Available online at http://scik.org
Adv. Fixed Point Theory, 2026, 16:22
https://doi.org/10.28919/afpt/9054
ISSN: 1927-6303

SOME FIXED POINT THEOREMS OF α-CONVEX GENERALIZED NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

RAHUL SHUKLA*

Department of Mathematical Sciences and Computing, Walter Sisulu University, Mthatha, 5117, South Africa

Abstract. This study focuses on α-convex generalized nonexpansive (α-CGNE) mappings, a generalization of nonexpansive mappings defined by an additional parameter $\alpha \in (0,1)$. These mappings extend classical nonexpansive properties and are applicable to problems involving variational inequalities and monotone operators. We investigate the fixed point theory of α-CGNE mappings in hyperbolic metric spaces, emphasizing strong convergence results using projection mappings. Additionally, Δ-convergence theorems are established under specific conditions, highlighting the asymptotic behavior of iterative methods in nonlinear settings. These results expand fixed point theory beyond Banach spaces, offering new approaches for studying nonlinear phenomena.

Keywords: nonexpansive mapping; fixed point; hyperbolic spaces.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Fixed point theory is a cornerstone of mathematical analysis with applications spanning numerous fields, including functional analysis, optimization, and applied mathematics. The study of fixed points of mappings dates back to seminal works by Browder [2], Göhde [8], and Kirk [10], who independently established fundamental existence results for fixed points of nonexpansive mappings in Banach spaces. These results have since been extended and generalized

*Corresponding author

E-mail address: rshukla@wsu.ac.za

Received December 03, 2024

in many directions, leading to a vast body of literature on fixed point theory. In particular, Suzuki [22] introduced a generalization of nonexpansive mappings that significantly broadened the scope of fixed point results in Banach spaces, particularly in uniformly convex settings, see also [6, 14].

In metric spaces $(\mathscr{Y}, \Theta)$, a mapping $\psi : \mathscr{Y} \to \mathscr{Y}$ is called Lipschitzian if there exists a constant $k \geq 0$ such that for all $\upsilon, y \in \mathscr{Y}$, the inequality

$$\Theta(\psi(\upsilon), \psi(y)) \leq k\Theta(\upsilon, y)$$

holds. The smallest such k is the Lipschitz constant of ψ. Nonexpansive mappings are a special case of Lipschitzian mappings where the Lipschitz constant is equal to one, preserving distances by ensuring that the distance between the images of two points does not exceed the distance between the points themselves. These properties are central to fixed point theory.

A fundamental result in this area is Browder-Göhde's Theorem [2, 8], which asserts that if $\mathscr{C}$ is a nonempty, bounded, closed, and convex subset of a uniformly convex Banach space E, and $\psi : \mathscr{C} \to \mathscr{C}$ is a nonexpansive mapping, then ψ has a fixed point. This theorem established a foundation for further exploration into the existence of fixed points under broader conditions.

Suzuki [22] extended the understanding of fixed point theory by introducing Suzuki-generalized nonexpansive mappings, a weaker condition than nonexpansiveness. For a mapping $\psi : \mathscr{C} \to \mathscr{C}$ and all $\upsilon, y \in \mathscr{C}$, the Suzuki condition is given as:

$$\frac{1}{2}\Theta(\upsilon, \psi(\upsilon)) \leq \Theta(\upsilon, y) \quad \Rightarrow \quad \Theta(\psi(\upsilon), \psi(y)) \leq \Theta(\upsilon, y).$$

These mappings are not necessarily continuous but still satisfy the fixed point property in uniformly convex Banach spaces, providing a broader framework for analyzing fixed points.

In fixed point theory, the fixed point property (FPP) of a Banach space is significant. A Banach space is said to possess the FPP if every nonexpansive self-mapping of a nonempty, closed, convex, bounded subset has a fixed point. The geometric structure of Banach spaces, especially uniform convexity, is closely linked to fixed point properties.

The study of nonexpansive mappings has naturally led to more generalized classes of mappings. Among these are α-convex generalized nonexpansive (α-CGNE) mappings, introduced by Shukla et al. [20]. Such a mapping $\psi : \mathscr{C} \to \mathscr{C}$, defined on a convex subset $\mathscr{C}$ of a Banach space $\mathscr{B}$, satisfies:

$$\|\psi((1-\alpha)\upsilon + \alpha\psi(\upsilon)) - \psi(\sigma)\| \leq (1-\alpha)\|\upsilon - \sigma\| + \alpha\|\psi(\upsilon) - \sigma\|,$$

for all $\upsilon, \sigma \in \mathscr{C}$ and $\alpha \in (0,1)$. This generalization includes mappings derived from variational inequalities and monotone operators, significantly expanding the scope of fixed point theory.

The role of convexity and iterative methods in approximating fixed points has been extensively studied, including by Takahashi [23] and Goebel and Kirk [7]. Methods like the Krasnosel'skiĭ-Mann iteration, initially developed for nonexpansive mappings, have been extended to broader classes. Leuştean [13] advanced fixed point results within geodesic spaces, focusing on uniform convexity and the asymptotic regularity of iterative processes. Shukla and Panicker [19] contributed by extending existence results for nonexpansive type mappings in geodesic spaces and analyzing the asymptotic behavior of Picard iterates. Shukla's work [17] further explored methods for solving optimization problems in nonlinear spaces. Shukla and Panicker [18] extended enriched nonexpansive mappings in geodesic spaces and obtained existence and convergence results. Nanjaras et al. [15] extended Suzuki's results to CAT(0) spaces, generalizing Hilbert spaces in nonlinear settings.

This study builds upon these advancements to investigate the behavior of α-CGNE mappings in hyperbolic metric spaces. Geodesic spaces, which generalize Banach spaces by replacing linearity with shortest paths (geodesics), are naturally suited to nonlinear phenomena. This work derives Δ- and strong convergence results using projection mappings, providing new insights into iterative methods and extending fixed point theory to nonlinear settings.

2. Preliminaries

The concept of hyperbolic spaces, as presented by Kohlenbach [12], follows this framework.

Definition 1. [12]. *A metric space $(\mathscr{Y}, \Theta)$ is said to be hyperbolic metric space if the function $W : \mathscr{Y} \times \mathscr{Y} \times [0,1] \to \mathscr{Y}$ satisfies the following conditions for all $\upsilon, \vartheta, z, w \in \mathscr{Y}$ and $\beta, t \in [0,1]$:*

(W1) $\Theta(z,W(\upsilon,\vartheta,\beta)) \leq (1-\beta)\Theta(z,\upsilon)+\beta\Theta(z,\vartheta)$;

(W2) $\Theta(W(\upsilon,\vartheta,\beta),W(\upsilon,\vartheta,t)) = |\beta-t|\Theta(\upsilon,\vartheta)$;

(W3) $W(\upsilon,\vartheta,\beta) = W(\vartheta,\upsilon,1-\beta)$;

(W4) $\Theta(W(\upsilon,z,\beta),W(\vartheta,w,\beta)) \leq (1-\beta)\Theta(\upsilon,\vartheta)+\beta\Theta(z,w)$.

We will employ the notation

$$W(\zeta,\vartheta,\beta) := (1-\beta)\zeta \oplus \beta\vartheta$$

to represent a point $W(\zeta,\vartheta,\beta)$ within a hyperbolic metric space.

Definition 2. [13]. *A hyperbolic metric space $(\mathscr{Y},\Theta)$ is uniformly convex (UC) if, for every $\varepsilon \in (0,2]$ and any $b>0$, there exists a $\delta \in (0,1]$ such that*

$$\left.\begin{array}{l}\Theta(\upsilon,z) \leq b\\ \Theta(\vartheta,z) \leq b\\ \Theta(\upsilon,\vartheta) \geq \varepsilon b\end{array}\right\} \Rightarrow \Theta\left(\frac{1}{2}\upsilon \oplus \frac{1}{2}\vartheta, z\right) \leq (1-\delta)b$$

for all $\upsilon,\vartheta,z \in \mathscr{Y}$.

In UC hyperbolic metric space, let $\upsilon,\vartheta \in \mathscr{Y}$ and any $\beta \in [0,1]$, there is an element $w \in \mathscr{Y}$ which is unique (say $w = W(\upsilon,\vartheta,\beta)$) such that

$$\Theta(\upsilon,w) = \beta\Theta(\upsilon,\vartheta) \text{ and } \Theta(\vartheta,w) = (1-\beta)\Theta(\upsilon,\vartheta). \tag{1}$$

Consider a W-hyperbolic space $(\mathscr{Y},\Theta)$ and let $\mathscr{C} \subset \mathscr{Y}$ with $\mathscr{C} \neq \emptyset$. A sequence $\{\upsilon_n\}$ in $\mathscr{Y}$ is said to be Fejér monotone with respect to $\mathscr{C}$ if

$$\Theta(\upsilon^\dagger,\upsilon_{n+1}) \leq \Theta(\upsilon^\dagger,\upsilon_n), \quad \forall \quad n \geq 0, \text{ and } \forall \quad \upsilon^\dagger \in \mathscr{C}.$$

Let $\psi : \mathscr{Y} \to \mathscr{Y}$ be a mapping. We denote the set of fixed points of ψ as

$$F(\psi) := \{\upsilon \in \mathscr{Y} : \psi(\upsilon) = \upsilon\}.$$

Definition 3. [3]. *A mapping $\psi : \mathscr{C} \to \mathscr{C}$ is called quasi-nonexpansive (QNE) if*

$$\Theta(\psi(\upsilon),\upsilon^\dagger) \leq \Theta(\upsilon,\upsilon^\dagger), \quad \forall \upsilon \in \mathscr{C} \quad \textit{and} \quad \upsilon^\dagger \in F(\psi).$$

Definition 4. [5]. *Let $\mathscr{C}$ be a nonempty subset of a metric space $\mathscr{Y}$. A mapping $\psi : \mathscr{C} \to \mathscr{C}$ is said to fulfill condition (E) if there exists $\mu \geq 1$ such that*

$$\Theta(\upsilon, \psi(\vartheta)) \leq \mu\Theta(\upsilon, \psi(\upsilon)) + \Theta(\upsilon, \vartheta) \ \forall \ \upsilon, \vartheta \in \mathscr{C}.$$

The nearest point projection $P_{\mathscr{C}} : \mathscr{Y} \to 2^{\mathscr{C}}$ is defined as

$$P_{\mathscr{C}}(\upsilon) = \{c \in \mathscr{C} : \Theta(\upsilon, c) = \inf\{\Theta(\upsilon, c) : c \in \mathscr{C}\}\}.$$

A subset $\mathscr{C}$ of a metric space $(\mathscr{Y}, \Theta)$ is called a Chebyshev set if, for each point $\upsilon \in \mathscr{Y}$, there exists a unique point $z \in \mathscr{C}$ such that

$$\Theta(\upsilon, z) = \Theta(\upsilon, \mathscr{C}) = \inf\{\Theta(\upsilon, y) : y \in \mathscr{C}\}.$$

If $\mathscr{C}$ is a Chebyshev set, then the nearest point projection $P : \mathscr{Y} \to \mathscr{C}$ can be defined by assigning z to υ.

Proposition 1. [13] . *Let $(\mathscr{Y}, \Theta)$ be a complete UC hyperbolic metric space. Every closed convex subset $\mathscr{C}$ of $\mathscr{Y}$ is a Chebyshev set.*

Definition 5. [21]. *Consider a bounded sequence $\{\upsilon_n\}$ in a hyperbolic metric space $(\mathscr{Y}, \Theta)$. The sequence $\{\upsilon_n\}$ is said to Δ-converge to υ if υ is the unique asymptotic center for every subsequence $\{\rho_n\}$ derived from $\{\upsilon_n\}$. We use the notation $\upsilon_n \xrightarrow{\omega} \upsilon$.*

Definition 6. [16]. *A function $\psi : \mathscr{C} \to \mathscr{C}$, where $F(\psi) \neq \emptyset$, satisfies Condition (I) if there exists a function $f : [0, +\infty) \to [0, +\infty)$ such that the following hold:*

(1) $f(r) > 0$ for $r \in (0, +\infty)$ and $f(0) = 0$.

(2) $\Theta(\upsilon, \psi(\upsilon)) \geq f(\Theta(\upsilon, F(\psi)))$ for all $\upsilon \in \mathscr{C}$,

where $\Theta(\upsilon, F(\psi)) = \inf\{\Theta(\upsilon, \vartheta) : \vartheta \in F(\psi)\}$.

Definition 7. *Consider a metric space $(\mathscr{Y}, \Theta)$ and let $\mathscr{C} \subset \mathscr{Y}$ with $\mathscr{C} \neq \emptyset$. A mapping $\psi : \mathscr{C} \to \mathscr{C}$ is called compact if the closure of $\psi(\mathscr{C})$ is compact.*

Recently, Khamsi and Shukri [9] broadened the Gromov geometric definition of CAT(0) metric spaces to include cases where the comparison triangles are located within a general Banach space. Specifically, this includes cases where the Banach space is ℓ_p with $p \geq 2$.

Definition 8. [9]. *A geodesic metric space $(\mathscr{Y},\Theta)$ is called a CAT$_p(0)$ metric space if, for any geodesic triangle Δ in $\mathscr{Y}$, there exists a corresponding comparison triangle $\bar{\Delta}$ in ℓ_p that satisfies the comparison axiom. This means that for all points $\upsilon, y \in \Delta$ and their respective comparison points $\bar{\upsilon}, \bar{y} \in \bar{\Delta}$, the inequality*

$$\Theta(\upsilon, y) \leq \|\bar{\upsilon} - \bar{y}\|$$

holds.

It is clear that CAT$_2(0)$ metric spaces are precisely the traditional CAT(0) metric spaces. Moreover, while ℓ_p spaces for $p \neq 2$ are examples of CAT$_p(0)$ metric spaces, they do not qualify as CAT(0) metric spaces [9].

Lemma 1. [7]. *$(\mathscr{Y},\Theta)$ be a complete UC hyperbolic metric space. Let $\{\upsilon_n\}$ and $\{\vartheta_n\}$ be two bounded sequences in $\mathscr{Y}$ and $\lambda \in (0,1)$. Suppose that $\upsilon_{n+1} = (1-\lambda)\upsilon_n \oplus \lambda \vartheta_n$ with $\Theta(\vartheta_{n+1}, \vartheta_n) \leq \Theta(\upsilon_{n+1}, \upsilon_n)\ \forall \quad n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \Theta(\upsilon_n, \vartheta_n) = 0.$$

3. α-Convex Generalized Nonexpansive Mapping

In this section, we extend the class of α-CGNE mappings in nonlinear spaces:

Definition 9. *Let $(\mathscr{Y},\Theta)$ be a hyperbolic metric space, and let $\mathscr{C}$ be a convex subset of $\mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$. A mapping $\psi : \mathscr{C} \to \mathscr{C}$ is said to be α-convex generalized nonexpansive (α-CGNE) if there exists $\alpha \in (0,1)$ such that*

$$\Theta\left(\psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon)), \psi(\sigma)\right) \leq (1-\alpha)\Theta(\upsilon, \sigma) + \alpha\Theta(\psi(\upsilon), \sigma) \tag{2}$$

for all $\upsilon, \sigma \in \mathscr{C}$.

The following example shows that the α-CGNE is more general class of mapping than Condition (C).

Example 1. [20]. *Let $\mathscr{C} = [0,4] \subset \mathbb{R}$ with the usual norm. Define $\psi : \mathscr{C} \to \mathscr{C}$ by*

$$\psi(\upsilon) = \begin{cases} 0, & \text{if } \upsilon \neq 4 \\ \frac{36}{19}, & \text{if } \upsilon = 4 \end{cases}$$

The mapping ψ is α-convex generalized for $\alpha = \dfrac{9}{10}$. On the other hand, ψ does not satisfy condition (C). Hence ψ is not nonexpansive mapping.

We prove the following important Lemma:

Lemma 2. *Let $(\mathscr{Y}, \Theta)$ be a hyperbolic metric space. Let $\mathscr{C} \subset \mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$, $\mathscr{C}$ be a convex. Let $\psi : \mathscr{C} \to \mathscr{C}$ be an α-CGNE mapping Then*

$$\Theta(\upsilon, \psi(\sigma)) \leq (1+2\alpha)\Theta(\upsilon, \psi(\upsilon)) + \Theta(\upsilon, \sigma) \tag{3}$$

holds for all $\upsilon, \sigma \in \mathscr{C}$. That is, the mapping ψ satisfies condition (E).

Proof. By the triangle inequality, we have

$$\begin{aligned}
\Theta(\upsilon, \psi(\sigma)) &\leq \Theta(\upsilon, \psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon))) + \Theta(\psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon)), \psi(\sigma)) \\
&\leq \Theta(\upsilon, (1-\alpha)\upsilon \oplus \alpha\psi(\upsilon)) + \Theta((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon), \psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon))) \\
&\quad + \Theta(\psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon)), \psi(\sigma)) \\
&= \alpha\Theta(\upsilon, \psi(\upsilon)) + \Theta((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon), \psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon))) \\
&\quad + \Theta(\psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon)), \psi(\sigma)).
\end{aligned}$$

By the condition on the mapping ψ, we have

$$\begin{aligned}
\Theta(\upsilon, \psi(\sigma)) &\leq \alpha\Theta(\upsilon, \psi(\upsilon)) + \Theta((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon), \psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon))) \\
&\quad + (1-\alpha)\Theta(\upsilon, \sigma) + \alpha\Theta(\psi(\upsilon), \sigma) \\
&\leq \alpha\Theta(\upsilon, \psi(\upsilon)) + \Theta((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon), \psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon))) \\
&\quad + (1-\alpha)\Theta(\upsilon, \sigma) + \alpha\Theta(\upsilon, \sigma) + \alpha\Theta(\upsilon, \psi(\upsilon)) \\
&\leq 2\alpha\Theta(\upsilon, \psi(\upsilon)) + \Theta((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon), \psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon))) + \Theta(\upsilon, \sigma) \\
&\leq 2\alpha\Theta(\upsilon, \psi(\upsilon)) + \Theta((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon), \psi(\upsilon)) \\
&\quad + \Theta(\psi(\upsilon), \psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon))) + \Theta(\upsilon, \sigma) \\
&\leq 2\alpha\Theta(\upsilon, \psi(\upsilon)) + (1-\alpha)\Theta(\upsilon, \psi(\upsilon)) + \Theta(\psi(\upsilon), \psi((1-\alpha)\upsilon \oplus \alpha\psi(\upsilon))) \\
&\quad + \Theta(\upsilon, \sigma)
\end{aligned}$$

By the condition on the mapping ψ, we obtain

$$\begin{aligned}\Theta(\upsilon,\psi(\sigma)) &\leq (1+\alpha)\Theta(\upsilon,\psi(\upsilon))+\alpha\Theta(\upsilon,\psi(\upsilon)))+\Theta(\upsilon,\sigma)\\ &=(1+2\alpha)\Theta(\upsilon,\psi(\upsilon))+\Theta(\upsilon,\sigma).\end{aligned}$$

By taking $\mu=(1+2\alpha)$, we can conclude the proof. $\square$

Example 2. *Let ψ be a mapping on $[0,4]$ defined by*

$$\psi(\upsilon)=\begin{cases}0 & \textit{if } \upsilon\neq 4\\ 3 & \textit{if } \upsilon=4\end{cases}$$

Then the mapping ψ satisfies condition (E) and the mapping ψ is not α-CGNE mapping.

Proposition 2. *Let $(\mathscr{Y},\Theta)$ be a hyperbolic metric space. Let $\mathscr{C}\subset\mathscr{Y}$ such that $\mathscr{C}\neq\emptyset$, $\mathscr{C}$ be a convex and $\psi:\mathscr{C}\to\mathscr{C}$ be an α-CGNE mapping with $F(\psi)\neq\emptyset$. Then ψ is a QNE mapping.*

Lemma 3. *Let $(\mathscr{Y},\Theta)$ be a complete UC-hyperbolic metric space, and let $\mathscr{C}\subset\mathscr{Y}$ be a non-empty, bounded, convex subset. Suppose that $\psi:\mathscr{C}\to\mathscr{C}$ is an α-CGNE mapping. For a given $\upsilon_0\in\mathscr{C}$, define the sequence $\{\upsilon_n\}$ recursively by*

$$\upsilon_{n+1}=(1-\alpha)\upsilon_n\oplus\alpha\psi(\upsilon_n) \tag{4}$$

for each $n\in\mathbb{N}\cup\{0\}$. Then, the $\lim_{n\to\infty}\Theta(\upsilon_n,\psi(\upsilon_n))=0$ holds, indicating that $\{\upsilon_n\}$ is an approximate fixed point sequence (a.f.p.s.) of ψ.

Proof. From (4) and (1), we obtain

$$\Theta(\upsilon_n,\upsilon_{n+1})=\alpha\Theta(\upsilon_n,\psi(\upsilon_n)).$$

Since ψ is an α-CGNE mapping, it follows that

$$\begin{aligned}\Theta(\psi(\upsilon_{n+1}),\psi(\upsilon_n)) &= \Theta\left(\psi((1-\alpha)\upsilon_n\oplus\alpha\psi(\upsilon_n)),\psi(\upsilon_n)\right)\\ &\leq (1-\alpha)\Theta(\upsilon_n,\upsilon_n)+\alpha\Theta(\upsilon_n,\psi(\upsilon_n))\\ &= \alpha\Theta(\upsilon_n,\psi(\upsilon_n))\\ &= \Theta(\upsilon_n,\upsilon_{n+1})\end{aligned}$$

for all $n \in \mathbb{N}$. Applying Lemma 1, we conclude that $\lim_{n\to\infty} \Theta(\upsilon_n, \psi(\upsilon_n)) = 0$. $\square$

We now turn to examining the existence of fixed points for α-CGNE mappings in complete UC hyperbolic metric spaces.

Theorem 1. *Let $(\mathscr{Y}, \Theta)$ be a complete UC hyperbolic metric space, and let $\mathscr{C} \subset \mathscr{Y}$ be a nonempty, bounded, closed, and convex subset. Suppose $\psi : \mathscr{C} \to \mathscr{C}$ is an α-CGNE mapping. Then ψ has a fixed point in $\mathscr{C}$.*

Proof. Consider an initial point $\upsilon_0 \in \mathscr{C}$, and define a sequence $\{\upsilon_n\}$ by the recursive formula

$$\upsilon_{n+1} = (1-\alpha)\upsilon_n \oplus \alpha\psi(\upsilon_n) \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

According to [13, Proposition 3.3], the sequence $\{\upsilon_n\}$ has a unique asymptotic center with respect to $\mathscr{C}$. Let $\{z\} = A(\mathscr{C}, \{\upsilon_n\})$; by definition of the asymptotic center $A(\mathscr{C}, \{\upsilon_n\})$, it follows that $z \in \mathscr{C}$.

From Lemma 3, we obtain

$$\lim_{n\to\infty} \Theta(\psi(\upsilon_n), \upsilon_n) = 0. \tag{5}$$

Moreover, by (3), we have

$$\Theta(\upsilon_n, \psi(z)) \leq (1+2\alpha)\Theta(\psi(\upsilon_n), \upsilon_n) + \Theta(\upsilon_n, z).$$

From (5), it follows that

$$\limsup_{n\to\infty} \Theta(\upsilon_n, \psi(z)) \leq \limsup_{n\to\infty} \Theta(\upsilon_n, z).$$

Thus,

$$r(\psi(z), \{\upsilon_n\}) \leq r(z, \{\upsilon_n\}).$$

Since the asymptotic center of $\{\upsilon_n\}$ is unique, it follows that $z = \psi(z)$, proving that z is a fixed point of ψ. $\square$

Corollary 1 extends [4, Theorem 3.3] to a broader class of mappings.

Corollary 1. *Let $\mathscr{Y}$ be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let $\mathscr{C} \subset \mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$, $\mathscr{C}$ be a bounded closed convex. Let $\psi : \mathscr{C} \to \mathscr{C}$ be an α-CGNE mapping. Then ψ has a fixed point in $\mathscr{C}$.*

Utilizing Theorem 1 and Proposition 2 along with [1, Lemma 6.2], we obtain the following corollary.

Corollary 2. *Let $(\mathscr{Y},\Theta)$ be a complete UC hyperbolic metric space and $\mathscr{C} \subset \mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$, $\mathscr{C}$ be bounded closed convex. Let $\psi : \mathscr{C} \to \mathscr{C}$ be an α-CGNE mapping. Then $F(\psi)$ is nonempty closed and convex.*

Theorem 2. *Let $(\mathscr{Y},\Theta)$ be a complete UC hyperbolic metric space, and let $\mathscr{C} \subset \mathscr{Y}$ be a non-empty, bounded, closed, and convex subset. Suppose $\{\psi_j\}_{j=1}^{\infty}$ is a countable family of commuting α-CGNE mappings. Then $\{\psi_j\}_{j=1}^{\infty}$ has a common fixed point.*

Proof. For each $n \in \mathbb{N}$, define

$$\mathscr{C}_n := \bigcap_{j=1}^{n} F(\psi_j).$$

Thus, $\mathscr{C}_1 = F(\psi_1)$. By Corollary 2, $\mathscr{C}_1$ is a non-empty, closed, and convex subset of $\mathscr{Y}$, and since $\mathscr{C}_1 \subset \mathscr{C}$, it is also bounded. Now, let $k \in \mathbb{N}$ with $k \geq 2$ and assume that $\mathscr{C}_{k-1}$ is non-empty, closed, bounded, and convex. We claim that $\mathscr{C}_k$ is also non-empty, closed, bounded, and convex.

Since $\mathscr{C}_{k-1} \neq \emptyset$, take any $\upsilon \in \mathscr{C}_{k-1}$ and $j \in \mathbb{N}$ with $1 \leq j < k$. Given that $\{\psi_j\}_{j=1}^{\infty}$ is a family of commuting mappings, ψ_k and ψ_j commute, so we have

$$\psi_k(\upsilon) = \psi_k \circ \psi_j(\upsilon) = \psi_j \circ \psi_k(\upsilon).$$

Thus, $\psi_k(\upsilon)$ is a fixed point of ψ_j, implying that $\psi_k(\upsilon) \in \mathscr{C}_{k-1}$. Therefore, $\psi_k(\mathscr{C}_{k-1}) \subset \mathscr{C}_{k-1}$. By Theorem 1, ψ_k has a fixed point in $\mathscr{C}_{k-1}$, and hence

$$\mathscr{C}_k = \mathscr{C}_{k-1} \cap F(\psi_k) \neq \emptyset.$$

Using Corollary 2, we find that $\mathscr{C}_k$ is closed and convex. By induction, $\mathscr{C}_n$ is non-empty, closed, bounded, and convex for all $n \in \mathbb{N}$, and $\mathscr{C}_n \subset \mathscr{C}_{n-1}$ for all $n \in \mathbb{N}$.

Applying [11, Proposition 2.2], we conclude that

$$\bigcap_{j=1}^{\infty} F(\psi_j) = \bigcap_{n=1}^{\infty} \mathscr{C}_n \neq \emptyset.$$

This completes the proof. $\square$

Corollary 3. *Let $\mathscr{Y}$ be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let $\mathscr{C} \subset \mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$, $\mathscr{C}$ be a bounded closed convex. Let $\left\{\psi_j\right\}_{j=1}^{\infty}$ be a countable family of commuting an α-CGNE mapping. Then $\left\{\psi_j\right\}_{j=1}^{\infty}$ has a common fixed point.*

Lemma 4. *Let $(\mathscr{Y}, \Theta, W)$ be a complete UC hyperbolic metric space, and let $\mathscr{C} \subset \mathscr{Y}$ be a non-empty, bounded, closed, and convex subset. Suppose $\psi : \mathscr{C} \to \mathscr{C}$ is an α-CGNE mapping. For an initial point $\upsilon_0 \in \mathscr{C}$, define the sequence $\{\upsilon_n\}$ iteratively as follows:*

$$\upsilon_{n+1} = (1-\alpha)\upsilon_n \oplus \alpha\psi(\upsilon_n) \tag{6}$$

for each $n \in \mathbb{N} \cup \{0\}$. Then $\lim_{n\to\infty} \Theta(\upsilon_n, p)$ exists for any $\upsilon^\dagger \in F(\psi)$.

Proof. According to Theorem 1, $F(\psi) \neq \emptyset$. Take any $\upsilon^\dagger \in F(\psi)$. Using property (W1) and Proposition 2, we have

$$\begin{aligned}\Theta(\upsilon_{n+1}, \upsilon^\dagger) &= \Theta((1-\alpha)\upsilon_n \oplus \alpha\psi(\upsilon_n), \upsilon^\dagger) \\ &\leq (1-\alpha)\Theta(\upsilon_n, \upsilon^\dagger) + \alpha\Theta(\psi(\upsilon_n), \upsilon^\dagger) \\ &\leq \Theta(\upsilon_n, \upsilon^\dagger).\end{aligned}$$

This shows that

$$\Theta(\upsilon_{n+1}, \upsilon^\dagger) \leq \Theta(\upsilon_n, \upsilon^\dagger).$$

Thus, the sequence $\{\Theta(\upsilon_n, \upsilon^\dagger)\}$ is bounded and monotone for all $\upsilon^\dagger \in F(\psi)$. Therefore, the limit exists as required. □

Now we prove our Δ convergence theorem.

Theorem 3. *Let $(\mathscr{Y}, \Theta)$ be a complete UC hyperbolic metric space, and let $\mathscr{C} \subset \mathscr{Y}$ be a non-empty, bounded, closed, and convex set. Suppose $\psi : \mathscr{C} \to \mathscr{C}$ is an α-CGNE mapping. Let $\upsilon_0 \in \mathscr{C}$, and define the sequence $\{\upsilon_n\}$ by the iterative relation*

$$\upsilon_{n+1} = (1-\alpha)\upsilon_n \oplus \alpha\psi(\upsilon_n) \tag{7}$$

for all $n \in \mathbb{N} \cup \{0\}$. Then the sequence $\{\upsilon_n\}$ Δ-converges to a point in $F(\psi)$.

Proof. By Theorem 1, we know that $F(\psi) \neq \emptyset$. Let $z^\dagger \in F(\psi)$. According to Lemma 4, the sequence $\{\Theta(\upsilon_n, z^\dagger)\}$ satisfies the inequality

$$\Theta(\upsilon_{n+1}, z^\dagger) \leq \Theta(\upsilon_n, z^\dagger),$$

for all $z^\dagger \in F(\psi)$. This implies that the sequence $\{\upsilon_n\}$ is Fejér monotone with respect to $F(\psi)$. Since $F(\psi)$ is closed and convex by Corollary 2, and using [13, Proposition 3.3], the sequence $\{\upsilon_n\}$ has a unique asymptotic center, denoted by $w^\dagger$, with respect to $F(\psi)$.

Next, let $\{\rho_n\}$ be a subsequence of $\{\upsilon_n\}$. Again, by [13, Proposition 3.3], the subsequence $\{\rho_n\}$ has a unique asymptotic center, denoted by $\rho^\dagger$, with respect to $F(\psi)$. Now, applying the inequality from Lemma 3, we get

$$\Theta(\rho_n, \psi(\rho^\dagger)) \leq (1+2\alpha)\Theta(\psi(\rho_n), \rho_n) + \Theta(\rho_n, \rho^\dagger).$$

Using the fact that $\lim_{n\to\infty} \Theta(\upsilon_n, \psi(\upsilon_n)) = 0$, we obtain

$$\Theta(\rho_n, \psi(\rho^\dagger)) \leq \Theta(\rho_n, \rho^\dagger).$$

By [13, Lemma 3.2], we know that $\psi(\rho^\dagger) = \rho^\dagger$. Finally, using [1, Lemma 6.1 (iii)], we conclude that the sequence $\{\upsilon_n\}$ Δ-converges to a point in $F(\psi)$. □

Corollary 4. *Let $\mathscr{Y}$ be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let $\mathscr{C} \subset \mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$, $\mathscr{C}$ be a bounded closed convex. Let $\psi : \mathscr{C} \to \mathscr{C}$ be an α-CGNE mapping. Let $\upsilon_0 \in \mathscr{C}$ and consider the sequence $\{\upsilon_n\}$ by the successive iteration*

$$\upsilon_{n+1} = (1-\alpha)\upsilon_n \oplus \alpha\psi(\upsilon_n) \tag{8}$$

for any $n \in \mathbb{N} \cup \{0\}$. Then the sequence $\{\upsilon_n\}$ Δ-converges to a point in $F(\psi)$.

The following result is analogous to the demi-closed principle established by Göhde [8] in uniformly convex Banach spaces.

Proposition 3. *Let $(\mathscr{Y}, \Theta)$ be a complete UC hyperbolic metric space and $\mathscr{C} \subset \mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$, $\mathscr{C}$ be bounded closed convex. Let $\psi : \mathscr{C} \to \mathscr{C}$ be an α-CGNE mapping. If $\{\upsilon_n\} \subset \mathscr{C}$ is an a.f.p.s. of ψ, i.e., $\lim\limits_{n\to\infty} \Theta(\upsilon_n, \psi(\upsilon_n)) = 0$, such that $\upsilon_n \xrightarrow{\omega} \upsilon$, then υ is a fixed point of ψ.*

Proof. Since $\upsilon_n \xrightarrow{\omega} \upsilon$, the point υ serves as the asymptotic center for every subsequence $\{u_n\}$ of $\{\upsilon_n\}$. Additionally, $\{u_n\}$ forms an a.f.p.s. for ψ. By closely following the argument in Theorem 3, we conclude that the asymptotic center of $\{u_n\}$ is a fixed point of ψ. Hence, the result follows. $\square$

Theorem 4. *Let $\mathscr{Y}$, $\mathscr{C}$, ψ, and $\{\upsilon_n\}$ be as defined in Theorem 3. If ψ is a compact mapping, then the sequence $\{\upsilon_n\}$ strongly converges to a point in $F(\psi)$.*

Proof. From Lemma 4, it is clear that the sequence $\{\upsilon_n\}$ is bounded. Furthermore, applying Lemma 3, we obtain

$$\lim_{n\to\infty} \Theta(\upsilon_n, \psi(\upsilon_n)) = 0. \tag{9}$$

Since ψ is a compact mapping, the image of $\mathscr{C}$ under ψ is contained within a compact set. Therefore, there exists a subsequence $\{\psi(\upsilon_{n_j})\}$ of $\{\psi(\upsilon_n)\}$ that converges strongly to some point $\upsilon^\dagger \in \mathscr{C}$. Given (9), we deduce that the subsequence $\{\upsilon_{n_j}\}$ must strongly converge to $\upsilon^\dagger$.

Next, using the inequality from Lemma 3 and (9), we have

$$\Theta(\upsilon_{n_j}, \psi(\upsilon^\dagger)) \leq (1+2\alpha)\Theta(\upsilon_{n_j}, \psi(\upsilon_{n_j})) + \Theta(\upsilon_{n_j}, \upsilon^\dagger) \leq \Theta(\upsilon_{n_j}, \upsilon^\dagger).$$

Thus, the subsequence $\{\upsilon_{n_j}\}$ strongly converges to $\psi(\upsilon^\dagger)$, which implies that $\psi(\upsilon^\dagger) = \upsilon^\dagger$.

Finally, since $\lim_{n\to\infty} \Theta(\upsilon_n, \upsilon^\dagger)$ exists, we conclude that the entire sequence $\{\upsilon_n\}$ strongly converges to a point in $F(\psi)$. $\square$

Corollary 5. *Let $\mathscr{Y}$ be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let $\mathscr{C} \subset \mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$, $\mathscr{C}$ be a bounded closed convex. Let $\psi : \mathscr{C} \to \mathscr{C}$ be an α-CGNE mapping. Let $\upsilon_0 \in \mathscr{C}$ and consider the sequence $\{\upsilon_n\}$ by the successive iteration*

$$\upsilon_{n+1} = (1-\alpha)\upsilon_n \oplus \alpha\psi(\upsilon_n) \tag{10}$$

for any $n \in \mathbb{N} \cup \{0\}$. If ψ is a compact mapping, then the sequence $\{\upsilon_n\}$ strongly converges to a point in $F(\psi)$.

By assuming compactness on the set and relaxing the compactness condition on the mapping, Theorem 3.7 leads to the following result.

Corollary 6. *Let $(\mathscr{Y},\Theta)$ be a complete UC hyperbolic metric space and $\mathscr{C}\subset\mathscr{Y}$ such that $\mathscr{C}\neq\emptyset$, $\mathscr{C}$ be a compact convex. Let $\psi:\mathscr{C}\to\mathscr{C}$ be an α-CGNE mapping. Let $\upsilon_0\in\mathscr{C}$ and define the sequence $\{\upsilon_n\}$ by the successive iteration*

$$\upsilon_{n+1}=(1-\alpha)\upsilon_n\oplus\alpha\psi(\upsilon_n) \tag{11}$$

for any $n\in\mathbb{N}\cup\{0\}$. Then the sequence $\{\upsilon_n\}$ strongly converges to a point in $F(\psi)$.

Theorem 5. *Let $\mathscr{Y}$, $\mathscr{C}$, ψ, and $\{\upsilon_n\}$ be as defined in Theorem 3. If the mapping ψ satisfies condition (I), then the sequence $\{\upsilon_n\}$ strongly converges to a point in $F(\psi)$.*

Proof. By Lemma 4, the sequence $\{\Theta(\upsilon_n,z^\dagger)\}$ is monotonically non-increasing for all $z^\dagger\in F(\psi)$. Hence, the sequence $\{\Theta(\upsilon_n,F(\psi))\}$ is also monotonic and non-increasing, implying that the limit

$$\lim_{n\to\infty}\Theta(\upsilon_n,F(\psi))$$

exists. According to Lemma 3, we obtain the following result:

$$\lim_{n\to\infty}\Theta(\upsilon_n,\psi(\upsilon_n))=0. \tag{12}$$

Since ψ satisfies condition (I), we have

$$\Theta(\upsilon_n,\psi(\upsilon_n))\geq f(\Theta(\upsilon_n,F(\psi))).$$

From the above, it follows that

$$\lim_{n\to\infty}f(\Theta(\upsilon_n,F(\psi)))=0,$$

and consequently,

$$\lim_{n\to\infty}\Theta(\upsilon_n,F(\psi))=0.$$

Now, we demonstrate that the sequence $\{\upsilon_n\}$ is Cauchy. For any $\varepsilon>0$, from the equation $\lim_{n\to\infty}\Theta(\upsilon_n,F(\psi))=0$, there exists an $n_0\in\mathbb{N}$ such that for all $n\geq n_0$,

$$\Theta(\upsilon_n,F(\psi))<\frac{\varepsilon}{4}.$$

Thus, there exists $z^\dagger\in F(\psi)$ such that

$$\Theta(\upsilon_{n_0},z^\dagger)<\frac{\varepsilon}{2}.$$

Now, for all $m,n \geq n_0$, we have

$$\Theta(\upsilon_{n+m},\upsilon_n) \leq \Theta(\upsilon_{n+m},z^\dagger) + \Theta(z^\dagger,\upsilon_n) \leq 2\Theta(\upsilon_{n_0},z^\dagger) < \varepsilon.$$

Thus, the sequence $\{\upsilon_n\}$ is Cauchy. Since $\mathscr{C}$ is closed in $\mathscr{Y}$, the sequence $\{\upsilon_n\}$ converges to some $\upsilon^\dagger \in \mathscr{C}$.

Next, by condition on mapping, we have

$$\begin{aligned} \Theta(\upsilon^\dagger,\psi(\upsilon^\dagger)) &\leq \Theta(\upsilon^\dagger,\upsilon_n) + \Theta(\upsilon_n,\psi(\upsilon^\dagger)) \\ &\leq \Theta(\upsilon^\dagger,\upsilon_n) + (1+2\alpha)\Theta(\upsilon_n,\psi(\upsilon_n)) + \Theta(\upsilon^\dagger,\upsilon_n). \end{aligned}$$

From (12), we obtain that $\upsilon^\dagger = \psi(\upsilon^\dagger)$. Therefore, the sequence $\{\upsilon_n\}$ strongly converges to a point in $F(\psi)$. $\square$

Corollary 7. *Let $\mathscr{Y}$ be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let $\mathscr{C} \subset \mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$, $\mathscr{C}$ be a bounded closed convex. Let $\psi : \mathscr{C} \to \mathscr{C}$ be an α-CGNE mapping. Let $\upsilon_0 \in \mathscr{C}$ and consider the sequence $\{\upsilon_n\}$ by the successive iteration*

$$\upsilon_{n+1} = (1-\alpha)\upsilon_n \oplus \alpha\psi(\upsilon_n) \tag{13}$$

for any $n \in \mathbb{N} \cup \{0\}\}$. If ψ is a compact mapping, then the sequence $\{\upsilon_n\}$ strongly converges to a point in $F(\psi)$.

Lemma 5. *Let $(\mathscr{Y},\Theta)$ be a complete UC hyperbolic metric space. Let $\mathscr{C} \subset \mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$, $\mathscr{C}$ be a bounded closed convex. Let $\psi : \mathscr{C} \to \mathscr{C}$ be an α-CGNE mapping. Then $F(\psi)$ is Chebyshev subset.*

Proof. The proof can be completed in view of Corollary 2 and Proposition 1 $\square$

Finally, we investigate the convergence of fixed-point theorems for α-CGNE mappings in complete UC hyperbolic metric spaces, utilizing the concept of the nearest point projection. We then establish the following convergence result.

Theorem 6. *Let $(\mathscr{Y},\Theta)$ be a complete UC hyperbolic metric space, and let $\mathscr{C} \subset \mathscr{Y}$ be a non-empty, bounded, closed, and convex set. Let $\psi : \mathscr{C} \to \mathscr{C}$ be an α-CGNE mapping. Define the* *sequence* $\{\upsilon_n\}$ *by* $\upsilon_0 \in \mathscr{C}$ *and*

$$\upsilon_{n+1} = (1-\alpha)\upsilon_n \oplus \alpha\psi(\upsilon_n)$$

for all $n \in \mathbb{N} \cup \{0\}$. *Let* $P_{F(\psi)}$ *be the nearest point projection of* $\mathscr{C}$ *onto* $F(\psi)$. *Then, the sequence* $\{P_{F(\psi)}(\upsilon_n)\}$ *converges strongly to a fixed point of* ψ.

Proof. By Theorem 3, the sequence $\{\upsilon_n\}$ converges weakly to a point $\upsilon \in F(\psi)$, with υ serving as the unique asymptotic center of the sequence. Additionally, by Lemma 5, the sequence $\{P_{F(\psi)}(\upsilon_n)\}$ is well-defined. Our goal is to show that $\{P_{F(\psi)}(\upsilon_n)\}$ strongly converges to υ.
We proceed by contradiction. Suppose there exists $\varepsilon > 0$ and a subsequence $\{P_{F(\psi)}(\upsilon_{n_i})\}$ such that $\Theta(P_{F(\psi)}(\upsilon_{n_i}), \upsilon) \geq \varepsilon$ for all $n_i \geq 1$.
Since $\mathscr{Y}$ is a UC hyperbolic metric space, there exists a function $\eta(s, \varepsilon) > 0$ (for every $s \geq 0$ and $\varepsilon > 0$) such that $\delta(r, \varepsilon) > \eta(s, \varepsilon)$ for any $r > s$. Let $R = \Theta(\upsilon_1, \upsilon) > 0$ (otherwise, $\{\upsilon_n\}$ would be constant). It follows that $\delta\left(\Theta(\upsilon_{n_i}, \upsilon), \frac{\varepsilon}{R}\right) > \eta$ for all $n_i \geq 1$.
Now consider the inequality

$$\Theta\left(\upsilon_{n_i}, \frac{1}{2}\upsilon \oplus \frac{1}{2}P_{F(\psi)}(\upsilon_{n_i})\right) \leq \Theta(\upsilon_{n_i}, \upsilon)\left(1 - \delta\left(\Theta(\upsilon_{n_i}, \upsilon), \frac{\varepsilon}{R}\right)\right),$$

which simplifies to

$$\Theta\left(\upsilon_{n_i}, \frac{1}{2}\upsilon \oplus \frac{1}{2}P_{F(\psi)}(\upsilon_{n_i})\right) \leq \Theta(\upsilon_{n_i}, \upsilon)(1-\eta) \tag{14}$$

for all $n_i \geq 1$. By Corollary 2, the convexity of $F(\psi)$ ensures that $\frac{1}{2}\upsilon \oplus \frac{1}{2}P_{F(\psi)}(\upsilon_{n_i}) \in F(\psi)$. Using the properties of the projection $P_{F(\psi)}$, we obtain

$$\Theta(\upsilon_{n_i}, P_{F(\psi)}(\upsilon_{n_i})) \leq \Theta\left(\upsilon_{n_i}, \frac{1}{2}\upsilon \oplus \frac{1}{2}P_{F(\psi)}(\upsilon_{n_i})\right)$$

for each $n_i \geq 1$. Using (14), we have

$$\Theta(\upsilon_{n_i}, P_{F(\psi)}(\upsilon_{n_i})) \leq \Theta(\upsilon_{n_i}, \upsilon)(1-\eta) \tag{15}$$

for each $n_i \geq 1$. Furthermore,

$$\begin{aligned}
\Theta(\upsilon_{n_i+1}, P_{F(\psi)}(\upsilon_{n_i})) &= \Theta((1-\alpha)\upsilon_{n_i} \oplus \alpha\psi(\upsilon_{n_i}), P_{F(\psi)}(\upsilon_{n_i})) \\
&\leq (1-\alpha)\Theta(\upsilon_{n_i}, P_{F(\psi)}(\upsilon_{n_i})) + \alpha\Theta(\psi(\upsilon_{n_i}), P_{F(\psi)}(\upsilon_{n_i}))
\end{aligned}$$

$$\leq \quad \Theta(\upsilon_{n_i}, P_{F(\psi)}(\upsilon_{n_i}))$$

by induction on $m \geq 1$, we have

$$\Theta(\upsilon_{n_i+m}, P_{F(\psi)}(\upsilon_{n_i})) \leq \Theta(\upsilon_{n_i}, P_{F(\psi)}(\upsilon_{n_i})).$$

Thus, from (15)

$$\Theta(\upsilon_{n_i+m}, P_{F(\psi)}(\upsilon_{n_i})) \leq \Theta(\upsilon_{n_i}, \upsilon)(1-\eta)$$

for all $n_i \geq 1$. Since $P_{F(\psi)}(\upsilon_{n_i}) \in F(\psi)$, the sequence $\{\Theta(\upsilon_n, P_{F(\psi)}(\upsilon_{n_i}))\}$ is non-increasing in n for fixed n_i. Therefore,

$$\limsup_{m\to\infty} \Theta(\upsilon_{n_i+m}, P_{F(\psi)}(\upsilon_{n_i})) = \lim_{n\to\infty} \Theta(\upsilon_n, P_{F(\psi)}(\upsilon_{n_i})) \leq \Theta(\upsilon_{n_i}, \upsilon)(1-\eta)$$

for all $n_i \geq 1$. Given that υ is the asymptotic center of $\{\upsilon_n\}$, we also have

$$\lim_{n\to\infty} \Theta(\upsilon_n, \upsilon) \leq \lim_{n\to\infty} \Theta(\upsilon_n, P_{F(\psi)}(\upsilon_{n_i})) \leq \Theta(\upsilon_{n_i}, \upsilon)(1-\eta).$$

Taking the limit as $n_i \to \infty$, we obtain

$$\lim_{n\to\infty} \Theta(\upsilon_n, \upsilon) \leq \lim_{n\to\infty} \Theta(\upsilon_n, \upsilon)(1-\eta),$$

which leads to a contradiction because $1 \leq 1-\eta$ is impossible. Therefore, the assumption that there exists a subsequence with $\Theta(P_{F(\psi)}(\upsilon_{n_i}), \upsilon) \geq \varepsilon$ must be false. Hence, the sequence $\{P_{F(\psi)}(\upsilon_n)\}$ strongly converges to υ. □

Corollary 8. *Let $\mathscr{Y}$ be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let $\mathscr{C} \subset \mathscr{Y}$ such that $\mathscr{C} \neq \emptyset$, $\mathscr{C}$ be a bounded closed convex. Let $\psi : \mathscr{C} \to \mathscr{C}$ be an α-CGNE mapping. Let $\upsilon_0 \in \mathscr{C}$ and define the sequence $\{\upsilon_n\}$ by*

$$\upsilon_{n+1} = (1-\alpha)\upsilon_n \oplus \alpha\psi(\upsilon_n) \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

Then, the sequence $\{P_{F(\psi)}(\upsilon_n)\}$ converges strongly to a fixed point of ψ.

Authors' Contributions

The authors contributed equally to this work.

Funding information

This work was supported by Directorate of Research and Innovation, Walter Sisulu University, South Africa.

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] D. Ariza-Ruiz, L. Leuştean, G. López-Acedo, Firmly Nonexpansive Mappings in Classes of Geodesic Spaces, Trans. Am. Math. Soc. 366 (2014), 4299–4322. https://doi.org/10.1090/s0002-9947-2014-05968-0.

[2] F.E. Browder, Nonexpansive Nonlinear Operators in a Banach Space, Proc. Natl. Acad. Sci. 54 (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041.

[3] A. Cegielski, Iterative Methods for Fixed Point Problems in Hilbert Spaces, Springer Berlin, 2013. https://doi.org/10.1007/978-3-642-30901-4.

[4] A.A. Darweesh, S. Shukri, Fixed Points of Suzuki-Generalized Nonexpansive Mappings in $CAT_p(0)$ Metric Spaces, Arab. J. Math. 13 (2024), 227–236. https://doi.org/10.1007/s40065-024-00455-2.

[5] J. García-Falset, E. Llorens-Fuster, T. Suzuki, Fixed Point Theory for a Class of Generalized Nonexpansive Mappings, J. Math. Anal. Appl. 375 (2011), 185–195. https://doi.org/10.1016/j.jmaa.2010.08.069.

[6] K. Goebel, W.A. Kirk, A Fixed Point Theorem for Asymptotically Nonexpansive Mappings, Proc. Am. Math. Soc. 35 (1972), 171–174. https://doi.org/10.1090/s0002-9939-1972-0298500-3.

[7] K. Goebel, W.A. Kirk, Iteration Processes for Nonexpansive Mappings, in: S.P. Singh, S. Thomeier, B. Watson (Eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 1983: pp. 115–123. https://doi.org/10.1090/conm/021/729507.

[8] D. Göhde, Zum Prinzip Der Kontraktiven Abbildung, Math. Nachr. 30 (1965), 251–258. https://doi.org/10.1002/mana.19650300312.

[9] M.A. Khamsi, S.A. Shukri, Generalized CAT(0) Spaces, Bull. Belg. Math. Soc. - Simon Stevin 24 (2017), 417–426. https://doi.org/10.36045/bbms/1506477690.

[10] W.A. Kirk, A Fixed Point Theorem for Mappings Which Do Not Increase Distances, Am. Math. Mon. 72 (1965), 1004–1006. https://doi.org/10.2307/2313345.

[11] U. Kohlenbach, L. Leuştean, Asymptotically Nonexpansive Mappings in Uniformly Convex Hyperbolic Spaces, J. Eur. Math. Soc. 12 (2009), 71–92. https://doi.org/10.4171/jems/190.

[12] U. Kohlenbach, Some Logical Metatheorems with Applications in Functional Analysis, Trans. Am. Math. Soc. 357 (2004), 89–128. https://doi.org/10.1090/s0002-9947-04-03515-9.

[13] L. Leuştean, Nonexpansive Iterations in Uniformly Convex W-Hyperbolic Spaces, in: A. Leizarowitz, B.S. Mordukhovich, I. Shafrir, A.J. Zaslavski (Eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 2010: pp. 193–210. https://doi.org/10.1090/conm/513/10084.

[14] E. Llorens Fuster, E. Moreno Gálvez, The Fixed Point Theory for Some Generalized Nonexpansive Mappings, Abstr. Appl. Anal. 2011 (2011), 435686. https://doi.org/10.1155/2011/435686.

[15] B. Nanjaras, B. Panyanak, W. Phuengrattana, Fixed Point Theorems and Convergence Theorems for Suzuki-Generalized Nonexpansive Mappings in $CAT(0)$ Spaces, Nonlinear Anal.: Hybrid Syst. 4 (2010), 25–31. https://doi.org/10.1016/j.nahs.2009.07.003.

[16] H.F. Senter, W.G. Dotson, Approximating Fixed Points of Nonexpansive Mappings, Proc. Am. Math. Soc. 44 (1974), 375–380. https://doi.org/10.1090/s0002-9939-1974-0346608-8.

[17] R. Shukla, Approximating Solutions of Optimization Problems via Fixed Point Techniques in Geodesic Spaces, Axioms 11 (2022), 492. https://doi.org/10.3390/axioms11100492.

[18] R. Shukla, R. Panicker, Approximating Fixed Points of Enriched Nonexpansive Mappings in Geodesic Spaces, J. Funct. Spaces 2022 (2022), 6161839. https://doi.org/10.1155/2022/6161839.

[19] R. Shukla, R. Panicker, Some New Fixed Point Theorems for Nonexpansive-Type Mappings in Geodesic Spaces, Open Math. 20 (2022), 1246–1260. https://doi.org/10.1515/math-2022-0497.

[20] R. Shukla, R. Panicker, D. Vijayasenan, Approximating Fixed Points of α-Convex Generalized Nonexpansive Mappings, Adv. Fixed Point Theory 14 (2024), 9. https://doi.org/10.28919/afpt/8421.

[21] E.N. Sosov, On Analogues of Weak Convergence in a Special Metric Space, Izv. Vyssh. Uchebn. Zaved. Mat. 5 (2004), 84–89.

[22] T. Suzuki, Fixed Point Theorems and Convergence Theorems for Some Generalized Nonexpansive Mappings, J. Math. Anal. Appl. 340 (2008), 1088–1095. https://doi.org/10.1016/j.jmaa.2007.09.023.

[23] W. Takahashi, A Convexity in Metric Space and Nonexpansive Mappings. I., Kodai Math. J. 22 (1970), 142–149. https://doi.org/10.2996/kmj/1138846111.