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NEW GENERALIZATION OF STRONG-COMPOSED METRIC TYPE SPACES WITH SPECIAL (ψ, ϕ)-CONTRACTION

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Abstract. The study aims to propose several generalizations of a strong b-metric space which is called strong-composed metric spaces. Therefore, to illustrate the concept, the study provides examples of a Strong-composed metric space, which are not a Strong-controlled metric type space, it is also not a Strong b-metric space. Finally, the investigation demonstrates the uniqueness of some fixed-point results involving some general structure contractions with applications in nonlinear integral and fractional differential equations.

Keywords: double-composed metric space; fixed point; strong b-metric space; strong controlled metric type space; strong composed metric space.

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1. Introduction

In recent years, there has been a surge in interest in the fixed-point theorem (FPT). Its modification depends on tools of triangular inequality of metric space via important contractions in extension of concept of the fixed point theorem with the application. In 1989, Bakhtin [1] and Czerwik [2] represented the b-metric space (bMS), which is a generalization to the metric space. Many previous works in this area deal with the important properties of bMS, see [3, 4],

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whereas others focus their attention on (SbMS) Kirk in [5]. Definitely, every SbM is a bMS anywhere the constant coefficient $\mathfrak{s} \geq 1$, but the reverse is not sufficiently true. Extended SbMS via some fixed point theorems in [6]. In 2023 Santina, D. et al., introduced a new generalization of SbMS called strong-controlled b-metric type space (CSbMS) through some fixed-point theorems with famous applications [7], inspired this extended from Mliaki, N. which obtained controlled metric space and double-controlled metric space, Ref. ([8,9]), go head to the SCbMS generalized to SbMS, that is; the controlled function as a constant. Many authors endowed various fixed-point results linked to bMS; see [10–13]. Despite all of these studies, there is much work concerning the application of special contractions to SbMS see [14, 15].
Hence, the paper establishes an extended concept of CSbMS called strong-composed metric space (SCMS). The triangle inequality is constituted as $\mathscr{S}_{\psi}(a,c) \leq \mathscr{S}_{\psi}(a,b) + \psi(\mathscr{S}_{\psi}(b,c))$ for all $a,b,c \in \mathfrak{S}$, and $\psi : [0,\infty) \to [0,\infty)$, the reverse is not necessarily true. Subsequently, CSbMS does not imply SbMS with non-trivial examples. Following that, the paper displays the concept in Hardy-Rogers type contraction with notice in terms of the particular types contractive, pass into Matkowski [16]. The main result shows a new general of (ψ,ϕ)-contraction for two maps. For more, see [17–21]. Finally, we focus on Fisher contractions on SCMS with a common fixed point, based on Ref. [10], last but not least, the research provides some corollaries and applications about the work through an example that has satisfied the current results.

2. PRELIMINARIES

The following explanation introduces some basic concepts of SbMS, which are due to Kirk, W. [5].

Definition 2.1. ([5]) Let $\mathfrak{S}$ be a nonempty set, and $\mathfrak{s} \geq 1$. The mapping $d_{\mathfrak{s}} : \mathfrak{S} \times \mathfrak{S} \to [0,\infty)$ is said to be a strong b-metric on $\mathfrak{S}$ if for all $a,b,c \in \mathfrak{S}$ the following conditions hold:

(S1) $d_{\mathfrak{s}}(a,b) = 0$ if and only if $a = b$,

(S2) $d_{\mathfrak{s}}(a,b) = d_{\mathfrak{s}}(b,a)$,

(S3) $d_{\mathfrak{s}}(a,b) \leq d_{\mathfrak{s}}(a,c) + \mathfrak{s} d_{\mathfrak{s}}(c,b)$.

The pair $(\mathfrak{S}, d_{\mathfrak{s}})$ is called an SbMS.

In the following, Santina, D. et al. give an extended concept of SbMS, which is called CSbMS [7].

Definition 2.2. ([7]) Suppose $\mathfrak{S}$ is a nonempty set and, $\eta : \mathfrak{S}\times\mathfrak{S} \to [1,\infty)$. A mapping $\Delta_\eta : \mathfrak{S}\times\mathfrak{S} \to [0,\infty)$ is a controlled-strong b-metric if for each $a,b,c \in \mathfrak{S}$, the following conditions hold:

(C1) $\Delta_\eta(a,b) = 0$ if and only if $a = b$,

(C2) $\Delta_\eta(a,b) = \Delta_\eta(b,a)$,

(C3) $\Delta_\eta(a,c) \leq \Delta_\eta(a,b) + \eta(b,c)\Delta_\eta(b,c)$.

Then the triple $(\mathfrak{S},\Delta_\eta,\eta)$ is called a CSbMS.

Clearly, every SbMS is a CSbMS, just take $\eta(b,c) = \mathfrak{s}$; however, the reverse is not necessarily true (see, e.g. [7]). We next establish a new notion that is a generalization of CSbMS and is referred to as SCMS.

Definition 2.3. Suppose $\mathfrak{S}$ is nonempty. A mapping $\mathscr{S}_\psi : \mathfrak{S}\times\mathfrak{S} \to [0,\infty)$ is a strong-composed metric if there is a $\psi : [0,\infty) \to [0,\infty)$, such that, for all $a,b,c \in \mathfrak{S}$, the following conditions hold:

(SC1) $\mathscr{S}_\psi(a,b) \geq 0$ and $\mathscr{S}_\psi(a,b) = 0$ if and only if $a = b$,

(SC2) $\mathscr{S}_\psi(a,b) = \mathscr{S}_\psi(b,a)$,

(SC3) $\mathscr{S}_\psi(a,c) \leq \mathscr{S}_\psi(a,b) + \psi\left(\mathscr{S}_\psi(b,c)\right)$.

Then the triple $(\mathfrak{S},\mathscr{S}_\psi,\psi)$ is called an SCMS.

Obviously, every CSbMS is an SCMS, wherever $\psi(t) = \eta(b,c)t, t \geq 0$, but the converse is not true, in general. An example of a SCMS that is not a CSbMS is provided below to highlight the observation:

Example 2.4. Let $(\mathfrak{S},d_\mathfrak{s})$ be an SbMS via $\mathfrak{s} \geq 1$ and let $\mathscr{S}_\psi(a,b) = \sinh^{-1}(d_\mathfrak{s}(a,b))$. We show that $\mathscr{S}_\psi$ is an SCMS via $\psi(t) = \sinh^{-1}(\mathfrak{s}\sinh(t))$, for all $t \geq 0$. Obviously, conditions (SC1) and (SC2) of Definition 2.3 are satisfied. Since $\sinh^{-1}(t)$ is an increasing function, hence for all $a_1,a_2 \geq 0$, we undergo,

$$\sinh^{-1}(a_1+a_2) \leq \sinh^{-1}(a_1) + \sinh^{-1}(a_2). \tag{2.1}$$

$$\begin{aligned}\mathscr{S}_{\psi}(a,b)=\sinh^{-1}(d_{\mathfrak{s}}(a,b)) &\leq \sinh^{-1}(d_{\mathfrak{s}}(a,c)+\mathfrak{s}d_{s}(c,b))\\ &=\sinh^{-1}(d_{\mathfrak{s}}(a,c)+\mathfrak{s}\sinh(\sinh^{-1}(d_{\mathfrak{s}}(c,b))))\\ &\leq \sinh^{-1}(d_{\mathfrak{s}}(a,c))+\sinh^{-1}(\mathfrak{s}\sinh(\sinh^{-1}(d_{\mathfrak{s}}(c,b))))\\ &=\mathscr{S}_{\psi}(a,c)+\psi(\mathscr{S}_{\psi}(c,b)).\end{aligned}$$

Thus, $(\mathfrak{S},\mathscr{S}_{\psi})$ is an SCMS.

Notice that if assumed $(\mathfrak{S},d)$ is a metric space, then $\mathscr{S}_{\psi}(a,b)=\sinh^{-1}(d(a,b))$ is an SCMS via $\psi(t)=\sinh(t)$ for all $t\geq 0$.

Remark 2.5. Consider that a mapping $\Delta_{\eta}:\mathfrak{S}\times\mathfrak{S}\to[0,\infty)$ with a nonempty set $\mathfrak{S}$ is SbMS or CSbMS, which implies that SCMS of the mapping $\mathscr{S}_{\psi}$, defined by $\mathscr{S}_{\psi}(a,b)=\psi^{-1}(\Delta_{\eta}(a,b))$ with respect to $\hat{\psi}(t)=\psi^{-1}(\eta(b,c)\psi(t))$, wherever $\hat{\psi},\psi:[0,\infty)\to[0,\infty)$ and $\eta:\mathfrak{S}\times\mathfrak{S}\to[1,\infty)$. In (SC3), that satisfies $\mathscr{S}_{\psi}(a,b)\leq\mathscr{S}_{\psi}(a,c)+\hat{\psi}(\mathscr{S}_{\psi}(c,b))$.

In the following example, we notice another formulas, also we illustrate that every metric space is an SCMS.

Example 2.6. Let $\mathfrak{S}$ be a nonempty set, and define $\mathscr{S}_{\psi}:\mathfrak{S}\times\mathfrak{S}\to[0,\infty)$ by $\mathscr{S}_{\psi}(a,b)=|a-b|$. Then $(\mathfrak{S},\mathscr{S}_{\psi})$ is an (SCMS) with $\psi(t)=e^{t}-1$. That is, enough to prove the inequality (SC3). Hence $t\leq e^{t}-1$ for all $t\in\mathbb{R}$, also $|a-b|\leq|a|+|b|$.
Thus, $\mathscr{S}_{\psi}(a,b)\leq\mathscr{S}_{\psi}(a,c)+\psi(\mathscr{S}_{\psi}(c,b))$. Therefore, $(\mathfrak{S},\mathscr{S}_{\psi})$ is an (SCMS).

We then go over some topological characteristics of SCMS.

Definition 2.7. Let $(\mathfrak{S},\mathscr{S}_{\psi})$ be an SCMS. A sequence $\{a_n\}$ in $\mathfrak{S}$ is said to be:

(1) Cauchy if, for any $\varepsilon>0$, there exists a positive integer $\aleph$ such that for all $m,n>\aleph,\mathscr{S}_{\psi}(a_n,a_m)<\varepsilon$.
(2) Convergence to point $a_0\in\mathfrak{S}$, for any $\varepsilon>0$, there exists a positive integer $\aleph$ such that for all $n>\aleph,\mathscr{S}_{\psi}(a_n,a_0)<\varepsilon$.

An SCMS is called complete if every Cauchy sequence converges in $\mathscr{S}$.

Definition 2.8. Consider $(\mathfrak{S},\mathscr{S}_{\psi})$ be an (SCMS). Take $a_0\in\mathfrak{S}$ through $\varepsilon>0$.

(1) The set $\mathscr{B}(a_0,\varepsilon)=\{b\in\mathfrak{S}:\mathscr{S}_\psi(a_0,b)<\varepsilon\}$ is called an open ball with center a_0 and radius ε.

(2) The mapping $T:\mathfrak{S}\to\mathfrak{S}$ is called continuous at $a_0\in\mathfrak{S}$ if $\forall\varepsilon>0,\exists\varsigma>0$, satisfying $T(\mathscr{B}(a_0,\varsigma))\subseteq\mathscr{B}^{'}(Ta_0,\varepsilon)$.

Obviously, if T is continuous at a_0 in the SCMS of $(\mathfrak{S},\mathscr{S}_\psi)$, then for any $\{a_n\}\to a_0$ it yields $\{Ta_n\}\to Ta_0$, as $n\to\infty$.

Remark 2.9. Consider $(\mathfrak{S},\mathscr{S}_\psi)$ as an SCMS. If the sequence $\{a_n\}$ in $\mathfrak{S}$ converges to a_0. Then a_0 is unique, consequently the Cauchy sequence in $\mathfrak{S}$.

Let Ψ be the family of all mappings $\psi:[0,\infty)\to[0,\infty)$ satisfying the conditions; $t\le\psi(t)$ for each $t\in[0,\infty)$, and $\psi^{'}$ (derivative of ψ) increases in [11].

Lemma 2.10. *Let $(\mathfrak{S},\mathscr{S}_\psi)$ be an SCMS. If $\psi\in\Psi$, then for all $a,b\in[0,\infty)$, we get:*

$$|\psi^{-1}(a)-\psi^{-1}(b)|\le\psi^{-1}(|a-b|)\le|a-b|\le\psi(|a-b|)\le|\psi(a)-\psi(b)|.$$

In particular, if $b=0$, that is, $|\psi^{-1}(a)|\le\psi^{-1}(|a|)\le|a|\le\psi(|a|)\le|\psi(a)|$.

3. The Main Results

In this section, we prove some FPT by aiding various contraction mappings such as Hardy-Rogers-type contraction, Matkowski contraction, the main result of special (ψ,ϕ)-contraction, and their related consequences on SCMS. At this point, we are ready to look into the primary outcome related to the BCP with generalizations of Hardy-Rogers type contraction, as follows.

Theorem 3.1. *([18]) Presume $(\mathfrak{S},D_C)$ be a complete double-composed metric space regarding to ψ_1,ψ_2 and $D_C:\mathfrak{S}\times\mathfrak{S}\to\mathbb{R}^+$. Let $T:\mathfrak{S}\to\mathfrak{S}$ be a mapping satisfying,*

$$D_C(Ta,Tb)\le K_1D_C(a,b)+K_2D_C(a,Ta)+K_3D_C(b,Tb)+K_4D_C(a,Tb)+K_5D_C(b,Ta),$$

$\forall a,b\in\mathscr{S}$, where $K_i\in[0,1),i=1,2,\dots,5$, and $\sum_{i=1}^{5}K_i<1$. For any $a_0\in\mathscr{S}$, choose $a_n=T^na_0$. Suppose that,

(1) Let ψ_1,ψ_2 be continuous, non-decreasing and ψ_2 is a sub-additive and comparison function, and ψ_1 is an in-comparison function.

(2) $\lim_{n,m\to\infty}\sum_{i=m}^{n-2}\psi_2^{i-m}\psi_1\left(R^i\psi_1^i\left(D_C(a_0,a_1)\right)\right)+\psi_2^{n-m-1}\left(R^{n-1}\psi_1^{n-1}\left(D_C(a_0,a_1)\right)\right)\to 0$ *(as $n,m\to\infty$), where* $R=\frac{K_1+K_2+K_4}{1-K_3-K_4}$.

Then T has a unique fixed point.

The subsequent findings provide the SCMS of Hardy-Rogers type contraction of fixed-point theorems.

Corollary 3.2. *Let $(\mathfrak{S},\mathscr{S}_\psi)$ be a complete SCMS regarding to ψ and $\mathscr{S}_\psi:\mathfrak{S}\times\mathfrak{S}\to\mathbb{R}^+$. Let $T:\mathfrak{S}\to\mathfrak{S}$ be a mapping satisfying,*

$$\mathscr{S}_\psi(Ta,Tb)\le K_1\mathscr{S}_\psi(a,b)+K_2\mathscr{S}_\psi(a,Ta)+K_3\mathscr{S}_\psi(b,Tb)+K_4\mathscr{S}_\psi(a,Tb)+K_5\mathscr{S}_\psi(b,Ta),$$

$\forall a,b\in\mathscr{S}$, where $K_i\in[0,1), i=1,2,\dots,5$, and $\sum_{i=1}^5 K_i<1$. For any $a_0\in\mathscr{S}$, choose $a_n=T^na_0$. Suppose that,

$$\lim_{n,m\to\infty}\sum_{i=m}^{n-2}\psi\left(R^i\psi^i\left(\mathscr{S}_\psi(a_0,a_1)\right)\right)+R^{n-1}\psi^{n-1}\left(\mathscr{S}_\psi(a_0,a_1)\right)\to 0,$$

(as $n,m\to\infty$), where $R=\frac{K_1+K_2+K_4}{1-K_3-K_4}$. Then T has a unique fixed point.

Proof. In Theorem 3.1 just take $\psi_1(t)=\psi(t)$ and $\psi_2(t)=t$, since $\psi\in\Psi$ is continuous and non-decreasing map, and $t\le\psi(t)$ for all t, we get a completeness SCMS. □

Remark 3.3. Consider $(\mathfrak{S},\mathscr{S}_\psi)$ to complete SCMS and $T:\mathfrak{S}\to\mathfrak{S}$ be a mapping, for any $a,b\in\mathfrak{S}$. Then, we get the fixed point theory of the following contraction is obvious that a particular Hardy-Rogers type contraction is as follows:

(1) Banach Type: $\mathscr{S}_\psi(Ta,Tb)\le K_1\mathscr{S}_\psi(a,b)$, wherever $K_1\in(0,1)$.

(2) Kannan Type: $\mathscr{S}_\psi(Ta,Tb)\le K_2\mathscr{S}_\psi(a,Ta)+K_3\mathscr{S}_\psi(b,Tb)$, wherever $K_2+K_3<1$ and $K_2,K_3\in[0,1)$.

(3) Chatterjee Type: $\mathscr{S}_\psi(Ta,Tb)\le K_4\mathscr{S}_\psi(a,Tb)+K_5\mathscr{S}_\psi(b,Ta)$, wherever $K_4+K_5<1$ and $K_4,K_5\in[0,1)$.

(4) Reich Type: $\mathscr{S}_\psi(Ta,Tb)\le K_1\mathscr{S}_\psi(a,b)+K_2\mathscr{S}_\psi(a,Ta)+K_3\mathscr{S}_\psi(b,Tb)$, wherever $K_1+K_2+K_3<1$ and $K_1,K_2,K_3\in[0,1)$.

Let Φ denote the class of all functions $\phi:[0,\infty)\to[0,\infty)$ such that ϕ is non-decreasing, continuous and $\sum_{i=1}^\infty\phi^i(t)<+\infty$, for all $t>0$. It is clear that for any $t>0$, $\phi^i(t)\to 0$ as $i\to\infty$, and hence $\phi(t) < t$, for all $t > 0$. In order to reach a fixed point in the nonlinear contraction, we introduce a control function specified by Matkowski [16] in the following theorem.

Theorem 3.4. *Assume $(\mathfrak{S}, \mathscr{S}_\psi)$ is a complete SCMS via the function ψ. Suppose that $T : \mathfrak{S} \to \mathfrak{S}$ be a mapping $\forall a, b \in \mathfrak{S}$,*

$$\mathscr{S}_\psi(Ta, Tb) \le \phi(\rho(a,b)), \quad \rho(a,b) = Max\{\mathscr{S}_\psi(a,b), \mathscr{S}_\psi(a,Ta), \mathscr{S}_\psi(b,Tb)\}, \tag{3.1}$$

where $\phi \in \Phi$. For any $a_0 \in \mathfrak{S}$, we obtain

$$\lim_{n,m\to\infty} \sum_{i=m}^{n-1} \psi^{i-m}\left(\phi^i\left(\mathscr{S}_\psi(a_0,a_1)\right)\right) \to 0, \tag{3.2}$$

and $a_n = T^n a_0, \forall n \ge 0$. If the mapping T is continuous, then there is a fixed point unique to T (say a^*). That is, $T^n a \to a^*, \forall a \in \mathfrak{S}$.

Proof. In the sequence $\{a_n\}$ and a_0 to be the same as in the hypothesis of Theorem 3.4. If $a_{m+1} = Ta_m$ for any arbitrary m. So, assume that $a_{n+1} \ne a_n, \forall n$. Utilizing the condition 3.1,

$$\mathscr{S}_\psi(a_n, a_{n+1}) = \mathscr{S}_\psi(Ta_{n-1}, Ta_n) \le \phi(\rho(a_{n-1}, a_n)), \tag{3.3}$$

where $\rho(a_{n-1},a_n) = Max\{\mathscr{S}_\psi(a_{n-1},a_n), \mathscr{S}_\psi(a_n,a_{n+1})\}$. If for any arbitrary n, we show that $\rho(a_{n-1},a_n) = \mathscr{S}_\psi(a_n,a_{n+1})$, then using 3.3, since $\phi(t) < t$, for all $t > 0$, we undergo

$$0 < \mathscr{S}_\psi(a_n,a_{n+1}) < \phi(\mathscr{S}_\psi(a_n,a_{n+1})) < \mathscr{S}_\psi(a_n,a_{n+1}),$$

Clearly implies that is a contradiction. Also, for every n it should be expressed as $\rho(a_{n-1},a_n) = \mathscr{S}_\psi(a_{n-1},a_n)$. By conclusion, it means that $0 < \mathscr{S}_\psi(a_n,a_{n+1}) \le \phi(\mathscr{S}_\psi(a_{n-1},a_n))$. If we repeat process, we conclude that for each $n \in \mathbb{R}^+$, we get

$$0 < \mathscr{S}_\psi(a_n,a_{n+1}) \le \phi^n(\mathscr{S}_\psi(a_0,a_1)) \tag{3.4}$$

So, as $\lim\limits_{n\to\infty} \mathscr{S}_\psi(a_n,a_{n+1}) = 0$. For $m < n$ where n,m are two integers, we obtain

$$\begin{aligned}
\mathscr{S}_\psi(a_m,a_n) &\le \mathscr{S}_\psi(a_m,a_{m+1}) + \psi(\mathscr{S}_\psi(a_{m+1},a_n)) \\
&\le \mathscr{S}_\psi(a_m,a_{m+1}) + \psi\big(\mathscr{S}_\psi(a_{m+1},a_{m+2}) + \psi(\mathscr{S}_\psi(a_{m+2},a_n))\big) \\
&\ \ \vdots
\end{aligned}$$

$$\leq \sum_{i=m}^{n-1} \psi^{i-m}\left(\mathscr{S}_{\psi}(a_i,a_{i+1})\right) \tag{3.5}$$

By utilizing 3.4 in 3.5, yields that $\mathscr{S}_{\psi}(a_m,a_n) \leq \sum_{i=m}^{n-1} \psi^{i-m}\left(\phi^i(\mathscr{S}_{\psi}(a_0,a_1))\right)$, again applying it to the condition 3.2, it holds that $\{a_n\}$ is Cauchy. In completeness $(\mathfrak{S},\mathscr{S}_{\psi})$, hence, there is $a^* \in \mathfrak{S}$ satisfies $\lim_{n\to\infty} \mathscr{S}_{\psi}(a_n,a^*) = 0$. Thus, but T is continuous, we get $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} Ta_n = T \lim_{n,m\to\infty} a_n = Ta^*$ and a^* is a fixed point of T. Further, Let b satisfy the $Tb = b$, and $a^* \neq b$. By 3.1, we reach

$$0 < \mathscr{S}_{\psi}(a^*,b) = \mathscr{S}_{\psi}(Ta^*,Tb) \leq \phi(\rho(a^*,b)) = \phi(\mathscr{S}_{\psi}(a^*,b)) < \mathscr{S}_{\psi}(a^*,b),$$

Obviously it implies that is a contradiction. □

Next, we introduce the extended special concepts of (ψ,ϕ)-contraction as follows.

Theorem 3.5. *Let $(\mathfrak{S}_1,\mathscr{S}_{\psi})$ and $(\mathfrak{S}_2,\mathscr{T}_{\psi})$ be two complete SCMS. Let $\mathscr{P}_1 : \mathfrak{S}_1 \to \mathfrak{S}_2$ and $\mathscr{P}_2 : \mathfrak{S}_2 \to \mathfrak{S}_1$ satisfying the inequalities:*

$$\psi\left(\mathscr{S}_{\psi}(\mathscr{P}_2\mathscr{P}_1 a, \mathscr{P}_2\mathscr{P}_1 b)\right) \leq \phi\left(Max\{\mathscr{S}_{\psi}(a,b), \psi^{-1}(\mathscr{T}_{\psi}(\mathscr{P}_1 a, \mathscr{P}_1 b))\}\right) \tag{3.6}$$

$$\psi\left(\mathscr{T}_{\psi}(\mathscr{P}_1\mathscr{P}_2 a, \mathscr{P}_1\mathscr{P}_2 b)\right) \leq \phi\left(Max\{\psi^{-1}(\mathscr{T}_{\psi}(c,d)), \mathscr{S}_{\psi}(\mathscr{P}_2 c, \mathscr{P}_2 d)\}\right), \tag{3.7}$$

for all $a,b \in \mathfrak{S}_1$ and $c,d \in \mathfrak{S}_2$, where $\phi \in \Phi$ and $\psi \in \Psi$. Let for each $m \geq n$,

$$\psi^{m-n-1} \sum_{i=n}^{\infty} \psi^{-(m-1)}\left(\phi^i(t)\right) \to 0 \quad as\ n \to \infty.$$

One of the mappings $\mathscr{P}_1$ and $\mathscr{P}_2$ is continuous, then $\mathscr{P}_2\mathscr{P}_1$ has a unique fixed point a^* in $\mathfrak{S}_1$ and $\mathscr{P}_1\mathscr{P}_2$ has a unique fixed point c^* in $\mathfrak{S}_2$. In addition, $\mathscr{P}_1 a^* = c^*$ and $\mathscr{P}_2 c^* = a^*$.

Proof. Let a_0 be an arbitrary point in $\mathfrak{S}_1$ and $\mathscr{P}_1 a_0 = c_0, \mathscr{P}_2 c_0 = a_1, \mathscr{P}_1 a_1 = c_1, \mathscr{P}_2 c_1 = a_2$ and in general let $\mathscr{P}_1 a_n = c_n, \mathscr{P}_2 c_n = a_{n+1}, n \geq 0$.

Denote $\pi_n = \mathscr{S}_{\psi}(a_n,a_{n+1})$ and $\hat{\pi}_n = \mathscr{T}_{\psi}(c_n,c_{n+1})$. Then in general we have,

$$\begin{aligned}
\pi_n = \mathscr{S}_{\psi}(a_n,a_{n+1}) &= \mathscr{S}_{\psi}(\mathscr{P}_2\mathscr{P}_1 a_{n-1}, \mathscr{P}_2\mathscr{P}_1 a_n) \\
&\leq \psi^{-1} \circ \phi\left(Max\{\mathscr{S}_{\psi}(a_{n-1},a_n), \psi^{-1}\left(\mathscr{T}_{\psi}(\mathscr{P}_1 a_{n-1}, \mathscr{P}_1 a_n)\right)\}\right) \\
&= \psi^{-1} \circ \phi\left(Max\{\mathscr{S}_{\psi}(a_{n-1},a_n), \psi^{-1}\left(\mathscr{T}_{\psi}(c_{n-1},c_n)\right)\}\right) \\
&= \psi^{-1} \circ \phi\left(Max\{\pi_{n-1}, \psi^{-1}\left(\hat{\pi}_{n-1}\right)\}\right).
\end{aligned}$$

Hence, we have $\pi_n = \psi^{-1}\circ\phi\big(Max\{\pi_{n-1},\psi^{-1}\big(\hat{\pi}_{n-1}\big)\}\big), \forall n\geq 1$.

By same way, utilizing the inequality 3.7, we undergo

$$\begin{aligned}
\hat{\pi}_n &= \mathscr{T}_\psi(c_n,c_{n+1}) = \mathscr{T}_\psi(\mathscr{P}_1\mathscr{P}_2c_{n-1},\mathscr{P}_1\mathscr{P}_2c_n)\\
&\leq \psi^{-1}\circ\phi\big(Max\{\psi^{-1}\big(\mathscr{T}_\psi(c_{n-1},c_n)\big),\mathscr{S}_\psi(\mathscr{P}_2c_{n-1},\mathscr{P}_2c_n)\}\big)\\
&= \psi^{-1}\circ\phi\big(Max\{\psi^{-1}\big(\mathscr{T}_\psi(c_{n-1},c_n)\big),\mathscr{S}_\psi(a_n,a_{n+1})\}\big)\\
&= \psi^{-1}\circ\phi\big(Max\{\psi^{-1}(\hat{\pi}_{n-1}),\pi_n\}\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi^{-1}(\hat{\pi}_n) \leq \hat{\pi}_n &\leq \psi^{-1}\circ\phi\big(Max\{\psi^{-1}(\hat{\pi}_{n-1}),\pi_n\}\big)\\
&\leq \psi^{-1}\circ\phi\big(Max\{\psi^{-1}(\hat{\pi}_{n-1}),\psi^{-1}\circ\phi\big(Max\{\pi_{n-1},\psi^{-1}(\hat{\pi}_{n-1})\}\big)\}\big)\\
&\leq \psi^{-1}\circ\phi\big(Max\{\psi^{-1}(\hat{\pi}_{n-1}),Max\{\pi_{n-1},\psi^{-1}(\hat{\pi}_{n-1})\}\}\big)\\
&= \psi^{-1}\circ\phi\big(Max\{\psi^{-1}(\hat{\pi}_{n-1}),\pi_{n-1}\}\big), \forall n\geq 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
t_{n+1} := Max\{\pi_{n+1},\psi^{-1}(\hat{\pi}_{n+1})\} &\leq \psi^{-1}\circ\phi\big(Max\{\pi_n,\pi^{-1}(\hat{\pi}_n)\}\big) = \psi^{-1}(\phi(t_n))\\
&\leq \psi^{-1}(\phi(\psi^{-1}(\phi(t_{n-1})))) = \psi^{-2}(\phi^2(t_{n-1})) \leq \cdots \leq \psi^{-(n+1)}(\phi^{n+1}(t_0)),
\end{aligned}$$

hence from the previous inequality we get

$$\pi_{n+1} = \mathscr{S}_\psi(a_{n+1},a_{n+2}) \leq t_{n+1} \leq \psi^{-(n+1)}(\phi^{n+1}(t_0)).$$

Therefore, for any $t>0$ and $m\geq n$ we get,

$$\begin{aligned}
&\psi^{-(m-n-1)}\mathscr{S}_\psi(a_n,a_m) \leq \psi^{-(m-n-1)}\big[\mathscr{S}_\psi(a_n,a_{n+1})+\psi(\mathscr{S}_\psi(a_{n+1},a_m))\big]\\
&\leq \psi^{-(m-n-1)}(\mathscr{S}_\psi(a_n,a_{n+1}))+\psi^{-(m-n-1)+1}\big[\mathscr{S}_\psi(a_{n+1},a_{n+2})+\psi(\mathscr{S}_\psi(a_{n+2},a_m))\big]\\
&\leq \psi^{-(m-n-1)}(\mathscr{S}_\psi(a_n,a_{n+1}))+\psi^{-(m-n-1)+1}(\mathscr{S}_\psi(a_{n+1},a_{n+2}))+\psi^{-(m-n-1)+2}(\mathscr{S}_\psi(a_{n+2},a_m))\\
&\quad\vdots\\
&\leq \psi^{-m+n+1}(\mathscr{S}_\psi(a_n,a_{n+1}))+\psi^{-m+n+2}(\mathscr{S}_\psi(a_{n+1},a_{n+2}))+\cdots+\psi^{-1}(\mathscr{S}_\psi(a_{m-2},a_{m-1}))\\
&\quad+\mathscr{S}_\psi(a_{m-1},a_m)
\end{aligned}$$

$$\leq \psi^{-m+n+1}(\psi^{-n}(\phi^{n}(t_0))) + \psi^{-m+n+2}(\psi^{-(n+1)}(\phi^{n+1}(t_0))) + \cdots + \psi^{n}(\psi^{-(m-1)}(\phi^{m-1}(t_0)))$$
$$= \psi^{-(m-1)}(\phi^{n}(t_0)) + \psi^{-(m-1)}(\phi^{n+1}(t_0)) + \cdots + \psi^{-(m-1)}(\phi^{m-1}(t_0))$$
$$\leq \sum_{i=n}^{\infty} \psi^{-(m-1)}\left(\phi^{i}(t_0)\right) \to 0, \quad \text{as}\, n \to \infty.$$

So as, $\psi^{m-n-1}\sum_{i=n}^{\infty} \psi^{-(m-1)}\left(\phi^{i}(t_0)\right) \to 0$ as $n \to \infty$.
Thus, $\{a_n\}$ is a Cauchy in $(\mathfrak{S}_1, \mathscr{S}_\psi)$. By completeness space $(\mathfrak{S}_1, \mathscr{S}_\psi)$, so there exists $a^* \in \mathfrak{S}_1$ such that $a_n \to a^*$, as $n \to \infty$, that is; $\lim\limits_{n\to\infty} \mathscr{S}_\psi(a_{n+1}, a^*) = \lim\limits_{n\to\infty} \mathscr{S}_\psi(\mathscr{P}_2 c_n, a^*) = 0$.
Similarly, the sequence $\{c_n\}$ is a Cauchy. Since $\mathfrak{S}_2$ is complete, there is $c^* \in \mathfrak{S}_2$ such that, $c_n \to c^*$, as $n \to \infty$, that is; $\lim\limits_{n\to\infty} \mathscr{T}_\psi(c_n, c^*) = \lim\limits_{n\to\infty} \mathscr{S}_\psi(\mathscr{P}_1 a_n, c^*) = 0$. Now, presume that $\mathscr{P}_1$ is continuous. Then $\lim\limits_{n\to\infty} \mathscr{P}_1 a_n = \mathscr{P}_1 a^* = \lim\limits_{n\to\infty} c_n = c^*$, so $\mathscr{P}_1 a^* = c^*$. Utilizing inequality 3.6, we undergo

$$\begin{aligned}\mathscr{S}_\psi(a_n, \mathscr{P}_2\mathscr{P}_1 a^*) = \mathscr{S}_\psi(\mathscr{P}_2\mathscr{P}_1 a_{n-1}, \mathscr{P}_2\mathscr{P}_1 a^*) &\leq \psi(\mathscr{S}_\psi(\mathscr{P}_2\mathscr{P}_1 a_{n-1}, \mathscr{P}_2\mathscr{P}_1 a^*))\\ &\leq \phi\left(Max\left(\mathscr{S}_\psi(a_{n-1}, a^*), \mathscr{T}_\psi(\mathscr{P}_1 a_{n-1}, \mathscr{P}_1 a^*)\right)\right)\\ &= \phi\left(Max\left(\mathscr{S}_\psi(a_{n-1}, a^*), \mathscr{T}_\psi(c_{n-1}, c^*)\right)\right).\end{aligned}$$

Letting $n \to \infty$, we get $\lim\limits_{n\to\infty} \mathscr{S}_\psi(a_n, \mathscr{P}_2\mathscr{P}_1 a^*) = 0$. Since $\lim\limits_{n\to\infty} \mathscr{S}_\psi(a_n, a^*) = 0$, hence it is results $\mathscr{P}_2\mathscr{P}_1 a^* = a^*$. Thus, $\mathscr{P}_2 c^* = a^*$ and $\mathscr{P}_1\mathscr{P}_2 c^* = \mathscr{P}_1 a^* = c^*$. By same way, if the function $\mathscr{P}_2$ is continuous. The uniqueness of the fixed point follows easily from 3.6, 3.7. Indeed, if a, b and c, d are two fixed points of $\mathscr{P}_2\mathscr{P}_1$ and $\mathscr{P}_1\mathscr{P}_2$ respectively, then from 3.6, 3.7, we get

$$\begin{aligned}\mathscr{S}_\psi(a,b) = \mathscr{S}_\psi(\mathscr{P}_2\mathscr{P}_1 a, \mathscr{P}_2\mathscr{P}_1 b) &\leq \psi\left(\mathscr{S}_\psi(\mathscr{P}_2\mathscr{P}_1 a, \mathscr{P}_2\mathscr{P}_1 b)\right)\\ &\leq \phi\left(Max\{\mathscr{S}_\psi(a,b), \mathscr{T}_\psi(\mathscr{P}_1 a, \mathscr{P}_1 b)\}\right) = \phi\left(Max\{\mathscr{S}_\psi(a,b), \mathscr{T}_\psi(c,d)\}\right),\end{aligned}$$

and

$$\begin{aligned}\mathscr{T}_\psi(c,d) = \mathscr{T}_\psi(\mathscr{P}_1\mathscr{P}_2 c, \mathscr{P}_1\mathscr{P}_2 d) &\leq \psi\left(\mathscr{T}_\psi(\mathscr{P}_1\mathscr{P}_2 c, \mathscr{P}_1\mathscr{P}_2 d)\right)\\ &\leq \phi\left(Max\{\mathscr{T}_\psi(c,d), \mathscr{S}_\psi(\mathscr{P}_2 c, \mathscr{P}_2 d)\}\right) = \phi\left(Max\{\mathscr{T}_\psi(c,d), \mathscr{S}_\psi(a,c)\}\right).\end{aligned}$$

Therefore,

$$Max\{\mathscr{S}_\psi(a,b), \mathscr{T}_\psi(c,d)\} \leq \phi\left(Max\{\mathscr{S}_\psi(a,b), \mathscr{T}_\psi(c,d)\}\right). \tag{3.8}$$

Hence, if at least one of the $\mathscr{S}_\psi(a,b)$ or $\mathscr{T}_\psi(c,d)$ is not zero, by 3.8 and property of ϕ we yields a contradiction. Thus, $\mathscr{S}_\psi(a,b)=\mathscr{T}_\psi(c,d)=0$. Similarly, we reach $\mathscr{S}_\psi(a,a)=\mathscr{T}_\psi(c,c)=0$, implies that $a=b$ and $c=d$. □

Corollary 3.6. *Let $(\mathfrak{S}_1,\mathscr{S}_\psi)$ and $(\mathfrak{S}_2,\mathscr{T}_\psi)$ be two complete SCMS. Let $\mathscr{P}_1:\mathfrak{S}_1\to\mathfrak{S}_2$ and $\mathscr{P}_2:\mathfrak{S}_2\to\mathfrak{S}_1$ satisfy the inequalities:*

$$\mathfrak{s}\mathscr{S}_\psi(\mathscr{P}_2\mathscr{P}_1a,\mathscr{P}_2\mathscr{P}_1b)\le\phi\big(Max\{\mathscr{S}_\psi(a,b),\frac{1}{\mathfrak{s}}\mathscr{T}_\psi(\mathscr{P}_1a,\mathscr{P}_1b)\}\big) \tag{3.9}$$

$$\mathfrak{s}\mathscr{T}_\psi(\mathscr{P}_1\mathscr{P}_2a,\mathscr{P}_1\mathscr{P}_2b)\le\phi\big(Max\{\frac{1}{\mathfrak{s}}\mathscr{T}_\psi(c,d),\mathscr{S}_\psi(\mathscr{P}_2c,\mathscr{P}_2d)\}\big), \tag{3.10}$$

for all $a,b\in\mathfrak{S}_1$ and $c,d\in\mathfrak{S}_2$, where $\phi\in\Phi$ and $\psi\in\Psi$. If one of the mappings $\mathscr{P}_1$ and $\mathscr{P}_2$ is continuous, then $\mathscr{P}_2\mathscr{P}_1$ has a unique fixed point a^* in $\mathfrak{S}_1$ and $\mathscr{P}_1\mathscr{P}_2$ has a unique fixed point c^* in $\mathfrak{S}_2$. Moreover, $\mathscr{P}_1a^*=c^*$ and $\mathscr{P}_2c^*=a^*$.

Proof. It is a enough set $\psi(t)=\mathfrak{s}t$ for every $\mathfrak{s}\ge1$ and $\phi(t)=kt$ for every $0<k<1$. Hence

$$\psi^{m-n-1}\sum_{i=n}^{\infty}\psi^{-(m-1)}(\phi^i(t))=\mathfrak{s}^{m-n-1}\sum_{i=n}^{\infty}\mathfrak{s}^{-(m-1)}k^it=\frac{t}{\mathfrak{s}^n}\sum_{i=n}^{\infty}k^i\le\frac{t}{\mathfrak{s}^n}\frac{k^n}{k-1}\to0,$$

as $n\to\infty$. Hence, since all the conditions of Theorem 3.5 hold, $\mathscr{P}_2\mathscr{P}_1$ and $\mathscr{P}_1\mathscr{P}_2$ have a unique fixed point $a^*\in\mathfrak{S}_1$ and $c^*\in\mathfrak{S}_2$ respectively. □

Corollary 3.7. *Let $(\mathfrak{S},\mathscr{S}_\psi)$ be a complete SCMS. Let $\mathscr{P}:\mathfrak{S}\to\mathfrak{S}$ satisfying the inequalities:*

$$\psi\left(\mathscr{S}_\psi(\mathscr{P}^2a,\mathscr{P}^2b)\right)\le\phi\left(Max\{\mathscr{S}_\psi(a,b),\psi^{-1}\left(\mathscr{S}_\psi(\mathscr{P}a,\mathscr{P}b)\right)\}\right) \tag{3.11}$$

for all $a,b\in\mathfrak{S}$, where $\phi\in\Phi$ and $\psi\in\Psi$. Let for each $m\ge n$,

$$\psi^{m-n-1}\sum_{i=n}^{\infty}\psi^{-(m-1)}\left(\phi^i(t)\right)\to0\quad as\, n\to\infty.$$

If $\mathscr{P}$ is continuous, then $\mathscr{P}$ has a unique fixed point a^* in $\mathfrak{S}$.

Example 3.8. Let $\mathfrak{S}_1=\mathbb{R}$ and $\mathfrak{S}_2=\mathbb{R}^2$. Let for any $a,b,c,d\in\mathbb{R},\mathscr{S}_\psi:\mathfrak{S}_1\times\mathfrak{S}_1\to\mathbb{R}$. Define by

$$\mathscr{S}_\psi(a,b)=|a-b|,$$

and $\mathscr{T}_\psi:\mathfrak{S}_2\times\mathfrak{S}_2\to\mathbb{R}$, define by

$$\mathscr{T}_\psi\big((a,b),(c,d)\big)=Max\{|a-c|,2|b-d|-1\}$$

via $\psi(t)=\eta t$, where $\eta=\big(|a-c|+|b-d|+2\big)$. If define $\mathscr{P}_1(a)=\frac{1}{2\eta}(a,\sin a)$ and $\mathscr{P}_2(a,b)=\frac{1}{2}a$, then $(\mathfrak{S}_1,\mathscr{S}_\psi)$ and $(\mathfrak{S}_2,\mathscr{T}_\psi)$ are two SCMS. Hence, we have

$$\begin{aligned}\eta\mathscr{S}_\psi(\mathscr{P}_2\mathscr{P}_1a,\mathscr{P}_2\mathscr{P}_1c)&\leq\eta\mathscr{S}_\psi\left(\mathscr{P}_2\left(\frac{1}{2\eta}a,\frac{1}{2\eta}\sin a\right),\mathscr{P}_2\left(\frac{1}{2\eta}c,\frac{1}{2\eta}\sin c\right)\right)\\&\leq\eta\mathscr{S}_\psi\left(\frac{1}{4\eta}a,\frac{1}{4\eta}c\right)\leq\frac{1}{4}|a-c|.\end{aligned}$$

While,

$$\eta\mathscr{T}_\psi(\mathscr{P}_1\mathscr{P}_2(a,b),\mathscr{P}_1\mathscr{P}_2(c,d))\leq\eta\mathscr{T}_\psi\left(\left(\frac{1}{4\eta}a,\frac{1}{2\eta}\sin a\right),\left(\frac{1}{4\eta}c,\frac{1}{2\eta}\sin c\right)\right)=\frac{1}{4}|a-c|.$$

Therefore, if take $\phi(t)=\frac{t}{2}$ then all the conditions of Theorem 3.5 hold, then $a^*=0$ and $C^*=(0,0)$.

4. Rational-Type Contractions of SCMS

In this section, we provide a unique fixed point in a Rational-type contraction map of SCMS. Inspire [22], suppose $\mathscr{K}$ symbol of all maps $K:\mathfrak{S}\times\mathfrak{S}\to[0,1)$ that satisfies the following conditions:

(1) $K(\mathscr{P}a,b)\leq K(a,b)$ for each $a,b\in\mathfrak{S}$ and $\mathscr{P}:\mathfrak{S}\to\mathfrak{S}$ a mapping.
(2) $K(a,\mathscr{P}b)\leq K(a,b)$ for each $a,b\in\mathfrak{S}$.

It is clear that iterative $K^i(a,b)\to 0$ is as $i\to\infty$.

Theorem 4.1. *Let $(\mathfrak{S},\mathscr{S}_\psi)$ be a complete SCMS. Consider $\mathscr{P}_1,\mathscr{P}_2:\mathfrak{S}\to\mathfrak{S}$ a maps, and there is $K_1,K_2\in\mathscr{K}$ such that*

$$\mathscr{S}_\psi(\mathscr{P}_1a,\mathscr{P}_2b)\leq K_1(a,b)\mathscr{S}_\psi(a,b)+K_2(a,b)\frac{\mathscr{S}_\psi(a,\mathscr{P}_1a)\mathscr{S}_\psi(b,\mathscr{P}_2b)}{1+\mathscr{S}_\psi(a,b)},\tag{4.1}$$

for all $a,b\in\mathfrak{S}$. For $a_0\in\mathfrak{S}$, define as $a_{2n+1}=\mathscr{P}_1a_{2n}$ and $a_{2n+2}=\mathscr{P}_2a_{2n+1}$ for every $n\geq 0$. Suppose that,

$$\lim_{n,m\to\infty}\sum_{i=m}^{n-1}\psi^{i-m}\left(\mu^i\left(\mathscr{S}_\psi(a_0,a_1)\right)\right)\to 0,\tag{4.2}$$

wherever $\mu=\frac{K_1(a_0,a_1)}{1+K_2(a_0,a_1)}<1$. Then there is a unique fixed point (say) $a^*\in\mathfrak{S}$ such that $\mathscr{P}_1a^*=\mathscr{P}_2a^*=a^*$.

Proof. Let $a_0 \in \mathfrak{S}$. We construct $\{a_n\}$ in $\mathfrak{S}$ by $a_{2n+1} = \mathscr{P}_1 a_{2n}$ and $a_{2n+2} = \mathscr{P}_2 a_{2n+1}$ for all $n \in N$. If $\exists n_0 \in N$ for which $a_{n_0+1} = a_{n_0}$, then $\mathscr{P}_1 a_{n_0} = a_{n_0}$. Thus, there is nothing to prove. Similarly for $\mathscr{P}_2$. So, we assume that $a_{n+1} \neq a_n$ for all $n \in N$. By aid of 4.1, we obtain

$$\begin{aligned}
&\mathscr{S}_\psi(a_{2n+1},a_{2n+2}) = \mathscr{S}_\psi(\mathscr{P}_1 a_{2n}, \mathscr{P}_2 a_{2n+1})\\
&\leq K_1(a_{2n},a_{2n+1})\mathscr{S}_\psi(a_{2n},a_{2n+1}) + K_2(a_{2n},a_{2n+1})\frac{\mathscr{S}_\psi(a_{2n},\mathscr{P}_1 a_{2n})\mathscr{S}_\psi(a_{2n+1},\mathscr{P}_2 a_{2n+1})}{1+\mathscr{S}_\psi(a_{2n},a_{2n+1})}\\
&\leq K_1(\mathscr{P}_1\mathscr{P}_2 a_{2n},a_{2n+1})\mathscr{S}_\psi(a_{2n},a_{2n+1}) + K_2(\mathscr{P}_1\mathscr{P}_2 a_{2n},a_{2n+1})\frac{\mathscr{S}_\psi(a_{2n},a_{2n+1})\mathscr{S}_\psi(a_{2n+1},a_{2n+2})}{1+\mathscr{S}_\psi(a_{2n},a_{2n+1})}\\
&\leq K_1(a_{2n-2},a_{2n+1})\mathscr{S}_\psi(a_{2n},a_{2n+1}) + K_2(a_{2n-2},a_{2n+1})\mathscr{S}_\psi(a_{2n+1},a_{2n+2})\\
&= K_1(\mathscr{P}_2\mathscr{P}_1 a_{2n-4},a_{2n+1})\mathscr{S}_\psi(a_{2n},a_{2n+1}) + K_2(\mathscr{P}_2\mathscr{P}_1 a_{2n-4},a_{2n+1})\mathscr{S}_\psi(a_{2n+1},a_{2n+2})\\
&\leq K_1(a_{2n-4},a_{2n+1})\mathscr{S}_\psi(a_{2n},a_{2n+1}) + K_2(a_{2n-4},a_{2n+1})\mathscr{S}_\psi(a_{2n+1},a_{2n+2})\\
&\leq \cdots \leq K_1(a_0,a_{2n+1})\mathscr{S}_\psi(a_{2n},a_{2n+1}) + K_2(a_0,a_{2n+1})\mathscr{S}_\psi(a_{2n+1},a_{2n+2})\\
&= K_1(a_0,\mathscr{P}_1\mathscr{P}_2 a_{2n+1})\mathscr{S}_\psi(a_{2n},a_{2n+1}) + K_2(a_0,\mathscr{P}_1\mathscr{P}_2 a_{2n+1})\mathscr{S}_\psi(a_{2n+1},a_{2n+2})\\
&\leq K_1(a_0,a_{2n+1})\mathscr{S}_\psi(a_{2n},a_{2n+1}) + K_2(a_0,a_{2n+1})\mathscr{S}_\psi(a_{2n+1},a_{2n+2})\\
&\leq \cdots \leq K_1(a_0,a_1)\mathscr{S}_\psi(a_{2n},a_{2n+1}) + K_2(a_0,a_1)\mathscr{S}_\psi(a_{2n+1},a_{2n+2})
\end{aligned}$$

This yields that

$$\mathscr{S}_\psi(a_{2n+1},a_{2n+2}) \leq \left(\frac{K_1(a_0,a_1)}{1-K_2(a_0,a_1)}\right)\mathscr{S}_\psi(a_{n-1},a_n) = \mu\mathscr{S}_\psi(a_{2n},a_{2n+1}).$$

Continuing in the same way, we undergo

$$\mathscr{S}_\psi(a_n,a_{n+1}) \leq \mu\mathscr{S}_\psi(a_{n-1},a_n) \leq \mu^2\mathscr{S}_\psi(a_{n-2},a_{n-1}) \leq \cdots \leq \mu^n\mathscr{S}_\psi(a_0,a_1).$$

Thus,

$$\mathscr{S}_\psi(a_n,a_{n+1}) \leq \mu^n\mathscr{S}_\psi(a_0,a_1). \tag{4.3}$$

For all $n,m \in N$ and $m < n$, similarity with Eq. 3.5, giving

$$\mathscr{S}_\psi(a_m,a_n) \leq \sum_{i=m}^{n-1}\psi^{i-m}\left(\mathscr{S}_\psi(a_i,a_{i+1})\right) \tag{4.4}$$

By means of 4.3 in 4.4, yields that $\mathscr{S}_\psi(a_m,a_n) \leq \sum_{i=m}^{n-1}\psi^{i-m}(\mu^i\mathscr{S}_\psi(a_0,a_1))$, ensuring that limit exists as $n,m \to \infty$ by condition Eq.4.2, we conclude that $\lim_{n,m\to\infty}\mathscr{S}_\psi(a_m,a_n) = 0$, therefore,

$\{a_n\}$ is a Cauchy sequence in the complete SCMS of $(\mathfrak{S},\mathscr{S}_\psi)$, so, there is $a^*\in\mathfrak{S}$ such that $\lim\limits_{n\to\infty}\mathscr{S}_\psi(a_n,a^*)=0$. That is, $a_n\to a^*$. So, by 4.1 and condition (SC3), we obtain

$$\begin{aligned}
&\mathscr{S}_\psi(a^*,\mathscr{P}_1a^*)\le \mathscr{S}_\psi(a^*,a_{2n+2})+\psi\left(\mathscr{S}_\psi(a_{2n+2},\mathscr{P}_1a^*)\right)\\
&=\mathscr{S}_\psi(a^*,a_{2n+2})+\psi\left(\mathscr{S}_\psi(\mathscr{P}_1a^*,\mathscr{P}_2a_{2n+1})\right)\\
&\le \mathscr{S}_\psi(a^*,a_{2n+2})\\
&\quad+\psi\left(K_1(a^*,a_{2n+1})\mathscr{S}_\psi(a^*,a_{2n+1})+K_2(a^*,a_{2n+1})\frac{\mathscr{S}_\psi(a^*,\mathscr{P}_1a^*)\mathscr{S}_\psi(a_{2n+1},\mathscr{P}_2a_{2n+1})}{1+\mathscr{S}_\psi(a^*,a_{2n+1})}\right)\\
&=\mathscr{S}_\psi(a^*,a_{2n+2})\\
&\quad+\psi\left(K_1(a^*,a_{2n+1})\mathscr{S}_\psi(a^*,a_{2n+1})+K_2(a^*,a_{2n+1})\frac{\mathscr{S}_\psi(a^*,\mathscr{P}_1a^*)\mathscr{S}_\psi(a_{2n+1},a_{2n+2})}{1+\mathscr{S}_\psi(a^*,a_{2n+1})}\right)
\end{aligned}$$

Limit as $n\to\infty$ and $\lim\limits_{n\to\infty}\mathscr{S}_\psi(a^*,a_n)=0$, which contradicts $\mathscr{S}_\psi(a^*,\mathscr{P}_1a^*)>0$. Therefore, $\mathscr{P}_1a^*=a^*$. By the same process, we can explore the result is: $\mathscr{P}_2a^*=a^*$. Hence, $\mathscr{P}_1$ and $\mathscr{P}_2$ has a common fixed point a^*.

Let $a^*,b^*\in\mathfrak{S}$ as two fixed points of $\mathscr{P}_1,\mathscr{P}_2$ and $a^*\neq b^*$, we reach

$$\begin{aligned}
&\mathscr{S}_\psi(a^*,b^*)=\mathscr{S}_\psi(\mathscr{P}_1a^*,\mathscr{P}_2b^*)\\
&\le K_1(a^*,b^*)\mathscr{S}_\psi(a^*,b^*)+K_2(a^*,b^*)\frac{\mathscr{S}_\psi(a^*,\mathscr{P}_1a^*)\mathscr{S}_\psi(b^*,\mathscr{P}_2b^*)}{1+\mathscr{S}_\psi(a^*,b^*)}\\
&=K_1(a^*,b^*)\mathscr{S}_\psi(a^*,b^*)+K_2(a^*,b^*)\frac{\mathscr{S}_\psi(a^*,a^*)\mathscr{S}_\psi(b^*,b^*)}{1+\mathscr{S}_\psi(a^*,b^*)}\\
&=K_1(a^*,b^*)\mathscr{S}_\psi(a^*,b^*).
\end{aligned}$$

Hence, $\mathscr{S}_\psi(a^*,b^*)=0$, because $K_1(a^*,b^*)\in[0,1)$, then $a^*=b^*$, we see that a^* is unique. □

Corollary 4.2. *Suppose $(\mathfrak{S},\mathscr{S}_\psi)$ is a complete SCMS. Consider $\mathscr{P}:\mathfrak{S}\to\mathfrak{S}$, and there is $K_1,K_2\in\mathscr{K}$ such that*

$$\mathscr{S}_\psi(\mathscr{P}a,\mathscr{P}b)\le K_1(a,b)\mathscr{S}_\psi(a,b)+K_2(a,b)\frac{\mathscr{S}_\psi(a,\mathscr{P}a)\mathscr{S}_\psi(b,\mathscr{P}b)}{1+\mathscr{S}_\psi(a,b)},$$

for all $a,b\in\mathfrak{S}$. For $a_0\in\mathfrak{S}$, aid of $a_{n+1}=\mathscr{P}^na_0, n\ge 0$. Suppose that,

$$\lim_{n,m\to\infty}\sum_{i=m}^{n-1}\psi^{i-m}\left(\mu^i\left(\mathscr{S}_\psi(a_0,a_1)\right)\right)\to 0,$$

wherever $\mu = \frac{K_1(a_0,a_1)}{1+K_2(a_0,a_1)} < 1$. *Then there is a unique fixed point (say)* $a^* \in \mathfrak{S}$ *such that* $\mathscr{P}a^* = a^*$.

Corollary 4.3. *Let* $(\mathfrak{S},\mathscr{S}_\psi)$ *be a complete SCMS. Consider* $\mathscr{P}_1,\mathscr{P}_2 : \mathfrak{S} \to \mathfrak{S}$ *a maps, and there is* $K_1,K_2 \in [0,1)$*, where* $K_1+K_2 < 1$ *such that*

$$\mathscr{S}_\psi(\mathscr{P}_1a,\mathscr{P}_2b) \le K_1\mathscr{S}_\psi(a,b) + K_2\frac{\mathscr{S}_\psi(a,\mathscr{P}_1a)\mathscr{S}_\psi(b,\mathscr{P}_2b)}{1+\mathscr{S}_\psi(a,b)},$$

for all $a,b \in \mathfrak{S}$*. For* $a_0 \in \mathfrak{S}$*, define as* $a_{2n+1} = \mathscr{P}_1a_{2n}$ *and* $a_{2n+2} = \mathscr{P}_2a_{2n+1}$ *for every* $n \ge 0$*. Suppose that,*

$$\lim_{n,m\to\infty}\sum_{i=m}^{n-1}\psi^{i-m}\left(\mu^i\left(\mathscr{S}_\psi(a_0,a_1)\right)\right) \to 0,$$

wherever $\mu = \frac{K_1}{1+K_2} < 1$*. Then there is a unique fixed point* $a^* \in \mathfrak{S}$ *such that* $\mathscr{P}_1a^* = \mathscr{P}_2a^* = a^*$.

Proof. Immediately by taking $K_1(a,b) = K_1$and $K_2(a,b) = K_2$ in Theorem 4.1. □

Corollary 4.4. *Suppose* $(\mathfrak{S},\mathscr{S}_\psi)$ *is a complete SCMS. Consider* $\mathscr{P} : \mathfrak{S} \to \mathfrak{S}$*, and there is* $K_1,K_2 \in [0,1)$*, where* $K_1+K_2 < 1$ *such that*

$$\mathscr{S}_\psi(\mathscr{P}a,\mathscr{P}b) \le K_1\mathscr{S}_\psi(a,b) + K_2\frac{\mathscr{S}_\psi(a,\mathscr{P}a)\mathscr{S}_\psi(b,\mathscr{P}b)}{1+\mathscr{S}_\psi(a,b)},$$

for all $a,b \in \mathfrak{S}$*. For* $a_0 \in \mathfrak{S}$*, define it as* $a_{n+1} = \mathscr{P}^na_0, n \ge 0$*. Suppose that,*

$$\lim_{n,m\to\infty}\sum_{i=m}^{n-1}\psi^{i-m}\left(\mu^i\left(\mathscr{S}_\psi(a_0,a_1)\right)\right) \to 0,$$

wherever $\mu = \frac{K_1}{1+K_2} < 1$*. Then there is a unique fixed point* $a^* \in \mathfrak{S}$ *such that* $\mathscr{P}a^* = a^*$.

Example 4.5. Let $\mathfrak{S} = 0,1,2$. Define $\mathscr{S}_\psi : \mathfrak{S}\times\mathfrak{S} \to [0,\infty)$ a symmetrical metric as
$\mathscr{S}_\psi(a,a) = 0$ for each $a \in \mathfrak{S}$ and $\mathscr{S}_\psi(0,1) = \frac{1}{2}, \mathscr{S}_\psi(0,2) = 2, \mathscr{S}_\psi(1,2) = \frac{3}{2}$.
Defining the map $\psi(t) = t^4$ for all $t \ge 0$. Obviously, $(\mathfrak{S},\mathscr{S}_\psi)$ is an SCMS. Given $\mathscr{P} : \mathfrak{S} \to \mathfrak{S}$ as $\mathscr{P}(0) = \mathscr{P}(1) = \mathscr{P}(2) = 1$, and assume that $K_1 = \frac{1}{2}, K_2 = \frac{1}{4}$, we obtain
Case 1. If $a = b = 0, a = b = 1, a = b = 2$, we get

$$\mathscr{S}_\psi(\mathscr{P}a,\mathscr{P}b) = 0 \le \frac{1}{2}(0) + \frac{1}{4}\frac{\mathscr{S}_\psi(a,\mathscr{P}a)\mathscr{S}_\psi(b,\mathscr{P}b)}{1+\mathscr{S}_\psi(a,b)}.$$

Case 2. If $a = 1, b = 2$, we get $\mathscr{S}_\psi(\mathscr{P}(1),\mathscr{P}(2)) = 0 \le \frac{1}{2}(\frac{3}{2}) + \frac{1}{4}(0) = \frac{3}{4}$.

Case 3. If $a=0, b=1$, we get $\mathscr{S}_\psi(\mathscr{P}(0), \mathscr{P}(1)) = 0 \leq \frac{1}{2}(\frac{1}{2}) + \frac{1}{4}(0) = \frac{1}{4}$.

Case 4. If $a=0, b=2$, we get $\mathscr{S}_\psi(\mathscr{P}(0), \mathscr{P}(2)) = 0 \leq \frac{1}{2}(2) + \frac{1}{4}(\frac{1}{4}) = \frac{17}{16}$.

Therefore, each the conditions of Corollary 4.4 is valid, so $a^* = 1$ is a unique fixed point.

5. Application

Finally, we have given some applications that based on our theorems as follows that:

5.1. Nonlinear Integral Equations. We hypothesis existence of a solution for the below integral equation

$$\mathfrak{f}(\tau) = \int_0^1 \mathscr{K}(\tau,\eta)\mathfrak{L}(\eta,\mathfrak{f}(\eta))d\eta, \tag{5.1}$$

$\tau \in [0,1]$. Suppose that $\mathfrak{S} = C([0,1])$ is the space of all continuous functions from $[0,1]$ into $\mathbb{R}$, assume that $\mathfrak{S}$ is given with the SCMS as:

$\mathscr{S}_\psi(\mathfrak{f},\mathfrak{g}) = \sup_{\tau\in[0,1]} \log(|\mathfrak{f}(\tau) - \mathfrak{g}(\tau)| + 1)$ for all $\mathfrak{f},\mathfrak{g} \in \mathfrak{S}$ via $\psi(t) = \log(\gamma e^t - \gamma), \gamma = \mathfrak{f} + \mathfrak{g} + 2$. Clearly, $(\mathfrak{S}, \mathscr{S}_\psi)$ is a complete SCMS.

Theorem 5.1. *Assume that the conditions below satisfied:*

(1) $\mathfrak{L} : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a continuous map,* $\mathfrak{L}()\tau,\eta) \geq 0$ *and there is a constant* $0 \leq \alpha < 1$ *such that for all* $\mathfrak{f},\mathfrak{g} \in \mathfrak{S}$

$$|\mathfrak{L}(\eta,\mathfrak{f}(\eta)) - \mathfrak{L}(\eta,\mathfrak{g}(\eta))| < \alpha \log\left(|\mathfrak{f}(\eta) - \mathfrak{g}(\eta)| + 1\right), \tag{5.2}$$

(2) $\mathscr{K} : [0,1] \times [0,1] \to \mathbb{R}$ *is a continuous, for all* $\tau,\eta \in [0,1]$ *such as* $\mathscr{K}(\tau,\eta) \geq 0$ *and* $\int_0^1 \mathscr{K}(\tau,\eta)d\eta \leq \beta$.

Then the integral equation 5.1 has a unique solution in $\mathfrak{S}$.

Proof. Define a mapping $\mathscr{P} : \mathfrak{S} \to \mathfrak{S}$ by

$$\mathscr{P}\mathfrak{f}(\tau) = \int_0^1 \mathscr{K}(\tau,\eta)\mathfrak{L}(\eta,\mathfrak{f}(\eta))d\eta,$$

$\tau \in [0,1]$ and for each $\mathfrak{f},\mathfrak{g} \in \mathfrak{S}$, we undergo, (presume that $\alpha\beta \leq 1$)

$$\begin{aligned}|\mathscr{P}\mathfrak{f}(\tau) - \mathscr{P}\mathfrak{g}(\tau)| &= \Big|\int_0^1 \mathscr{K}(\tau,\eta)\mathfrak{L}(\eta,\mathfrak{f}(\eta))d\eta - \int_0^1 \mathscr{K}(\tau,\eta)\mathfrak{L}(\eta,\mathfrak{g}(\eta))d\eta\Big| \\ &\leq \Big(\int_0^1 |\mathscr{K}(\tau,\eta)||\mathfrak{L}(\eta,\mathfrak{f}(\eta)) - \mathfrak{L}(\eta,\mathfrak{g}(\eta))|d\eta\end{aligned}$$

$$\begin{aligned}
&\leq \alpha \int_0^1 \mathscr{K}(\tau,\eta)\log(|\mathfrak{f}(\eta)-\mathfrak{g}(\eta)|+1)d\eta \\
&\leq \alpha \mathscr{S}_\psi(\mathfrak{f}(\tau),\mathfrak{g}(\tau))\int_0^1 \mathscr{K}(\tau,\eta)d\eta \\
&\leq \alpha\beta \mathscr{S}_\psi(\mathfrak{f}(\tau),\mathfrak{g}(\tau)).
\end{aligned}$$

Hence,

$$\log\big(|\mathscr{P}\mathfrak{f}(\tau)-\mathscr{P}\mathfrak{g}(\tau)|+1\big) \leq |\mathscr{P}\mathfrak{f}(\tau)-\mathscr{P}\mathfrak{g}(\tau)| \leq \alpha\beta\mathscr{S}_\psi(\mathfrak{f}(\tau),\mathfrak{g}(\tau)).$$

Thus, $\mathscr{S}_\psi(\mathscr{P}\mathfrak{f}(\tau),\mathscr{P}\mathfrak{g}(\tau)) \leq \alpha\beta\mathscr{S}_\psi(\mathfrak{f}(\tau),\mathfrak{g}(\tau)) \leq \phi(\mathscr{S}_\psi(\mathfrak{f}(\tau),\mathfrak{g}(\tau)))$, where $\phi(t)=\alpha\beta t$, and $0 \leq \alpha\beta < 1$.

Hence Theorem 3.4 holds; and equation 5.1 has a unique solution in $\mathfrak{S}$. In addition, we know that for condition 5.2, we obtain $\log\big(|\mathfrak{f}(\eta)-\mathfrak{g}(\eta)|+1\big) \leq |\mathfrak{f}(\eta)-\mathfrak{g}(\eta)|$. □

5.2. Fractional Differential Equation. In this part exhibits the fractional differential equation FDE as

$$(5.3)\qquad \begin{aligned}
&{}^{\alpha}\mathscr{D}^{\beta}\omega(\eta)+\mathfrak{f}(\eta,\omega(\eta))=0, 0\leq\eta\leq 1; 1\leq\beta\leq 2,\\
&\omega(0)=\omega(1)=0,
\end{aligned}$$

where $\mathfrak{f}:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and ${}^{\alpha}\mathscr{D}^{\beta}$ denote the order of β as the Caputo FDE defined by

$${}^{\alpha}\mathscr{D}^{\beta}\omega(\eta)=\frac{1}{\Gamma(r-\beta)}\int_0^{\eta}\frac{\omega^r(\mu)d\mu}{(\eta-\mu)^{\beta-r+1}}.$$

This system of FDE in 5.3 equivalent to $\omega(\eta)=\int_0^1\mathscr{K}(\eta,\mu)\mathfrak{f}(\eta,\omega(\mu))d\mu$, for each $\eta,\mu\in[0,1]$, where Green function as

$$\mathscr{K}(\eta,\mu)=\begin{cases}\frac{(\eta(1-\mu))^{\beta-1}-(\eta-\mu)^{\beta-1}}{\Gamma(\beta)} & 0\leq\mu\leq\eta\leq 1,\\ \frac{(\eta(1-\mu))^{\beta-1}}{\Gamma(\beta)} & 0\leq\eta\leq\mu\leq 1.\end{cases}$$

Let $\mathfrak{S}=C([0,1],\mathbb{R})$, and $\mathscr{S}_\psi:\mathfrak{S}\times\mathfrak{S}\to[0,\infty)$ a SCMS, such that

$$\mathscr{S}_\psi(\omega,\nu)=Max_{\eta\in[0,1]}\Big(\sqrt{|\omega(\eta)-\nu(\eta)|+1}-1\Big),$$

for each $\omega,\nu\in\mathfrak{S}$, and $\psi(t)=\sqrt{\delta(t+1)^2+1-\delta}-1, t\geq 0$, $\delta=max\{\omega,\nu\}+2$. Then $(\mathfrak{S},\mathscr{S}_\psi)$ is a complete SCMS.

Theorem 5.2. *In the non-linear FDE 5.3. Assume that the following conditions are satisfied:*

(1) There is $\varpi \in [0,1]$ *and* $\omega, \nu \in \mathfrak{S}$*, such that*

$$|\mathfrak{f}(\eta,\omega(\eta)) - \mathfrak{f}(\eta,\nu(\eta))| \leq \varpi\big(\sqrt{|\omega(\eta)-\nu(\eta)|} - 1\big).$$

(2) $sup_{\eta\in[0,1]} \int_0^1 \mathscr{K}(\eta,\mu)d\mu < 1.$

Then, FDE has a unique solution in $\mathfrak{S}$*.*

Proof. Define a function $\mathscr{P} : \mathfrak{S} \to \mathfrak{S}$ by

$$\mathscr{P}\omega(\eta) = \int_0^1 \mathscr{K}(\eta,\mu)\mathfrak{f}(\eta,\omega(\mu))d\mu.$$

For all $\omega, \nu \in \mathfrak{S}$, we reach,

$$\begin{aligned}
|\mathscr{P}\omega(\eta) - \mathscr{P}\nu(\eta)| &= |\int_0^1 \mathscr{K}(\eta,\mu)\mathfrak{f}(\eta,\omega(\mu))d\mu - \int_0^1 \mathscr{K}(\eta,\mu)\mathfrak{f}(\eta,\nu(\mu))d\mu| \\
&\leq \int_0^1 \mathscr{K}(\eta,\mu)|\mathfrak{f}(\eta,\omega(\mu)) - \mathfrak{f}(\eta,\nu(\mu))|d\mu \\
&\leq \varpi \int_0^1 \mathscr{K}(\eta,\mu)\big(\sqrt{|\omega(\eta)-\nu(\eta)|} - 1\big)d\mu \\
&\leq \varpi \int_0^1 \mathscr{K}(\eta,\mu)\big(\sqrt{|\omega(\eta)-\nu(\eta)|+1} - 1\big)d\mu \\
&\leq \varpi \mathscr{S}_\psi(\omega(\eta),\nu(\eta)).
\end{aligned}$$

Hence,

$$\sqrt{|\mathscr{P}\omega(\eta) - \mathscr{P}\nu(\eta)|+1} - 1 \leq |\mathscr{P}\omega(\eta) - \mathscr{P}\nu(\eta)| \leq \varpi \mathscr{S}_\psi(\omega(\eta),\nu(\eta)).$$

Taking the maximum, which implies that

$$\mathscr{S}_\psi(\mathscr{P}\omega(\eta),\mathscr{P}\nu(\eta)) \leq \varpi \mathscr{S}_\psi(\omega(\eta),\nu(\eta)) \leq \phi\big(\mathscr{S}_\psi(\omega(\eta),\nu(\eta))\big),$$

where $\phi(t) = \varpi t$, and $0 \leq \varpi < 1$. Therefore, in Theorem 3.4 all the conditions are fulfilled and the equation 5.3 has a unique solution in $\mathfrak{S}$. □

6. CONCLUSIONS

The article develops a novel concept that is an SCMS, which is a generalization of CSbMS, while it is extended to SbMS. It provides some results for the special (ψ,ϕ)-contraction fixed-point theorems in SCMS with some particular results. Moreover, it illustrates the theorem of Hardy-Rogers type fixed point theorem, Matkowsik type, and nonlinear-rational contraction. In addition, it presents some applications of certain works to nonlinear integral equations and fractional differential equations. Future work will study the strong-composed cone metric space, and the generalization of F-contraction and Z-contraction with the establishment of some new applications of SCMS.

AUTHOR CONTRIBUTIONS

A. A. and L. Kh. wrote the main manuscript text. All authors reviewed the manuscript.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] I.A. Bakhtin, The Contraction Mapping Principle in Almost Metric Spaces, Funct. Anal., Gos. Ped. Inst. Unianowsk 30 (1989), 26–37.

[2] S. Czerwik, Contraction Mappings in b-Metric Spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11. http://dml.cz/dmlcz/120469.

[3] Z.D. Mitrović, S. Radenović, The Banach and Reich Contractions in $b_v(s)$-Metric Spaces, J. Fixed Point Theory Appl. 19 (2017), 3087–3095. https://doi.org/10.1007/s11784-017-0469-2.

[4] Z. Mitrovic, H. Işık, S. Radenovic, The New Results in Extended b-Metric Spaces and Applications, Int. J. Nonlinear Anal. Appl. 11 (2020), 473-482. https://doi.org/10.22075/ijnaa.2019.18239.1998.

[5] W. Kirk, N. Shahzad, Fixed Point Theory in Distance Spaces, Springer, Cham, 2014. https://doi.org/10.1007/978-3-319-10927-5.

[6] B. Alqahtani, E. Karapinar, F. Khojasteh, On Some Fixed Point Results in Extended Strong b-Metric Spaces, Bull. Math. Anal. Appl. 10 (2018), 25–35.

[7] D. Santina, W.A. Mior Othman, K.B. Wong, N. Mlaiki, New Generalization of Metric-Type Spaces—Strong Controlled, Symmetry 15 (2023), 416. https://doi.org/10.3390/sym15020416.

[8] N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled Metric Type Spaces and the Related Contraction Principle, Mathematics 6 (2018), 194. https://doi.org/10.3390/math6100194.

[9] T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double Controlled Metric Type Spaces and Some Fixed Point Results, Mathematics 6 (2018), 320. https://doi.org/10.3390/math6120320.

[10] D. Lateef, Fisher Type Fixed Point Results in Controlled Metric Spaces, J. Math. Comput. Sci. 20 (2020), 234–240. https://doi.org/10.22436/jmcs.020.03.06.

[11] A. Karami, S. Sedghi, Z.D. Mitrović, Solving Existence Problems via Contractions in Expanded b-Metric Spaces, J. Anal. 30 (2022), 895–907. https://doi.org/10.1007/s41478-021-00376-9.

[12] I. Ayoob, N.Z. Chuan, N. Mlaiki, Double-Composed Metric Spaces, Mathematics 11 (2023), 1866. https://doi.org/10.3390/math11081866.

[13] C.J. Kil, C.S. Yu, U.C. Han, Fixed Point Results for Some Rational Type Contractions in Double-Composed Metric Spaces and Applications, Informatica, 34 (2023), 105–130.

[14] H. Doan, A New Type of Kannan's Fixed Point Theorem in Strong b- Metric Spaces, AIMS Math. 6 (2021), 7895–7908. https://doi.org/10.3934/math.2021458.

[15] A. Šostak, T. Öner, İ.C. Duman, On Topological and Metric Properties of $\oplus - sb$-Metric Spaces. Mathematics 11 (2023), 4090. https://doi.org/10.3390/math11194090.

[16] J. Matkowski, Fixed Point Theorems for Mappings with a Contractive Iterate at a Point, Proc. Amer. Math. Soc. 62 (1977), 344–348. https://doi.org/10.1090/S0002-9939-1977-0436113-5.

[17] I. Ayoob, N.Z. Chuan, N. Mlaiki, Hardy-Rogers Type Contraction in Double Controlled Metric-like Spaces, AIMS Math. 8 (2023), 13623–13636. https://doi.org/10.3934/math.2023691.

[18] A.A. Hijab, L.K. Shaakir, S. Aljohani, N. Mlaiki, Fredholm Integral Equation in Composed-Cone Metric Spaces, Bound. Value Probl. 2024 (2024), 64. https://doi.org/10.1186/s13661-024-01876-w.

[19] N. Hussain, Z. Kadelburg, S. Radenović, F. Al-Solamy, Comparison Functions and Fixed Point Results in Partial Metric Spaces, Abstr. Appl. Anal. 2012 (2012), 605781. https://doi.org/10.1155/2012/605781.

[20] A.A. Hijab, L.K. Shaakir, S. Aljohani, N. Mlaiki, Double Composed Metric-like Spaces via Some Fixed Point Theorems, AIMS Math. 9 (2024), 27205–27219. https://doi.org/10.3934/math.20241322.

[21] F. Khojasteh, S. Moradi, S. Randenvic, θ-Metric Spaces and Extension of Banach and Caristi's Fixed Point Theorem, Preprint, arXiv:1109.3025 [math.FA], 2011. https://doi.org/10.48550/arXiv.1109.3025.

[22] K. Ahmad, G. Murtaza, S. Alshaikey, et al. Common Fixed Point Results on a Double-Controlled Metric Space for Generalized Rational-Type Contractions with Application, Axioms 12 (2023), 941. https://doi.org/10.3390/axioms12100941.