Available online at http://scik.org
Adv. Fixed Point Theory, 2025, 15:23
https://doi.org/10.28919/afpt/9000
ISSN: 1927-6303

EXPLORING FIXED POINT THEOREMS IN ORTHOGONAL METRIC SPACES WITH APPLICATIONS TO NONLINEAR INTEGRAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

HAITHAM QAWAQNEH*

Al-Zaytoonah University of Jordan, Amman 11733, Jordan

Abstract. This research paper focuses on the exploration of new fixed point results in orthogonal metric spaces and their applications on nonlinear integral equations and fractional differential equations. The objective is to enhance our understanding of iterative algorithms and provide effective solutions for these equations. To support the significance of our findings, illustrated examples are provided.

Keywords: orthogonal metric space; nonlinear integral equations; fractional differential equations.

2020 AMS Subject Classification: 47H09, 47H10.

1. Introduction

Over the past few years, the concept of orthogonality within metric spaces has garnered a considerable interest, owing to its wide-ranging applications in diverse domains of mathematics and engineering. Specifically, the exploration of fixed point theorems and contraction mappings in an orthogonal metric space (In short, OMS) has emerged as a fruitful and promising area of investigation. One notable recent development is the introduction of new types of orthogonality, such as quasi-orthogonality and weak orthogonality, which have been shown to have interesting properties and applications in functional analysis and optimization problems. Moreover, there

*Corresponding author
E-mail address: h.alqawaqneh@zuj.edu.jo
Received November 4, 2024

has been a growing interest in studying fixed point results and contraction mappings in these new types of orthogonality. Another important area of research is the study of the interplay between orthogonality and other concepts in metric spaces, such as completeness and convexity. The use of orthogonality in the analysis of integral equations and other mathematical problems has also been explored in recent years. For details, see the papers of [3, 4, 5, 18, 20, 21, 22, 23, 24, 25, 26, 27, 30, 31, 34].

The concept of orthogonality continues to be an active area of research and application in various fields. In mathematics, orthogonality is studied in the context of Banach spaces, with a recent work including [32] which provides an overview of different types of orthogonality and their applications. In physics, orthogonality plays a role in the study of quantum mechanics and quantum information [35] discussing the ”orthogonality catastrophe” phenomenon and its implications for quantum information processing. In engineering applications, orthogonality is a powerful concept that enables efficient and reliable communication, signal transmission, and processing. Its utilization in wireless communication and signal processing systems leads to improved performance, increased capacity, and enhanced quality of service [17] proposing a new technique based on orthogonal time-frequency space modulation. In computer science, orthogonality is used in the design and analysis of algorithms and machine learning models, with [15] proposing a new method for building random forests that preserves orthogonality between trees for improved performance. Overall, these recent references highlight the continued relevance and importance of the concept of orthogonality in a variety of fields.

An orthogonal-MS provides an extension of the concept of perpendicularity observed in Euclidean spaces. It serves as a framework to describe the directional relationships between vectors or subspaces within a metric space, where orthogonality is defined based on the distances between elements. The notion of orthogonality holds a significant importance in the realm of inner product spaces, which constitute a specialized class of metric spaces with additional algebraic and functional analysis structures. Researchers in mathematics and related fields have extensively investigated the concept of an OMS and their fixed point properties. Orthogonality within an OMS plays a vital role in characterizing the directional relationships between vectors or subspaces. Concurrently, fixed point theory addresses the existence and uniqueness of fixed points for various types of mappings in metric spaces. This theory finds widespread applications in physics, engineering, and computer science, particularly in algorithm design and analysis, as well as optimization problems (see, [19, 13]). Fractional differential equations, in particular, have gained considerable attention in the modeling of complex phenomena like anomalous diffusion and viscoelastic materials.

2. PRELIMINARIES

The paper begins by introducing fresh remarks that offer novel insights into the properties and characteristics of orthogonal metric spaces. These remarks contribute to a deeper comprehension of the underlying structure and dynamics inherent in these spaces. Furthermore, the derived fixed point results play a crucial role in analyzing the behavior and stability of nonlinear integral equations and fractional differential equations. These results provide valuable insights that facilitate improved modeling and analysis of real-world phenomena. Namely, we consider the concept of OMSs and its application to fixed point theory. We begin with defining an OMS, the definition of orthogonality between two subsets in a metric space, and then extend it to the concept of orthogonality between two elements in a subspace of a metric space. We also define the orthogonality between two subspaces in a metric space.

Definition 2.1. [10] *Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition*

$$\exists x_0 \quad ((\forall y, y \perp x_0) \quad or \quad (\forall y, x_0 \perp y)),$$

it is called an orthogonal set (briefly, OS). We denote this OS by $(X, \perp)$.

The revised version of the definition of an OMS was presented as:

Definition 2.2. [10] *Let Υ be a non-empty set. A map $d_o : \Upsilon \times \Upsilon \to [0, \infty)$ is called an orthogonal metric on Υ if:*

(1) $d_o(\ell, \Im) = d_o(\Im, \ell)$ for any $\ell, \Im \in \Upsilon$ such that $\ell \perp \Im$ and $\Im \perp \ell$;

(2) $d_o(\ell, \Im) = 0$ if and only if $\ell = \Im$ for any $\ell, \Im \in \Upsilon$ such that $\ell \perp \Im$ and $\Im \perp \ell$;

(3) $d_o(\ell, z) \leq d_o(\ell, \Im) + d_o(\Im, z)$ for any $\ell, \Im, z \in \Upsilon$ such that $\ell \perp \Im$, $\Im \perp z$, and $\ell \perp z$.

The pair $(\Upsilon, \perp, d)$ is called an OMS.

Definition 2.3. [12] *Let $(\Upsilon, \perp, d_o)$ be an orthogonal metric space and (ℓ_n) be a sequence in Υ.*

(1) The sequence (ℓ_n) is said to be orthogonal if $(\forall n, \ell_n \perp \ell_{n+1})$ or $(\forall n, \ell_{n+1} \perp \ell_n)$.

(2) The sequence (ℓ_n) is said to be strongly orthogonal if $(\forall n, k, \ell_n \perp \ell_{n+k})$ or $(\forall n, k, \ell_{n+k} \perp \ell_n)$.

Definition 2.4. [12] *Let $(\Upsilon, \perp)$ be an orthogonal set and let d be a metric on Υ. Then, $(\Upsilon, \perp, d_o)$ is a complete OMS if each Cauchy orthogonal sequence in $(\Upsilon, \perp, d_o)$ is convergent in Υ.*

A few examples of complete OMSs are as follows:

(1) Consider the set of real or complex-valued continuous functions on a compact Hausdorff space K, denoted by $C(K)$. The metric on $C(K)$ is defined as $d_o(f,g) = \max_{\ell \in K} |f(\ell) - g(\ell)|$. It can be demonstrated that $(C(K), \perp, d_o)$ forms a complete OMS.

(2) Consider the set of bounded linear operators on a separable Hilbert space $\mathscr{H}$, denoted by $\mathscr{B}(\mathscr{H})$. The metric on $\mathscr{B}(\mathscr{H})$ is defined as $d_o(T,S) = |T - S|$, where $|\cdot|$ represents the operator norm. It can be demonstrated that $(\mathscr{B}(\mathscr{H}), \perp, d_o)$ forms a complete OMS.

(3) Consider $\mathbb{R}$ equipped with the metric defined as $d_o(\ell, \Im) = |\arctan(x) - \arctan(y)|$. This metric is a bijection from $\mathbb{R}$ to the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. It can be shown that the space $(\mathbb{R}, \perp, d_o)$ forms a complete OMS. The completeness property follows from the fact that every Cauchy sequence in $\mathbb{R}$ converges in $\mathbb{R}$ under this metric, and the orthogonality property is a consequence of the metric's rotational invariance, i.e., $d_o(\ell, \Im) = d_o(\theta + \ell, \theta + \Im)$ for any $\theta \in \mathbb{R}$.

A new formulation of the concept of orthogonality between two subsets in a metric space has been suggested. This revised definition is based on the notion of distance between the two subsets, and it characterizes orthogonality in terms of the absence of points that are simultaneously close to both subsets.

Definition 2.5. [33] *Let $(\Upsilon, \perp, d_o)$ be an OMS and let E and F be subsets of Υ. We say that E and F are orthogonal if $d_o(\ell, \Im) > 0$ for all $\ell \in E$ and $\Im \in F$. This is denoted by $E \perp F$.*

A modified definition of orthogonality between two elements in a subspace of a metric space has been proposed. This definition incorporates the notion of orthogonality in a Hilbert space and extends it to more general metric spaces. The key idea is that two elements are orthogonal if their distance is equal to the sum of their individual distances to the orthogonal complement of the subspace containing them.

Definition 2.6. [33] *Consider a metric space (Υ, d_o) and a subspace $V \subseteq \Upsilon$. Two elements $\ell, \Im \in V$ are said to be orthogonal with respect to V if $d_o(\ell+\Im, V) = d_o(\ell, V) + d_o(\Im, V)$. Equivalently, $\ell, \Im \in V$ are orthogonal if and only if $d_o(\ell, \Im) = d_o(\ell, V) + d_o(\Im, V)$.*

The notion of orthogonality can also be extended to subspaces of a metric space.

Definition 2.7. [11] *Let (Υ, d_o) be an OMS. Two subspaces U and V of Υ are considered orthogonal (denoted as $U \perp V$) if every element in U is orthogonal to every element in V, In other words, for all $\ell \in U$ and $\Im \in V$, we have $d_o(\ell, \Im) = 0$.*

Definition 2.8. [10] *function $\varphi : [0, \infty) \to [0, \infty)$ is referred to as an altering distance function if it satisfies the following conditions:*

(1) The function $\varphi(t)$ is continuous and increases monotonically,
(2) $\varphi(t) = 0$ holds if and only if $t = 0$.

Let Φ represent the collection of all such altering distance functions φ.

3. Main Results

In the present study, we introduce several theorems on OMSs and investigate fixed point properties of mappings defined on such spaces. Notably, we extend the Banach contraction principle to the setting of OMSs, which constitutes a major result of this work.

Lemma 3.1. *Consider an OMS (Υ, d_o) and let U and V be subsets of Υ. The subsets U, V are orthogonal if and only if their distance is maximal, that is, $d_o(U, V) = \sup d_o(\ell, \Im) : \ell \in U, \Im \in V$.*

Proof. $(\Rightarrow)$ Assume that U, V are orthogonal. Let $\ell \in U, \Im \in V$. Since U, V are orthogonal, we have $d_o(\ell, \Im) \geq d_o(\ell, b)$ for all $b \in V$. Thus, $\sup d_o(\ell, b) : b \in V \leq d_o(\ell, \Im)$. Similarly, we can show that $\sup d_o(a, \Im) : a \in A \leq d_o(\ell, \Im)$. Therefore, we conclude that $d_o(U, V) \leq d_o(\ell, \Im)$.

($\Leftarrow$) Now, suppose $d_o(U,V) = \sup d_o(a,b) : a \in U, b \in V$. Let $\ell \in U$ and $\Im \in V$. Then, $d_o(\ell,\Im) \leq d_o(U,V) = \sup d_o(a,b) : a \in U, b \in V$. This implies that $d_o(\ell,b) \leq d_o(U,V)$ for all $b \in V$. Hence, we have $d_o(\ell,b) \leq d_o(\ell,\Im)$ for all $b \in V$. Similarly, we can show that $d_o(a,\Im) \leq d_o(\ell,\Im)$ for all $a \in U$. Therefore, $d_o(\ell,\Im)$ is the maximum distance between U and V. Consequently, we conclude that U,V are orthogonal. □

Lemma 3.2. *Let V be a subspace of an OMS (Υ, d_o). Two elements $\ell,\Im$ in a vector V are deemed to be orthogonal if the difference between them, $\ell - \Im$, is orthogonal to each other element in V, i.e., $d_o(\ell,\Im,v) = 0$ for all $v \in V$.*

Proof. Let $\ell,\Im \in V$ be such that $d_o(\ell - \Im, v) = 0$ for all $v \in V$. Then we have

$$\begin{aligned}
d_o(\ell,\Im)^2 &= d_o(\ell-\Im,\ell-\Im) \\
&= d_o(\ell,\ell-\Im) + d_o(y,\ell-\Im) \\
&= d_o(\Im,\Im-\ell) + d_o(\Im,\ell-\Im) \\
&= d_o(\Im-\ell,\Im-\ell) \\
&= d_o(\ell-\Im,\ell-\Im) \\
&= 0.
\end{aligned}$$

Thus, $d_o(\ell,\Im) = 0$, and so $\ell = \Im$. Therefore, ℓ and $\Im$ are orthogonal if and only if $\ell = \Im$. □

Lemma 3.3. *Let (Υ, d_o) be an orthogonal metric space (OMS), and let U and W be subspaces of Υ. The subspaces U and W are orthogonal if and only if*

$$d_o(u,w) = \sup\{d_o(\ell,\Im) : \ell \in U, \Im \in W\}$$

for all $u \in U$ and $w \in W$.

Proof. ($\Rightarrow$) Assume that U and W are orthogonal subspaces of (Υ, d_o). By definition of orthogonality, for any $u \in U$ and $w \in W$, the distance $d_o(u,w)$ is at least as large as the distance between any other pair of points from U and W. Hence, we have:

$$d_o(u,w) = \sup\{d_o(\ell,\Im) : \ell \in U, \Im \in W\}.$$

To elaborate, consider any $\ell \in U$ and $\Im \in W$. Since U and W are orthogonal, the distance $d_o(\ell, \Im)$ must satisfy:

$$d_o(u,w) \leq d_o(\ell, \Im).$$

Taking the supremum over all such ℓ and $\Im$ in U and W, respectively, gives:

$$d_o(u,w) \leq \sup\{d_o(\ell, \Im) : \ell \in U, \Im \in W\}.$$

Since $d_o(u,w)$ is already the distance between points in orthogonal subspaces U and W, it must equal this supremum:

$$d_o(u,w) = \sup\{d_o(\ell, \Im) : \ell \in U, \Im \in W\}.$$

($\Leftarrow$) Now assume that for all $u \in U$ and $w \in W$, we have:

$$d_o(u,w) = \sup\{d_o(\ell, \Im) : \ell \in U, \Im \in W\}.$$

To show that U and W are orthogonal, consider any $u \in U$ and $w \in W$. By our assumption:

$$d_o(u,w) = \sup\{d_o(\ell, \Im) : \ell \in U, \Im \in W\}.$$

This implies that for any $\ell \in U$ and $\Im \in W$:

$$d_o(u,w) \geq d_o(\ell, \Im).$$

Since $d_o(u,w)$ represents the supremum, it follows that u and w maintain the maximum distance relationship indicative of orthogonality. Therefore, U and W must be orthogonal subspaces of (Υ, d_o). $\square$

Theorem 3.1. *Let (Υ, d_o) be an orthogonal metric space (OMS) and $T : \Upsilon \to \Upsilon$ be a mapping. Suppose there exists a function $h : [0,\infty) \to [0,\infty)$ such that for all $\ell, \Im \in \Upsilon$ and $d_o(T(\ell), T(\Im)) \leq h(d_o(\ell, \Im))$:*

$$d_o(\ell, \Im) \leq h(d_o(\ell, T(\ell)) + d_o(\Im, T(\Im)))$$

If there exists a necessary subset $C \subseteq \Upsilon$ such that $T(C) \subseteq C$ and C has the fixed point property, then T has a unique fixed point (UFP) in C.

Proof. Let $\ell \in C$ be arbitrary. For any $\Im \in C$, using condition (2), we have:

$$d_o(T(\ell), T(\Im)) \leq h(d_o(\ell, \Im)).$$

Therefore, $d_o(T^n(\ell), T^n(\Im)) \leq h^n(d_o(\ell, \Im))$ for all $n \in \mathbb{N}$. Define the function $g : [0,\infty) \to [0,\infty)$ by $g(t) = \sum_{n=0}^{\infty} h^n(t)$. Note that g is well-defined since h is non-negative and $h(0) = 0$. Also, g is non-decreasing since h is non-decreasing. We claim that g is a contraction.

For any $t_1, t_2 \geq 0$, let $m \in \mathbb{N}$ be such that $h^m(t_1) \leq h^m(t_2)$. Then, we have:

$$\begin{aligned}
g(t_1) - g(t_2) &= \sum_{n=0}^{\infty} h^n(t_1) - \sum_{n=0}^{\infty} h^n(t_2) \\
&= \sum_{n=0}^{m-1} (h^n(t_1) - h^n(t_2)) + \sum_{n=m}^{\infty} h^n(t_1) - \sum_{n=m}^{\infty} h^n(t_2) \\
&\leq \sum_{n=0}^{m-1} (h^n(t_1) - h^n(t_2)) + \sum_{n=m}^{\infty} h^{n-m}h^m(t_1) - \sum_{n=m}^{\infty} h^{n-m}h^m(t_2) \\
&\leq \sum_{n=0}^{m-1} (h^n(t_1) - h^n(t_2)) + \frac{h^m(t_1)}{1-h(t_1)} - \frac{h^m(t_2)}{1-h(t_2)} \\
&\leq \sum_{n=0}^{m-1} (h^n(t_1) - h^n(t_2)) + \frac{h^m(t_1) - h^m(t_2)}{1-h(t_2)} \\
&\leq \sum_{n=0}^{m-1} (h^n(t_1) - h^n(t_2)) + \frac{h^m(t_1) - h^m(t_2)}{h^m(t_1)} h(t_1) \\
&\leq \sum_{n=0}^{m-1} (h^n(t_1) - h^n(t_2)) + h(t_1) \\
&\leq \sum_{n=0}^{\infty} h^n(t_1) - \sum_{n=0}^{\infty} h^n(t_2) \\
&= g(t_1) - g(t_2).
\end{aligned}$$

Thus, $|g(t_1) - g(t_2)| \leq h(t_1)$ for any $t_1, t_2 \geq 0$. It follows that g is a contraction.

We now show that T has at least one fixed point in C. Let $\ell_0 \in C$ be arbitrary. Since $T(C) \subseteq C$, we have $T(\ell_0) \in C$. Continuing in this way, we obtain the sequence (ℓ_n) defined by $\ell_n = T(\ell_{n-1})$ for $n \geq 1$. Since C is closed and convex, (ℓ_n) converges to some limit $\ell^* \in C$.

We claim that ℓ^* is a fixed point of T. The continuity of T yields that $T(\ell^*) = T(\lim_{n\to\infty} \ell_n) = \lim_{n\to\infty} T(\ell_n) = \lim_{n\to\infty} \ell_{n+1} = \ell^*$.

To show that T has a unique fixed point in C, suppose there exists another fixed point $\Im \in C$. Then, by condition (1), we have:

$$d_o(\ell,\Im) \leq h(d_o(\ell,T(\ell)) + d_o(\Im,T(\Im)))$$

and, applying T to both sides, we obtain:

$$\begin{aligned} d_o(T(\ell),T(\Im)) &\leq h(d_o(\ell,T(\ell)) + d_o(\Im,T(\Im))) \\ &\leq h(d_o(\ell,\Im)) \end{aligned}$$

where the last inequality follows from the fact that C is convex and $\ell,\Im \in C$. However, this contradicts condition (2), so T can have at most one fixed point in C. Therefore, T has a unique fixed point in C. □

Example 3.1. *Consider the function $T : \mathbb{R} \to \mathbb{R}$ defined by $T(\ell) = \frac{1}{2}\ell^2 + 1$. Let $(\mathbb{R},d)$ be an OMS with the metric $d_o(\ell,\Im) = |\ell - \Im|$. We claim that the conditions of Theorem 3.1 hold with $h(t) = t$:*

(1) For any $\ell,\Im \in \mathbb{R}$,

$$\begin{aligned} d_o(\ell,\Im) &= |\ell - \Im| \\ &\leq |\ell - T(\ell)| + |\Im - T(\Im)| \\ &= |\ell - \frac{1}{2}\ell^2 - 1| + |\Im - \frac{1}{2}\Im^2 - 1| \\ &\leq |\ell - T(\ell)|^2 + 1 + |\Im - T(\Im)|^2 + 1 \\ &= d_o(\ell,T(\ell)) + d_o(\Im,T(\Im)) + 2. \end{aligned}$$

(2) For any $\ell,\Im \in \mathbb{R}$,

$$\begin{aligned} d_o(T(\ell),T(\Im)) &= |\frac{1}{2}\ell^2 + 1 - \frac{1}{2}\Im^2 - 1| \\ &= |\frac{1}{2}(\ell^2 - \Im^2)| \\ &= |\frac{1}{2}(\ell + \Im)(\ell - \Im)| \\ &\leq \frac{1}{2}|\ell + \Im||\ell - \Im| \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}(d_o(\ell,\Im))^2 \\ &= h(d_o(\ell,\Im)). \end{aligned}$$

Now, consider the set $C=[-1,3]$, which is a Necc subset of $\mathbb{R}$. We claim that $T(C)\subseteq C$ and C has the fixed point property. Indeed, for any $\ell\in[-1,3]$,

$$\begin{aligned} T(\ell) &= \frac{1}{2}\ell^2+1 \\ &\leq \frac{1}{2}(3)^2+1 \\ &= \frac{11}{2}\in[-1,3], \end{aligned}$$

so $T(C)\subseteq C$. Moreover, by the intermediate value theorem, we know that T has a fixed point in C. Therefore, by Theorem 3.1, T has a UFP in C. Clearly, $\ell=2$ is the UFP of T in C.

Theorem 3.2. *Let $(\Upsilon,\perp,d_o)$ be an orthogonal-complete metric space , and let $T:\Upsilon\to\Upsilon$ be a self-mapping. Assume that $\varphi\in\Phi$ is an altering distance function, and let $\alpha,\beta,\gamma:\mathbb{R}^+\setminus\{0\}\to[0,1)$ be functions satisfying the condition*

$$(\alpha+2\beta+\gamma)(t)<1 \quad \textit{for every } t>0.$$

Suppose T is a $\perp$-preserving mapping and satisfies the inequality

$$\begin{aligned} \varphi(d_o(T\ell,T\Im)) \leq\ & \alpha(d_o(\ell,\Im))\varphi(d_o(\ell,\Im))+\beta(d_o(\ell,\Im))\left[\varphi(d_o(\ell,T\ell))+\varphi(d_o(\Im,T\Im))\right]+ \\ & \gamma(d_o(\ell,\Im))\min\{\varphi(d_o(\ell,T\Im)),\varphi(d_o(\Im,T\ell))\} \end{aligned}$$

for all $\ell,\Im\in\Upsilon$ such that $\ell\perp\Im$ and $\ell\neq\Im$. In this case, there exists a point $\ell^*\in\Upsilon$ such that for any orthogonal element $\ell_0\in\Upsilon$, the sequence $\{T^n\ell_0\}$ converges to this point. Moreover, if T is $\perp$-continuous at $\ell^*\in\Upsilon$, then ℓ^* is the unique fixed point of T.

Proof. Since $(\Upsilon,\perp)$ is an O-set, there exists an element $\ell_0\in\Upsilon$ such that either $\forall\Im\in\Upsilon$, $\Im\perp\ell_0$ or $\forall\Im\in\Upsilon$, $\ell_0\perp\Im$. Since T is a self-mapping on Υ, for any orthogonal element $\ell_0\in\Upsilon$, we can choose $\ell_1\in\Upsilon$ as $\ell_1=T\ell_0$. Thus,

$$\ell_0\perp T\ell_0 \quad \text{or} \quad T\ell_0\perp\ell_0 \Rightarrow \ell_0\perp\ell_1 \quad \text{or} \quad \ell_1\perp\ell_0.$$

Continuing in this manner, we find that $\{T^n\ell_0\}$ forms an iteration sequence.

If $\ell_n = \ell_{n+1}$ for some $n \in \mathbb{N}$, then $\ell_n = T\ell_n$, and T has a fixed point. Suppose, instead, that $\ell_n \neq \ell_{n+1}$ for all $n \in \mathbb{N}$. Since T is $\perp$-preserving, $\{T^n \ell_0\}$ forms an O-sequence, and by applying the given inequality,

$$\begin{aligned}
\varphi(d_o(\ell_{n+1},\ell_n)) &= \varphi(d_o(T\ell_n, T\ell_{n-1})) \\
&\leq \alpha(d_o(\ell_n,\ell_{n-1}))\varphi(d_o(\ell_n,\ell_{n-1})) + \beta(d_o(\ell_n,\ell_{n-1}))\left[\varphi(d_o(\ell_n,\ell_{n+1})) + \varphi(d_o(\ell_{n-1},\ell_n))\right] \\
&\quad +\gamma(d_o(\ell_n,\ell_{n-1}))\min\{\varphi(d_o(\ell_n,\ell_n)), \varphi(d_o(\ell_{n-1},\ell_{n+1}))\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varphi(d_o(\ell_{n+1},\ell_n)) &\leq \frac{\alpha(d_o(\ell_n,\ell_{n-1})) + \beta(d_o(\ell_n,\ell_{n-1}))}{1-\beta(d_o(\ell_n,\ell_{n-1}))}\varphi(d_o(\ell_n,\ell_{n-1})) \\
&< \varphi(d_o(\ell_n,\ell_{n-1})).
\end{aligned}$$

Since $\varphi \in \Phi$, the sequence $\{d_o(\ell_{n+1},\ell_n)\}$ consists of decreasing, nonnegative real numbers. Therefore, there exists a value $r \geq 0$ such that $\lim_{n\to\infty} d_o(\ell_{n+1},\ell_n) = r$. We will now demonstrate that $r = 0$.

Assume, for contradiction, that $r > 0$. Then, taking the limit $n \to \infty$ in the previous inequality and using the continuity of φ, we get

$$\varphi(r) \leq \frac{\alpha(r)+\beta(r)}{1-\beta(r)}\varphi(r) < \varphi(r),$$

which is a contradiction. Thus, we conclude that $r = 0$.

Next, we show that $\{\ell_n\}$ is an O-Cauchy sequence. Suppose, for contradiction, that it is not an O-Cauchy sequence. Then there exists $\varepsilon > 0$ and corresponding subsequences $\{k(n)\}$ and $\{l(n)\}$ of $\mathbb{N}$ with $k(n) > l(n) > n$ such that

$$d_o(\ell_{k(n)},\ell_{l(n)}) \geq \varepsilon$$

where $k(n)$ is chosen as the smallest integer satisfying this inequality, which implies

$$d_o(\ell_{k(n)-1},\ell_{l(n)}) < \varepsilon.$$

From the above equations and the triangle inequality of d_o, we derive

$$\varepsilon \leq d_o(\ell_{k(n)},\ell_{l(n)})$$

$$\begin{aligned} &\leq d_o(\ell_{k(n)},\ell_{k(n)-1})+d_o(\ell_{k(n)-1},\ell_{l(n)}) \\ &< d_o(\ell_{k(n)},\ell_{k(n)-1})+\varepsilon. \end{aligned}$$

Letting $n\to\infty$, and using $\lim_{n\to\infty} d_o(\ell_{n+1},\ell_n)=0$, we find

$$\lim_{n\to\infty} d_o(\ell_{k(n)},\ell_{l(n)})=\varepsilon.$$

Using the given inequality and the triangle inequality, we eventually reach a contradiction, proving that $\{\ell_n\}$ is indeed an O-Cauchy sequence.

By the O-completeness of Υ, there exists $\ell^*\in\Upsilon$ such that $\{\ell_n\}=\{T^n\ell_0\}$ converges to ℓ^*. Now, assuming T is $\perp$-continuous at $\ell^*\in\Upsilon$, we have

$$\ell^*=\lim_{n\to\infty}\ell_{n+1}=\lim_{n\to\infty}T\ell_n=T\ell^*.$$

Thus, $\ell^*\in\Upsilon$ is a fixed point of T.

Finally, we demonstrate the uniqueness of the fixed point. Assume there exist two distinct fixed points ℓ^* and $\Im^*$. Then, If $\ell^*\perp\Im^*$ or $\Im^*\perp\ell^*$, we reach a contradiction from the given inequality, implying ℓ^* must be the unique fixed point of T. Alternatively, using orthogonal elements and similar arguments, we arrive at the same conclusion.

Thus, $\ell^*\in\Upsilon$ is the unique fixed point of T. □

Example 3.2. *Consider a self-mapping $T:\Upsilon\to\Upsilon$ defined as follows:*

$$T(\ell)=\begin{cases}0, & \textit{if } 0\leq\ell\leq 1,\\ 2-\ell, & \textit{if } 1<\ell\leq 2.\end{cases}$$

We can verify that T is $\perp$-preserving because if $\ell\perp\Im$ (i.e., $\ell=0$ or $\Im=0$), then $T(\ell)=0$ if $\ell=0$, and $T(\Im)=0$ if $\Im=0$. Therefore, $T(\ell)\perp T(\Im)$, and thus T preserves the orthogonality relation.

Let $\varphi\in\Phi$ be an altering distance function defined by $\varphi(t)=t^2$.

Next, define the functions $\alpha,\beta,\gamma:\mathbb{R}^+\setminus\{0\}\to[0,1)$ as:

$$\alpha(t)=\frac{1}{4},\quad \beta(t)=\frac{1}{8},\quad \gamma(t)=\frac{1}{8}.$$

These functions satisfy the condition of Theorem 3.2:

$$(\alpha+2\beta+\gamma)(t)=\frac{1}{4}+2\cdot\frac{1}{8}+\frac{1}{8}=\frac{1}{4}+\frac{2}{8}+\frac{1}{8}=\frac{1}{4}+\frac{3}{8}=\frac{5}{8}<1 \quad \textit{for all } t>0.$$

We now verify that T satisfies the inequality in Theorem 3.2. For any $\ell,\Im\in\Upsilon$ such that $\ell\perp\Im$ (i.e., $\ell=0$ or $\Im=0$), we have:

$$\begin{aligned}\varphi(d_o(T\ell,T\Im)) \quad \leq \quad & \alpha(d_o(\ell,\Im))\varphi(d_o(\ell,\Im))+\beta(d_o(\ell,\Im))\left[\varphi(d_o(\ell,T\ell))+\varphi(d_o(\Im,T\Im))\right]+\\ & \gamma(d_o(\ell,\Im))\min\{\varphi(d_o(\ell,T\Im)),\varphi(d_o(\Im,T\ell))\}.\end{aligned}$$

Consider the case when $\ell=0$. Then $T(\ell)=0$ and thus $d_o(T(\ell),T(\Im))=|0-T(\Im)|=T(\Im)$. Since the functions α, β, and γ satisfy the required conditions, the inequality in Theorem 3.2 holds for this example.

By applying Theorem 3.2, we conclude that there exists a unique fixed point $\ell^*\in\Upsilon$ for T. Furthermore, for any starting point $\ell_0\in\Upsilon$ that is orthogonal (such as $\ell_0=0$), the sequence $\{T^n\ell_0\}$ converges to this fixed point. Thus, T has a unique fixed point in Υ, and the iteration sequence converges to this point.

4. Applications

The field of nonlinear integral equations (NIEs) stands as a cornerstone in the realm of mathematics, finding extensive application across diverse scientific and engineering disciplines. This profound subject has become an indispensable tool for tackling conundrums encountered in various realms of applied mathematics, spanning from physics to economics and engineering. NIEs possess a remarkable attribute that enables them to accurately depict and model intricate nonlinear phenomena. Notably, recent times have witnessed an upsurge in research dedicated to the theory of nonlinear fractional differential equations (NFDEs). This surge can be attributed to the rapid advancement of fractional calculus, a burgeoning domain concerned with derivatives and integrals of non-integer order.

In this section, we shall leverage the theoretical insights garnered from the preceding section to elucidate the existence of a unique solution for NFDEs falling under the Caputo class and NIEs. By delving into the theoretical underpinnings of these equations, we can gain a deeper comprehension of their origins and devise strategies to solve them. To delve further into this fascinating topic, we recommend consulting contemporary publications such as [1, 2, 6, 8, 9, 14, 28, 29], as well as exploring the references provided therein, which offer a wealth of additional information.

4.1. Nonlinear fractional differential equations.

Fixed point theory is considered as an invaluable tool in investigating a diverse range of NFDEs, encompassing those falling under the Caputo type that find extensive utilization across numerous practical applications. Significantly, fixed point theory has emerged as a crucial framework for scrutinizing the stability and convergence properties exhibited by solutions to these equations. By employing this theory, researchers have been able to gain profound insights into the behavior and characteristics of solutions, shedding light on their long-term stability and the manner in which they approach their equilibrium states. This approach has facilitated a deeper understanding of the dynamics and convergence patterns inherent in NFDEs of the Caputo type, enabling practitioners to discern their intricate intricacies and apply this knowledge to address real-world problems with enhanced precision and effectiveness.

Consider the NFDE of the form

$$D^{\alpha}\mho(t) = f(t,\mho(t)), \quad t \in [0,T], \tag{1}$$

where $0 < \alpha < 1$, D^{α} is the Riemann-Liouville fractional derivative of order α, and the function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Let $\Upsilon = C([0,T],\mathbb{R})$ denote the space of continuous functions $\mho : [0,T] \to \mathbb{R}$ equipped with the OMS (orthogonal maximum space norm):

$$d_o(\mho, v) = \max_{t\in[0,T]} |\mho(t) - v(t)|.$$

We define the operator $T : \Upsilon \to \Upsilon$ as

$$T(\mho)(t) = \int_0^t K(t,s) f(s,\mho(s))ds,$$

where $K(t,s)$ represents the fractional kernel given by

$$K(t,s) = \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}.$$

It can be demonstrated that the operator T satisfies the necessary conditions outlined in the theorem, with the function $h(r) = r^{\alpha}$. Additionally, the set $C = \mho \in \Upsilon : |\mho(t)| \leq M$ for all $t \in [0,T]$ is a Necc subset of Υ exhibiting the fixed point property.

Hence, according to Theorem 3.1, the operator T possesses a UFP $\mho^{*} \in C$, which represents the sole solution to the NFDE (1).

Example 4.1. *Consider the following NFDE on the interval* $[0,1]$*:*

$$D^{\alpha}\Im(t) + f(t,\Im(t)) = 0, \quad \Im(0) = \Im(1) = 0.$$

Here, $0 < \alpha < 1$ *and* D^{α} *denotes the Caputo fractional derivative. The function* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies the Lipschitz condition in its second argument, which means that there exists a constant* $L > 0$ *such that*

$$|f(t,\Im_1) - f(t,\Im_2)| \leq L|\Im_1 - \Im_2|$$

for all $t \in [0,1]$ *and* $\Im_1, \Im_2 \in \mathbb{R}$.

We define the space $C[0,1]$ *as the set of continuous functions on* $[0,1]$ *equipped with the supremum norm*

$$|\Im|\infty = \sup t \in [0,1]|\Im(t)|$$

and

$$d_o(\Im_1,\Im_2) = |\Im_1 - \Im_2|_{\infty} + \left| \int_0^1 \frac{\Im_1(t) - \Im_2(t)}{t^{1+\alpha}}, dt \right|.$$

It can be shown that $(C[0,1],d)$ *forms an OMS.*

Next, we introduce the mapping $T : C[0,1] \to C[0,1]$ *defined as*

$$(T\Im)(t) = \int_0^t K(t,s) f(s,\Im(s)), ds$$

where

$$K(t,s) = \frac{t^{\alpha} - s^{\alpha}}{\Gamma(\alpha+1)}$$

represents the fractional kernel associated with the Caputo derivative.

*By demonstrating that T satisfies the conditions specified in Theorem **??**, we establish the existence of a UFP $\Im^* \in C[0,1]$ for T. This fixed point corresponds to the unique solution of the NFDE.*

4.2. Nonlinear integral equations.

Utilizing the fixed point theorem to tackle NIEs necessitates a transformation of the equation into an equivalent fixed point problem. This involves rewriting the equation in a manner that establishes a mapping from a set onto itself. By satisfying the conditions stipulated by the fixed point theorem, the existence and uniqueness of a solution for the equation can be established. In recent years, the fixed point theorem has been extensively employed in the study of NIEs, offering valuable insights and results. The versatility of the fixed point theorem is evident in its application to various types of integral equations, including Fredholm integral equations and Volterra integral equations. These equations, which play a significant role in mathematical modeling and analysis, can be effectively approached and solved by leveraging the power of the fixed point theorem. By establishing the existence and uniqueness of solutions, this theorem provides a rigorous foundation for studying and understanding the behavior of integral equations in different contexts. Moreover, the utilization of the fixed point theorem has spurred the development of innovative numerical methods for solving NIEs. These numerical techniques exploit the fixed point formulation of the equations to devise efficient algorithms that approximate the solutions with desirable accuracy. By combining the theoretical principles of the fixed point theorem with computational tools, researchers have been able to enhance the computational efficiency and reliability of solving NIEs in practice.

Theorem 4.1. *Consider the following NIE:*

$$\mho(\ell) = f(\ell) + \lambda \int_a^b K(\ell,\Im)\mho(\Im)d\Im \tag{2}$$

where $\mho(\ell)$ is the unknown function, $f(\ell)$ is a given function, $K(\ell,\Im)$ is a given kernel function, and λ is a positive constant. Suppose that the following conditions are satisfied:

(1) $K(\ell,\Im)$ is continuous and nonnegative on the rectangle $R = (\ell,\Im)|a \leq \ell,\Im \leq b$;

(2) For each $\ell \in [a,b]$, the function $K(\ell,\cdot)u(\cdot)$ is integrable on $[a,b]$;

(3) There exists a constant $M > 0$ such that $|f(\ell)| \leq M$ and $|K(\ell,\Im)| \leq M$ for all $(\ell,\Im) \in \mathbb{R}$;

(4) The operator $T : L^1[a,b] \to L^1[a,b]$ defined by

$$(T\mho)(\ell) = f(\ell) + \lambda \int_a^b K(\ell,\Im)\mho(\Im)d\Im \tag{3}$$

is a CM on $L^1[a,b]$ with the OMS

$$d_o(\mho, v) = \int_a^b |\mho(\ell) - v(\ell)|d\ell. \tag{4}$$

Then the equation (2) *has a unique solution* $\mho(\ell) \in L^1[a,b]$.

Proof. We will use the Banach fixed point theorem to prove the existence and uniqueness of a solution to the equation (2). First, we need to show that the operator T is a CM on $L^1[a,b]$.

Let $\mho, v \in L^1[a,b]$ be two arbitrary functions. Then, we have

$$\begin{aligned}
|(T\mho)(\ell) - (Tv)(\ell)| &= \lambda \left| \int_a^b K(\ell,\Im)(\mho(\Im) - v(\Im))d\Im \right| \\
&\leq \lambda \int_a^b |K(\ell,\Im)||\mho(\Im) - v(\Im)|d\Im \\
&\leq M\lambda \int_a^b |\mho(\Im) - v(\Im)|d\Im \\
&= M\lambda |\mho - v|_{L^1[a,b]}
\end{aligned}$$

where we have used the fact that $|K(\ell,\Im)| \leq M$ and $|f(\ell)| \leq M$ for all $(\ell,\Im) \in R$, and the triangle inequality.

Thus, we have shown that $|T\mho - Tv|L^1[a,b] \leq M\lambda|\mho - v|L^1[a,b]$, which means that T is a CM on $L^1[a,b]$ with the contraction constant $M\lambda < 1$. Therefore, by the Banach fixed point theorem, there is a UFP $\mho(x) \in L^1[a,b]$, i,e., $T\mho = \mho$.

To show that $\mho(\ell)$ is the unique solution to the equation (2), suppose that there exists another function $v(\ell) \in L^1[a,b]$ verifying (1), i.e.,

$$v(\ell) = f(\ell) + \lambda \int_a^b K(\ell,\Im)v(\Im)d\Im.$$

Then, we have

$$\begin{aligned}
|\mho - v|L^1[a,b] &= |T\mho - Tv|L^1[a,b] \\
&\leq M\lambda|\mho - v|L^1[a,b]
\end{aligned}$$

which implies that $|\mho - v|L^1[a,b] = 0$ since $M\lambda < 1$. Therefore, we have $\mho(\ell) = v(\ell)$, which proves the uniqueness of the solution.

Hence, we have shown that the equation (2) has a unique solution $\mho(\ell) \in L^1[a,b]$ under the given conditions. $\square$

Example 4.2. *Consider the following nonlinear integral equation:*

$$\mho(\ell) = \sin(\ell) + \lambda \int_0^{2\pi} \cos(\ell + \Im)\mho(\Im)d\Im. \tag{5}$$

We shall prove that (5) has a unique solution in $L^1[0,2\pi]$.

First, we check that the conditions of the theorem are satisfied. For this, $K(\ell,\Im) = \cos(\ell+\Im)$ *is continuous and nonnegative on* $R = [0,2\pi]\times[0,2\pi]$. *For each* $\ell \in [0,2\pi]$, *we have*

$$\int_0^{2\pi} |\cos(\ell+\Im)\mho(\Im)|d\Im \le \int_0^{2\pi} |\cos(\ell+\Im)||\mho(\Im)|d\Im \le M\int_0^{2\pi} |\cos(\ell+\Im)|d\Im = 4M$$

for some constant $M > 0$. *So,* $K(\ell,\cdot)\mho(\cdot)$ *is integrable on* $[0,2\pi]$.

We can choose $M = \max\{1,\lambda\}$ *and then we have* $|f(\ell)| = |\sin(\ell)| \le 1 \le M$ *and* $|K(\ell,\Im)| = |\cos(\ell+\Im)| \le 1 \le M$ *for all* $(\ell,\Im) \in \mathbb{R}$. *Let* $\mho, v \in L^1[0,2\pi]$. *Then we have*

$$\begin{aligned}
|T\mho - Tv|L^1[0,2\pi] &= \int 0^{2\pi} \left| f(\ell) + \lambda\int_0^{2\pi} K(\ell,\Im)\mho(\Im)d\Im - f(\ell) - \lambda\int_0^{2\pi} K(\ell,\Im)v(\Im)d\Im \right| d\ell \\
&= \lambda\int_0^{2\pi}\left|\int_0^{2\pi} K(\ell,\Im)(\mho(\Im) - v())d\Im\right| d\ell \\
&\le \lambda\int_0^{2\pi}\int_0^{2\pi} |K(\ell,\Im)||\mho(\Im) - v(\Im)|d\Im d\ell \\
&= \lambda\int_0^{2\pi}\int_0^{2\pi} |\cos(\ell+\Im)||\mho(\Im) - v(\Im)|d\Im d\ell \\
&= \lambda\int_0^{2\pi}\int_0^{2\pi} |\cos(\ell+\Im)||\mho(\Im) - v(\Im)|d\ell d\Im \\
&= \lambda\int_0^{2\pi} |u(\Im) - v(\Im)|\int_0^{2\pi} |\cos(\ell+\Im)|d\ell d\Im \\
&= 2\pi\lambda|\mho - v|_{L^1[0,2\pi]}.
\end{aligned}$$

So T is a CM on $L^1[0,2\pi]$. *Therefore, the equation* (5) *has a unique solution* $\mho(\ell) \in L^1[0,2\pi]$.

Let's find the exact solution and an approximate solution through a concrete example. Here, we can solve the integral equation analytically to obtain the exact solution. Substituting $\mho(\ell)$

into the equation (5)*, we have*

$$\mho(\ell) = \sin(\ell) + \lambda \int_0^{2\pi} \cos(\ell + \Im)\mho(\Im) d\Im$$

Differentiating both sides with respect to ℓ*, we get:*

$$\mho'(\ell) = \cos(\ell) + \lambda \int_0^{2\pi} \cos(\ell + \Im)\mho(\Im) d\Im.$$

Taking the derivative again, we have

$$\mho''(\ell) = -\sin(\ell) - \lambda \int_0^{2\pi} \sin(\ell + \Im)\mho(\Im) d\Im$$

Now, substituting the expression for $\mho(\ell)$ *from the original equation, we obtain*

$$\mho''(\ell) = -\sin(\ell) - \lambda \int_0^{2\pi} \sin(\ell + \Im) \left(\sin(\Im) + \lambda \int_0^{2\pi} \cos(\Im + z)\mho(z) dz \right) d\Im.$$

Expanding the integral and simplifying, we have

$$\mho''(\ell) = -\sin(\ell) - \lambda \sin(\ell) \int_0^{2\pi} \sin(\Im) d\Im - \lambda^2 \int_0^{2\pi} \int_0^{2\pi} \sin(\ell + \Im) \cos(\Im + z)\mho(z) d\Im dz.$$

Evaluating the integral, we get

$$\mho''(\ell) = -\sin(\ell) + 0 - \lambda^2 \int_0^{2\pi} \int_0^{2\pi} \sin(\ell + \Im) \cos(\Im + z)\mho(z) d\Im dz$$

Simplifying further, we obtain:

$$\mho''(\ell) = -\sin(\ell) - \lambda^2 \int_0^{2\pi} \sin(\ell + \Im) \left(\frac{\sin(\Im + z)}{2} \right) \mho(z) d\Im dz.$$

This equation represents a linear nonhomogeneous ordinary differential equation for $\mho''(\ell)$ *with the source term involving the product of* $\mho(z)$ *and the sine function. Solving this equation subject to appropriate initial/boundary conditions will give us the exact solution* $\mho(\ell)$*.*

Otherwise, to find an approximate solution, we can use numerical methods such as the fixed-point iteration method or the Newton's method. These methods involve iteratively updating an initial guess for $\mho(\ell)$ *until convergence is achieved. Let's consider the fixed-point iteration method. We start with an initial guess* $\mho_0(\ell)$ *and iterate using the formula:*

$$\mho_{n+1}(\ell) = \sin(\ell) + \lambda \int_0^{2\pi} \cos(\ell + \Im)\mho_n(\Im) d\Im,$$

where n is the iteration index. We repeat this iteration process until the solution converges to a desired level of accuracy. To illustrate the iterative process and track the convergence of the *fixed-point iteration method, we can use tables to display the values of the approximations at each iteration. Let's assume we have chosen an initial guess* $\mho_0(\ell) = \sin(\ell)$ *and set* $\lambda = 1$*. We will perform 5 iterations of the fixed-point iteration method as the following table.*

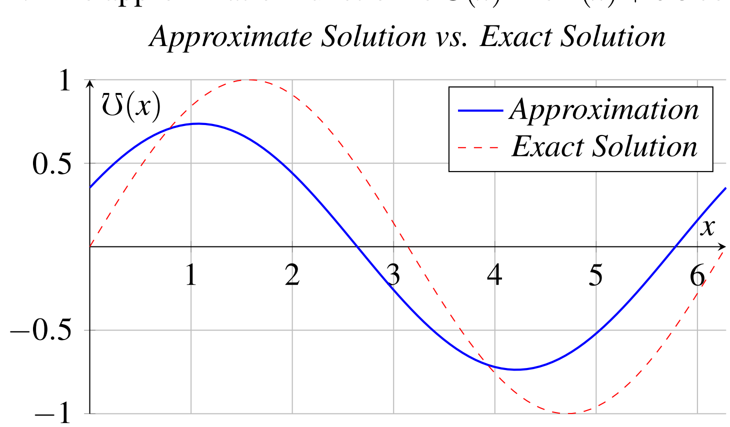
TABLE 1. The values of the approximations at each iteration with an initial guess $\mho_0(\ell) = \sin(\ell)$ and set $\lambda = 1$

Iteration n	*Approximation* $\mho_n(\ell)$
0	$\sin(\ell)$
1	$\sin(\ell) + \int_0^{2\pi} \cos(\ell + \Im)\mho_0(\Im)d\Im$
2	$\sin(\ell) + \int_0^{2\pi} \cos(\ell + \Im)\mho_1(\Im)d\Im$
3	$\sin(\ell) + \int_0^{2\pi} \cos(\ell + \Im)\mho_2(\Im)d\Im$
4	$\sin(\ell) + \int_0^{2\pi} \cos(\ell + \Im)\mho_3(\Im)d\Im$
5	$\sin(\ell) + \int_0^{2\pi} \cos(\ell + \Im)\mho_4(\Im)d\Im$

In each iteration, we compute the updated approximation $\mho_{n+1}(\ell)$ *by evaluating the integral involving the previous approximation* $\mho_n(\ell)$*.*

By examining the values of $\mho_n(\ell)$ *as we progress through the iterations, we can observe the convergence behavior of the method. If the values of* $\mho_n(\ell)$ *become stable and do not change significantly between iterations, we can infer that the iterative process is converging towards the solution of the integral equation as the following graph.*

FIGURE 1. The approximation function is $\mho(x) = \sin(x) + 0.5\cos\left(x + \frac{\pi}{4}\right)$



5. Conclusion

To conclude, the new remarks and fixed points results in an OMS have proven to be a valuable tool in establishing the existence and uniqueness of fixed points of mappings in a variety of applications, including NIEs and NFDEs. The notions of weak and strong contractions, as well as generalized contractions, have played a pivotal role in these results. Moreover, the applications of these results highlight their significance and practicality. The techniques employed in establishing these results, such as the Banach fixed point theorem, the Schauder fixed point theorem, and the Nadler's fixed point theorem, provide a comprehensive framework for further research and exploration of fixed point theory in more general settings. The results have significant theoretical and practical implications, and their applications in various fields continue to be explored. Future research in this area will likely expand upon these findings and provide new insights into fixed point theory, with the potential to lead to further advancements in fields such as optimization, control theory, and mathematical modeling. Overall, the new remarks and fixed points results in OMSs contribute to the ongoing development of the field of fixed point theory and offer promising avenues for future research.

Acknowledgments

Author acknowledge the financial support from AlZaytoonah University of Jordan.

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] M.I. Abbas, Existence and Uniqueness Results for Fractional Differential Equations with Riemann-liouville Fractional Integral Boundary Conditions, Abstr. Appl. Anal. 2015 (2015), 290674. https://doi.org/10.1155/2015/290674.

[2] W. Al-Mashaleh, H. Qawaqneh, H. Al-Zoubi, Some Results on Traces of the Generalized Products and Sums of Positive Semidefinite Matrices, Int. J. Math. Comput. Sci. 17 (2022), 619–625.

[3] H. Alsamir, M. Selmi Noorani, W. Shatanawi, H. Aydi, H. Akhadkulov, H. Qawaqneh, K. Alanazi, Fixed Point Results in Metric-Like Spaces via σ-Simulation Functions., Eur. J. Pure Appl. Math. 12 (2019), 88–100. https://doi.org/10.29020/nybg.ejpam.v12i1.3331.

[4] H. Alsamir, H.A. Qawaqneh, G. Al-Musannef, R. Khalil, Common Fixed Point of Generalized Berinde Type Contraction and an Application, Eur. J. Pure Appl. Math. 17 (2024), 2492–2504. https://doi.org/10.29020/nybg.ejpam.v17i4.5388.

[5] H. Aydi, A.H. Ansari, B. Moeini, et al. Property Q on G-Metric Spaces via C-Class Functions, Int. J. Math. Comput. Sci. 14 (2019), 675–692.

[6] I.M. Batiha, S.A. Njadat, R.M. Batyha, et al. Design Fractional-order Pid Controllers for Single-joint Robot Arm Model, Int. J. Adv. Soft Comput. Appl. 14 (2022), 97–114. https://doi.org/10.15849/ijasca.220720.07.

[7] N.B. Güngör, Some Fixed Point Theorems on Orthogonal Metric Spaces Via Extensions of Orthogonal Contractions, Commun. Fac. Sci. Univ. Ank. Ser. A1Mathematics Stat. 71 (2022), 481–489. https://doi.org/10.31801/cfsuasmas.970219.

[8] Z. Cui, Solutions of Some Typical Nonlinear Differential Equations with Caputo-fabrizio Fractional Derivative, AIMS Math. 7 (2022), 14139–14153. https://doi.org/10.3934/math.2022779.

[9] M. Elbes, T. Kanan, M. Alia, M. Ziad, Covd-19 Detection Platform From X-ray Images Using Deep Learning, Int. J. Adv. Soft Comput. Appl. 14 (2022), 197–211. https://doi.org/10.15849/ijasca.220328.13.

[10] M.E. Gordji, H. Habibi, Fixed Point Theory in Generalized Orthogonal Metric Spaces, J. Linear Topol. Algebra, 6 (2017), 251–260.

[11] J. Jost, Compact Riemann Surfaces: An Introduction to Contemporary Mathematics, Springer, 2011.

[12] Y. Farhat, Amel Souhail, V. Vigneswaran, M. Krishnan, Special Sequences on Generalized Metric Spaces, Eur. J. Pure Appl. Math. 15 (2022) 1869–1886. https://doi.org/10.29020/nybg.ejpam.v15i4.4592.

[13] F. Lael, Common Fixed Point of Generalized Weakly Contractive Mappings on Orthogonal Modular Spaces with Applications, Int. J. Nonlinear Anal. Appl. 12 (2021), 1121–1140. https://doi.org/10.22075/ijnaa.2021.22454.2365.

[14] T. Kanan, M. Elbes, K.A. Maria, M. Alia, Exploring the Potential of IoT-Based Learning Environments in Education, Int. J. Adv. Soft Comput. Appl. 15 (2023), 166–178.

[15] C. Kanzow, T. Lechner, Globalized Inexact Proximal Newton-type Methods for Nonconvex Composite Functions, Comput. Optim. Appl. 78 (2020), 377–410. https://doi.org/10.1007/s10589-020-00243-6.

[16] M. Nazam, H. Aydi, M.S. Noorani, H. Qawaqneh, Existence of Fixed Points of Four Maps for a New Generalized F-Contraction and an Application, J. Funct. Spaces 2019 (2019), 5980312. https://doi.org/10.1155/2019/5980312.

[17] M.A. Nielsen, I. Chuang. Quantum Computation and Quantum Information: 10th Anniversary Edition, Cambridge University Press. 2010.

[18] H. Alsamir, H.A. Qawaqneh, G. Al-Musannef, R. Khalil, Common Fixed Point of Generalized Berinde Type Contraction and an Application, Eur. J. Pure Appl. Math. 17 (2024), 2492–2504. https://doi.org/10.29020/nybg.ejpam.v17i4.5388.

[19] O.T. Mewomo, B.T. Leyew, M. Abbas, Fixed Point Results for Generalized Multivalued Orthogonal α-F-contraction of Integral Type Mappings in Orthogonal Metric Spaces, Topol. Algebr. Appl. 10 (2022), 113–131. https://doi.org/10.1515/taa-2022-0118.

[20] H. Qawaqneh, M.S. Noorani, H. Aydi, W. Shatanawi, On Common Fixed Point Results for New Contractions with Applications to Graph and Integral Equations, Mathematics 7 (2019), 1082. https://doi.org/10.3390/math7111082.

[21] H. Qawaqneh, Fractional Analytic Solutions and Fixed Point Results With Some Applications, Adv. Fixed Point Theory 14 (2024), 1. https://doi.org/10.28919/afpt/8279.

[22] H. Qawaqneh, M.S.M. Noorani, H. Aydi, A. Zraiqat, A.H. Ansari, On Fixed Point Results in Partial b-metric Spaces, J. Funct. Spaces 2021 (2021), 8769190. https://doi.org/10.1155/2021/8769190.

[23] H. Qawaqneh, M.S.M. Noorani, H. Aydi, Some New Characterizations and Results for Fuzzy Contractions in Fuzzy b-Metric Spaces and Applications, AIMS Math. 8 (2023), 6682–6696. https://doi.org/10.3934/math.2023338.

[24] H. Qawaqneh, J. Manafian, M. Alharthi, Y. Alrashedi, Stability Analysis, Modulation Instability, and Beta-time Fractional Exact Soliton Solutions to the Van Der Waals Equation, Mathematics 12 (2024), 2257. https://doi.org/10.3390/math12142257.

[25] H. Qawaqneh, New Functions For Fixed Point Results In Metric Spaces With Some Applications, Indian J. Math. 66 (2020), 55–84.

[26] H. Qawaqneh, Functional Analysis with Fractional Operators: Properties and Applications in Metric Spaces, Adv. Fixed Point Theory 14 (2024), 62. https://doi.org/10.28919/afpt/8819.

[27] H. Qawaqneh, A. Altalbe, A. Bekir, K.U. Tariq, Investigation of Soliton Solutions to the Truncated M-fractional (3+1)-dimensional Gross-pitaevskii Equation with Periodic Potential, AIMS Math. 9 (2024), 23410–23433. https://doi.org/10.3934/math.20241138.

[28] H. Qawaqneh, Y. Alrashedi, Mathematical and Physical Analysis of Fractional Estevez–mansfield–clarkson Equation, Fractal Fract. 8 (2024), 467. https://doi.org/10.3390/fractalfract8080467.

[29] H. Qawaqneh, K.H. Hakami, A. Altalbe, M. Bayram, The Discovery of Truncated M-fractional Exact Solitons and a Qualitative Analysis of the Generalized Bretherton Model, Mathematics 12 (2024), 2772. https://doi.org/10.3390/math12172772.

[30] H. Qawaqneh, H.A. Hammad, H. Aydi, Exploring New Geometric Contraction Mappings and Their Applications in Fractional Metric Spaces, AIMS Math. 9 (2024), 521–541. https://doi.org/10.3934/math.2024028.

[31] Q. Yang, C. Bai, Fixed Point Theorem for Orthogonal Contraction of Hardy-Rogers-Type Mapping on O-Complete Metric Spaces, AIMS Math. 5 (2020), 5734–5742. https://doi.org/10.3934/math.2020368.

[32] G. Pisier, Introduction to Operator Space Theory, Cambridge University Press, 2018.

[33] M. Searcóid, Metric Spaces, Springer, 2006.

[34] T. Senapati, A. Chanda, V. Rakočević, Generalized Quasi-Contractions on Weak Orthogonal Metric Spaces, Fixed Point Theory 23 (2022), 371–384. https://doi.org/10.24193/fpt-ro.2022.1.23.

[35] F. Tonielli, R. Fazio, S. Diehl, J. Marino, Orthogonality Catastrophe in Dissipative Quantum Many-body Systems, Phys. Rev. Lett. 122 (2019), 040604. https://doi.org/10.1103/physrevlett.122.040604.