,S'(y))dy\right| \\ &\leq \max_{x\in[0,\sigma]} \int_0^x |\mathscr{B}(x,y)|.|\mathscr{D}(y,S(y))|dy + \max_{x\in[0,\sigma]} \int_0^x |\mathscr{B}(x,y)|.|\mathscr{D}(y,S'(y))|dy \\ &= \max_{x\in[0,\sigma]} \left[\int_0^x |\mathscr{B}(x,y)|.[|\mathscr{D}(y,S(y))| + |\mathscr{D}(y,S'(y))|]dy\right] \\ &\leq \max_{x\in[0,\sigma]} \int_0^x |\mathscr{B}(x,y)|.\kappa[|S| + |S'|]dy \\ &\leq \kappa \max_{x\in[0,\sigma]} [|S(x)| + |S'(x)|] \\ &= \kappa m(S,S') \\ &\leq \kappa(m(S',\mathscr{A}S) + m(S,S') + m(S,\mathscr{A}S) + m(\mathscr{A}S',S') + m(S,\mathscr{A}S')), \end{aligned}$$

so, all the requirements of Theorem 1 with $\mathscr{A} = \mathscr{B}$ and $\mathscr{T} = \mathscr{S} = \mathscr{I}$ (the identity mapping on $\mathscr{M}$) are satisfied, hence, integral equation (1) has a unique solution in $\mathscr{M}$. □

4. Conclusion

In this work, as metric spaces yield partial metric spaces which yield metric domains, our presented results improve and extend some existing results found in fixed point literature, among them, Theorem 1 and Theorem 3 of Panthi and Subedi [24], Theorem 3.1 of Prudhvi [25], Theorem 2.1 and Theorem 2.2 of Reena and Singh [26], Theorem 2.2 and Theorem 2.7 of Karapinar and Yüksel [27], Theorem 3.1 of Mallesh and Srinivas [28] and others.

Conflict of Interests

The authors declare that there is no conflict of interests.

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