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α-ADMISSIBLE E-TYPE CONTRACTIONS WITH RESPECT TO ξ AND RELATED FIXED POINT THEOREMS IN SUPRA METRIC SPACE

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Abstract. In this article, we shall define α-admissible E-type contractions with the aid of simulation function and prove related fixed point theorems in complete supra metric space. Then, we shall provide a non-trivial example to show the real existence of proved results. With the help of proved result, we solve an integral equation.

Keywords: E-type contraction; fixed point; simulation function; supra metric space.

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1. INTRODUCTION

A self map $\mathfrak{F}$ on a metric space $(\mathfrak{M}, d_m)$ is said to be contraction map if $d_m(\mathfrak{F}a, \mathfrak{F}b) \leq k.d_m(a,b)$, for $k \in [0,1)$. In 1922, Banach [12] proved that a contraction map on a complete metric space has a unique fixed point. It provides solutions to the various non-linear problems of Physics as well as other branches of science. So, it becomes the major branch in pure and applied mathematics. Then, many authors tried to generalize the Banach contraction principle for example in 2008, Dutta and Choudhury [3] extended the notion of weak contractions in complete metric space which was proved by Alber and Guerre [11] in 1997 in the setting of

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Hilbert spaces. Zhang and Song [7] proved fixed point (fix. pt.) theorems for weak ϕ- contractions. Then in 2011, Karapinar gave cyclic weak ϕ-contractions [4] and defined the weak ϕ-contractions [5] and proved the quadruple fix. pt. theorems for it. In 2021, Wangwe and Kumar [6] generalize the notion of weak contractions in partial b- metric space. Few more results on weak contractions in ([9], [10]).

In 2023, Shukla [25] defined convex orbital (α,β)-contraction mapping and proved fix. pt. theorems in the geodesic space. In the same year, Shukla and Sinkala [26] solved a matrix equation with the aid of fix. pt. theorem for convex (α,β)-generalized contraction mapping.

2. Preliminaries

In this section, we give some notions from the literature that are essential to prove the results. The notion of simulation function was given by Khojasteh *et al.* in 2015 as follows:

Definition 1. [13] *A mapping* $\xi : [0,\infty)\times[0,\infty)\to\mathbb{R}$ *satisfying the following conditions:*

(1) $\xi(0,0)=0$*;*
(2) $\xi(a,b)<b-a$ *for all* $a,b>0$*;*
(3) If $\{a_n\},\{b_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n>0$ *then* $\limsup_{n\to\infty}\xi(a_n,b_n)<0$.

is termed as simulation function.

Family of simulation functions will be symbolized by $\mathscr{Z}$. The followings are typical examples of simulation functions which are provided in ([13], [15],[19]):

(1) $\xi_1(a,b)=\psi(b)-\phi(a)$ for all $a,b\in[0,\infty)$, where $\phi,\psi:[0,\infty)\to[0,\infty)$ are two continuous functions such that $\psi(a)=\phi(a)=0$ iff $a=0$ and $\psi(a)<a\leq\phi(a)$ for all $a>0$.
(2) $\xi_2(a,b)=b-\frac{f(a,b)}{g(a,b)}a$ for all $a,b\in[0,\infty)$, where $f,g:[0,\infty)\to[0,\infty)$ are two continuous functions with respect to each variable such that $f(a,b)>g(a,b)$ for all $a,b>0$.
(3) $\xi_3(a,b)=b-\phi(b)-a$ for all $a,b\in[0,\infty)$, where $\phi:[0,\infty)\to[0,\infty)$ is continuous function such that $\psi(a)=\phi(a)=0$.
(4) $\xi_4(a,b)=\frac{b}{b+1}-a$ for all $a,b\in[0,\infty)$.

(5) $\xi_5(a,b) = \lambda . b - a$ for all $a,b \in [0,\infty), \lambda \in [0,1)$.

(6) $\xi_6(a,b) = b\phi(b) - a$ for all $a,b \in [0,\infty)$, where $\phi : [0,\infty) \to [0,1)$ is a mapping such that $\limsup_{a\to r^+} \phi(a) < 1$ for all $r > 0$.

(7) $\xi_7(a,b) = b\eta(b) - a$ for all $a,b \in [0,\infty)$, where $\eta : [0,\infty) \to [0,1)$ is an upper semi continuous mapping such that $\eta(b) < b$ for all $b > 0$ and $\eta(0) = 0$.

In 2015 Argoubi *et al.* [18] found that condition 1 of Definition 1 is not used throughout the proof. So, they redefined the simulation function by removing it. Again in 2015, Nastasi and Vetro [14] proved fix. pt. theorems with the help of simulation function. Then, Karapinar *et al.* [15] proved the coincidence point theorems using simulation function in metric space.

Definition 2. [13] *Let $\mathfrak{F}$ be a self map on a metric space $(\mathfrak{M}, d_m)$. If there exists $\xi \in \mathscr{Z}$ such that*

$$\xi(d_m(\mathfrak{F}a, \mathfrak{F}b), d_m(a,b)) \geq 0 \tag{1}$$

for all $a,b \in \mathfrak{M}$, then $\mathfrak{F}$ is said to be $\mathscr{Z}$-contraction.

As $\mathfrak{F}$ is $\mathscr{Z}$-contraction, so using the property $\xi(t,s) < s - t$ for all $t,s > 0$, we get $d_m(\mathfrak{F}a, \mathfrak{F}b) < d_m(a,b)$.
Which shows that $\mathfrak{F}$ is a contractive map. Hence, $\mathfrak{F}$ is a continuous map.
Now, we are going to state the Theorem 2.8 of [13].

Theorem 1. *Let $(\mathfrak{M}, d_m)$ be a complete metric space and $\mathfrak{F} : \mathfrak{M} \to \mathfrak{M}$ be a $\mathscr{Z}$-contraction with respect to ξ. Then, $\mathfrak{F}$ has a fix. pt. in $\mathfrak{M}$ and for every $a_0 \in \mathfrak{M}$ the Picard sequence $\{a_n\} \in \mathfrak{M}$, where $a_n = \mathfrak{F}a_{n-1}$ for all $n \in \mathbb{N}$ converges to the fix. pt. of $\mathfrak{F}$.*

In 2016, Karapinar [17] defined the α-admissible $\mathscr{Z}$-contraction as explained below:

Definition 3. [17] *Let $\mathfrak{F}$ be a self map on a metric space $(\mathfrak{M}, d_m)$. If there exists $\xi \in \mathscr{Z}$ and $\alpha : \mathfrak{M} \times \mathfrak{M} \to [0,\infty)$ such that*

$$\xi(\alpha(a,b) d_m(\mathfrak{F}a, \mathfrak{F}b), d_m(a,b)) \geq 0$$

for all $a,b \in \mathfrak{M}$, then $\mathfrak{F}$ is said to be α-admissible $\mathscr{Z}$-contraction.

Then the following theorem is proved:

Theorem 2. [17] *Let $(\mathfrak{M}, d_m)$ be a complete metric space and $\mathfrak{F} : \mathfrak{M} \to \mathfrak{M}$ be a α-admissible $\mathscr{Z}$-contraction with respect to ξ. Now, assume*

(1) $\mathfrak{F}$ is triangular α-orbital admissible;

(2) there exists $a_0 \in \mathfrak{M}$ such that $\alpha(a_0, \mathfrak{F}a_0) \geq 1$;

(3) $\mathfrak{F}$ is continuous.

Then, $\mathfrak{F}$ has a fix. pt.

In 2016, Tchier *et al.* [16] provided best approximation and variational inequality problem containing simulation function. In 2017, Radenovic *et al.* [20] proved the following fix. pt. result for two maps with the use of simulation function:

Theorem 3. [20] *Let $(\mathfrak{M}, d_m)$ be a metric space and $S, \mathfrak{F} : \mathfrak{M} \to \mathfrak{M}$ be given maps. Assume that there exists $\xi \in \mathscr{Z}$ such that*

$$\xi(d_m(\mathfrak{F}a, \mathfrak{F}b), d_m(Sa, Sb)) \geq 0, \ for \ \ all \ a, b \in \mathfrak{M}. \tag{2}$$

If $\mathfrak{F}\mathfrak{M} \subset S\mathfrak{M}$ and $\mathfrak{F}\mathfrak{M}$ or $S\mathfrak{M}$ is a complete subset of $\mathfrak{M}$, then, S and $\mathfrak{F}$ have unique point of coincidence in $\mathfrak{M}$. Moreover, if $\mathfrak{F}$ and S are weakly compatible then, S and $\mathfrak{F}$ have unique common fix. pt. in $\mathfrak{M}$.

In 2021, Karapinar *et al.* [21] proved the fix. pt. theorems by providing two types of new contractions in the setting of partial b-metric space.

In 2012, Samet *et al.* [8] gave the notion of α-admissible mappings as follows:

Definition 4. *Let $\alpha : \mathfrak{M} \times \mathfrak{M} \to [0, \infty)$. A self mapping $\mathfrak{F} : \mathfrak{M} \to \mathfrak{M}$ is called α-admissible if the condition*

$$\alpha(a, b) \geq 1 \Rightarrow \alpha(\mathfrak{F}a, \mathfrak{F}b) \geq 1,$$

is satisfied for all $a, b \in \mathfrak{M}$.

In 2014, Popescu proposed the new notion of triangular α-orbital admissible maps.

Definition 5. [24] *Let $\alpha : \mathfrak{M} \times \mathfrak{M} \to [0,\infty)$. A self mapping $\mathfrak{F} : \mathfrak{M} \to \mathfrak{M}$ is called orbital α-admissible if the condition*

$$\alpha(a,\mathfrak{F}a) \geq 1 \Rightarrow \alpha(\mathfrak{F}a,\mathfrak{F}^2a) \geq 1,$$

is satisfied for all $a,b \in \mathfrak{M}$.

Furthermore, $\mathfrak{F}$ is called triangular α-orbital admissible if $\mathfrak{F}$ is α-orbital admissible and $\alpha(a,b) \geq 1$ and $\alpha(b,\mathfrak{F}b) \geq 1 \Rightarrow \alpha(a,\mathfrak{F}b) \geq 1$.
In the spade of generalization of notion of metric space, in 2022, Berzig [1] gave the notion of supra metric space as follows:

Definition 6. [1] *Let $\mathfrak{M}$ be a non-empty set and $\sigma_s : \mathfrak{M} \times \mathfrak{M} \to [0,\infty)$ be a map such that*

(1) $\sigma_s(a,b) = 0$ iff $a = b$;
(2) $\sigma_s(a,b) = \sigma_s(b,a)$;
(3) $\sigma_s(a,b) \leq \sigma_s(a,c) + \sigma_s(c,b) + \rho.\sigma_s(a,c)\sigma_s(c,b)$;

for all $a,b,c \in \mathfrak{M}$ and $\rho \in [0,\infty)$.

Then, σ_s is called supra metric and the ordered pair $(\mathfrak{M},\sigma_s)$ is called supra metric space (s. ms.).

Example 1. [1] *Suppose $(\mathfrak{M},\sigma)$ be a metric space and α,β are positive real numbers. Then*

(1) $\sigma_{s_1}^{\alpha}(a,b) = \sigma(a,b)(\sigma(a,b)+\alpha)$ are supra metrics with constant $\rho = \frac{2}{\alpha}$.
(2) $\sigma_{s_2}^{\alpha}(a,b) = \beta(e^{\sigma(a,b)} - 1)$ are supra metrics with constant $\rho = \frac{1}{\beta}$.

But, $\sigma_{s_1}^{\alpha}$ and $\sigma_{s_2}^{\beta}$ are not necessary usual metrics. For example for $\alpha = 2$ and $\sigma(a,b) = |a-b|$ $\sigma_{s_1}^2(4,0) > \sigma_{s_1}^2(4,2) + \sigma_{s_1}^2(2,0)$.

Definition 7. [1] *Let $(\mathfrak{M},\sigma_s)$ be a s. ms. The set*

$$B(a_0,r) := \{a \in \mathfrak{M} : \sigma_s(a_0,a) < r\},$$

where $r > 0$ and $a_0 \in \mathfrak{M}$, is called open of radius r and center a_0. A subset $\mathfrak{N}$ of $\mathfrak{M}$ is called open if for any point $a \in \mathfrak{N}$ there exists $r > 0$ such that $B(a,r) \subset \mathfrak{N}$. The family of all open subsets of $\mathfrak{M}$ will be denoted by τ.

Proposition 1. [1] *Let $(\mathfrak{M}, \sigma_s)$ be a s. ms. Then, each open ball is an open set.*

Definition 8. [1] *Let $(\mathfrak{M}, \sigma_s)$ be a s. ms. A sequence $\{a_n\} \in \mathfrak{M}$ converges to a if for all $\varepsilon > 0$ the ball $B(a, \varepsilon)$ contains all but a finite numbers of terms of the sequence. So, a is the limit of the sequence and we say that*
$\lim_{n\to\infty} \sigma_s(a_n, a) = 0$.

Proposition 2. [1] *Let $(\mathfrak{M}, \sigma_s)$ be a s. ms. If the sequence $\{a_n\}$ has a limit, then it is unique.*

Definition 9. [1] *Let $(\mathfrak{M}, \sigma_s)$ be a s. ms. A mapping $\mathfrak{F} : \mathfrak{M} \to \mathfrak{M}$ is said to be continuous at a if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\sigma_s(\mathfrak{F}b, \mathfrak{F}a) < \varepsilon$ whenever $\sigma_s(b, a) < \delta$. If $\mathfrak{F}$ is continuous at all the points of $\mathfrak{M}$ then, $\mathfrak{F}$ is continuous on $\mathfrak{M}$.*

Proposition 3. [1] *Let $(\mathfrak{M}, \sigma)$ be a metric space and $(\mathfrak{M}, \sigma_s)$ be a s. ms. Let $f : [0, \infty) \to [0, \infty)$ be a homeomorphism and $\sigma_s = f o \sigma$, then*

(1) A continuous mapping with respect to σ is continuous with respect to σ_s.
(2) A convergent sequence with respect to σ converges with respect to σ_s to the same point.

Definition 10. [1] *Let $(\mathfrak{M}, \sigma_s)$ be a s. ms. A sequence $\{a_n\} \in \mathfrak{M}$ is a Cauchy sequence if for all $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\sigma_s(a_n, a_m) < \varepsilon$ for all $m, n \geq k$.*

A supra metric is complete if every Cauchy sequence is convergent.

Proposition 4. [1] *Let $(\mathfrak{M}, \sigma)$ be a s. ms. Then the set $\mathfrak{M}$ with respect to supra metrics in Example 1 is complete s. ms.*

Remark 1. [1] *If a sequence $\{a_n\} \in \mathfrak{M}$ is a Cauchy sequence in $\mathfrak{M}$ then there exists $a \in \mathfrak{M}$ such that $\lim_{n\to\infty} \sigma_s(a_n, a) = 0$ and by property (σ_3) every subsequence converges to a.*

Lemma 1. [1] *Every supra metric is continuous.*

In 2023, Panda *et al.* [22] introduced the new notion of extended s. ms. and proved stone type theorems in it. Then in 2024, Panda *et al.* [23] again gave the new notion of complex valued s. ms. and proved Banach analogue in this setting also they solved volterra-Fredholm integral equations with it. In the same year Berzig [2] proved the fix. pt. results for non-linear contractions in the setting of supra b- metric space.

3. Main Results

In this section, we present our findings.

Definition 11. *Suppose $\mathfrak{F}$ be a self map on a s. ms. $(\mathfrak{M},\sigma_s)$. If there exists $\xi \in \mathscr{Z}$ and $\alpha : \mathfrak{M}\times\mathfrak{M} \to [0,\infty)$ such that*

$$\xi(\alpha(a,b)\sigma_s(\mathfrak{F}a,\mathfrak{F}b),E(a,b)) \geq 0, \tag{3}$$

where

$E(a,b) = \sigma_s(a,b)+|\sigma_s(a,\mathfrak{F}a)-\sigma_s(b,\mathfrak{F}b)|$, for all $a,b\in\mathfrak{M}$.

Then, we say that $\mathfrak{F}$ is an α-admissible E-type $\mathscr{Z}$ contraction with respect to ξ.

If $\alpha(a,b) = 1$, then $\mathfrak{F}$ is said to be E-type $\mathscr{Z}$ contraction with respect to ξ.

Remark 2. *If $\mathfrak{F}$ is an α-admissible E-type $\mathscr{Z}$ contraction with respect to ξ then*

$$\alpha(a,b)\sigma_s(\mathfrak{F}a,\mathfrak{F}b) < E(a,b), \tag{4}$$

for all $a,b\in\mathfrak{M}$.

Proof. With the help of condition 2 of Definition 1, we get

$$\begin{aligned} 0 &\leq \xi(\alpha(a,b)\sigma_s(\mathfrak{F}a,\mathfrak{F}b),E(a,b)),\\ &< E(a,b)-\alpha(a,b)\sigma_s(\mathfrak{F}a,\mathfrak{F}b),\\ \alpha(a,b)\sigma_s(\mathfrak{F}a,\mathfrak{F}b) &< E(a,b). \end{aligned}$$

So, the proof is done. □

Now, we are going to state and prove our main result.

Theorem 4. *Suppose $(\mathfrak{M},\sigma_s)$ be a complete s. ms. and $\mathfrak{F}$ is an α-admissible E-type $\mathscr{Z}$ contraction with respect to ξ. Assume*

(1) $\mathfrak{F}$ is α-admissible;

(2) there exists $a_0\in\mathfrak{M}$ such that $\alpha(a_0,\mathfrak{F}a_0)\geq 1$;

(3) $\mathfrak{F}$ is continuous.

Then, $\mathfrak{F}$ possesses a unique fix. pt.

Proof. Since there exists $a_0 \in \mathfrak{M}$ such that $\alpha(a_0, \mathfrak{F}a_0) \geq 1$, so, we can construct a Picard sequence in such a way that $\mathfrak{F}a_p = a_{p+1}$ for all $p \in \mathbb{N} \cup \{0\}$. If there exist p_0 such that $a_{p_0} = a_{p_0+1}$ then, $a_{p_0} = \mathfrak{F}a_{p_0}$. This implies that a_{p_0} act as a fix. pt. and proof is done.
So, we assume that $a_p \neq a_{p+1}$, hence $\sigma_s(a_p, a_{p+1}) > 0$ for all p. From the α-admissibility of $\mathfrak{F}$, we found that $\alpha(a_0, a_1) = \alpha(a_0, \mathfrak{F}a_0) \geq 1 \Rightarrow \alpha(\mathfrak{F}a_0, \mathfrak{F}a_1) = \alpha(a_1, a_2) \geq 1$.
Recursively, we found that

$$\alpha(a_p, a_{p+1}) \geq 1, \tag{5}$$

for all p.

From equations (3) and (5), we found that

$$\begin{aligned} o &\leq \xi(\alpha(a_{p-1}, a_p)\sigma_s(\mathfrak{F}a_{p-1}, \mathfrak{F}a_p), E(a_{p-1}, a_p)), \\ &< E(a_{p-1}, a_p) - \alpha(a_{p-1}, a_p)\sigma_s(\mathfrak{F}a_{p-1}, \mathfrak{F}a_p), \end{aligned}$$

$$\alpha(a_{p-1}, a_p)\sigma_s(\mathfrak{F}a_{p-1}, \mathfrak{F}a_p) \leq E(a_{p-1}, a_p). \tag{6}$$

$E(a_{p-1}, a_p) = \sigma_s(a_{p-1}, a_p) + |\sigma_s(a_{p-1}, Ta_{p-1}) - \sigma_s(a_p, \mathfrak{F}a_p)| = \sigma_s(a_{p-1}, a_p) + |\sigma_s(a_{p-1}, a_p) - \sigma_s(a_p, a_{p+1})|$.
If possible suppose that $\sigma_s(a_{p-1}, a_p) < \sigma_s(a_p, a_{p+1})$, then above becomes
$E(a_{p-1}, a_p) = \sigma_s(a_{p-1}, a_p) - \sigma_s(a_{p-1}, a_p) + \sigma_s(a_p, a_{p+1}) = \sigma_s(a_p, a_{p+1})$.
Equation (6) implies that

$$\sigma_s(a_p, a_{p+1}) \leq \alpha(a_{p-1}, a_p)\sigma_s(a_p, a_{p+1}) < \sigma_s(a_p, a_{p+1}),$$

for all p, which is a contradiction.

Thus, $\sigma_s(a_{p-1}, a_p) > \sigma_s(a_p, a_{p+1})$, for all p.

So, the sequence $\{\sigma_s(a_p, a_{p+1})\}$ is a monotonically decreasing sequence of positive real numbers that is bounded below by 0. Hence, we deduce that the sequence $\{\sigma_s(a_p, a_{p+1})\}$ converges to $a^* \geq 0$.

From this discussion, we can conclude that

$$\lim_{p \to \infty} E(a_p, a_{p+1}) = \lim_{p \to \infty} (2\sigma_s(a_{p-1}, a_p) - \sigma_s(a_p, a_{p+1})) = a^*. \tag{7}$$

Since,

$$\sigma_s(a_p,a_{p+1}) \leq \alpha(a_{p-1},a_p)\sigma_s(a_p,a_{p+1}) < \sigma_s(a_{p-1},a_p),$$

By taking limit as $p \to \infty$, we found that

$$\lim_{p\to\infty} \alpha(a_{p-1},a_p)\sigma_s(a_p,a_{p+1}) = a^*. \tag{8}$$

Now, taking $s_p = \alpha(a_{p-1},a_p)\sigma_s(a_p,a_{p+1}), t_p = \sigma_s(a_{p-1},a_p)$ and using Definition 1, we have

$$0 \leq \limsup_{p\to\infty} \xi(\alpha(a_{p-1},a_p)\sigma_s(a_p,a_{p+1}),\sigma_s(a_{p-1},a_p)) < 0 \tag{9}$$

which is a contradiction. Thus, we have $a^* = 0$.

Next we want to prove that $\{a_p\}$ is a Cauchy sequence. Otherwise there exist $\varepsilon > 0$ and subsequence $\{a_{p_r}\}$ and $\{a_{p_r}\}$ of $\{a_p\}$ such that for every positive integer r with $p_r > q_r > r$,

$$\sigma_s(a_{p_r},a_{q_r}) \geq \varepsilon, \tag{10}$$

and we have

$$\sigma_s(a_{p_r},a_{q_r-1}) < \varepsilon.$$

Therefore, using Definition 5 and inequality (10) for all $r \in \mathbb{N}$, we have

$$\begin{aligned} \sigma_s(a_{p_r},a_{q_r}) &\leq \sigma_s(a_{p_r},a_{q_r-1}) + \sigma_s(a_{q_r-1},a_{q_r}) + \rho\sigma_s(a_{q_r},a_{q_r-1})\sigma_s(a_{q_r-1},a_{q_r}) \\ &\leq \varepsilon + \sigma_s(a_{q_r-1},a_{q_r}) + \rho\varepsilon\sigma_s(a_{q_r-1},a_{q_r}), \end{aligned}$$

letting $r \to \infty$ in the above inequality and using equation (8), we obtain

$$\lim_{r\to\infty} \sigma_s(a_{p_r},a_{q_r}) = \varepsilon. \tag{11}$$

In addition, by using Definition 5, we have

$$\begin{aligned} \sigma_s(a_{p_r},a_{q_r}) &\leq \sigma_s(a_{q_r},a_{q_r+1}) + \sigma_s(a_{q_r+1},a_{p_r}) + \rho\sigma_s(a_{q_r},a_{q_r+1})\sigma_s(a_{q_r+1},a_{p_r}) \\ &\leq \sigma_s(a_{q_r},a_{q_r+1}) + \sigma_s(a_{q_r+1},a_{p_r+1}) + \sigma_s(a_{p_r+1},a_{p_r}) \\ &\quad +\rho\sigma_s(a_{q_r+1},a_{p_r+1})\sigma_s(a_{p_r},a_{p_r+1}) + \rho\sigma_s(a_{q_r},a_{q_r+1})\sigma_s(a_{q_r+1},a_{p_r}) \\ &\leq \sigma_s(a_{q_r},a_{q_r+1}) + \sigma_s(a_{q_r+1},a_{p_r+1}) + \sigma_s(a_{p_r+1},a_{p_r}) \\ &\quad +\rho\sigma_s(a_{q_r+1},a_{p_r+1})\sigma_s(a_{p_r},a_{p_r+1}) + \rho\sigma_s(a_{q_r},a_{q_r+1})(\sigma_s(a_{q_r},a_{q_r+1}) \\ &\quad +\sigma_s(a_{q_r},a_{p_r}) + \rho\sigma_s(a_{q_r},a_{q_r+1})\sigma_s(a_{q_r},a_{p_r})) \end{aligned}$$

or equivalently,

$$\begin{aligned}((1-\rho\sigma_s(a_{q_r},a_{q_r+1})-\rho^2\sigma_s(a_{q_r},a_{q_r+1})^2)\sigma_s(a_{q_r},a_{p_r}) &- \sigma_s(a_{q_r},a_{q_r+1})-\sigma_s(a_{p_r+1},a_{p_r})\\ &- \rho\sigma_s(a_{q_r},a_{q_r+1})^2)(1+\rho\sigma_s(a_{p_r},a_{p_r+1}))^{-1}\\ &\leq \sigma_s(a_{q_r+1},a_{p_r+1}).\end{aligned}$$

and

$$\begin{aligned}\sigma_s(a_{q_r+1},a_{p_r+1}) &\leq \sigma_s(a_{q_r+1},a_{q_r})+\sigma_s(a_{q_r},a_{p_r+1})+\rho\sigma_s(a_{q_r},a_{q_r+1})\sigma_s(a_{q_r},a_{p_r+1})\\ &\leq \sigma_s(a_{q_r+1},a_{q_r})+\sigma_s(a_{q_r},a_{p_r})+\sigma_s(a_{p_r},a_{p_r+1})\\ &\quad +\rho\sigma_s(a_{q_r},a_{p_r})\sigma_s(a_{p_r},a_{p_r+1})+\rho\sigma_s(a_{q_r},a_{q_r+1})\sigma_s(a_{q_r},a_{p_r+1})\\ &\leq \sigma_s(a_{q_r+1},a_{q_r})+\sigma_s(a_{q_r},a_{p_r})+\sigma_s(a_{p_r},a_{p_r+1})\\ &\quad +\rho\sigma_s(a_{q_r},a_{p_r})\sigma_s(a_{p_r},a_{p_r+1})+\rho\sigma_s(a_{q_r},a_{q_r+1})(\sigma_s(a_{q_r},a_{q_r+1})\\ &\quad +\sigma_s(a_{q_r+1},a_{p_r+1})+\rho\sigma_s(a_{q_r},a_{q_r+1})\sigma_s(a_{q_r+1},a_{p_r+1}))\end{aligned}$$

or equivalently,

$$\begin{aligned}((1-\rho\sigma_s(a_{q_r},a_{q_r+1}) &- \rho^2\sigma_s(a_{q_r},a_{q_r+1})^2)\sigma_s(a_{q_r+1},a_{p_r+1})\\ &\leq \sigma_s(a_{q_r},a_{q_r+1})+\sigma_s(a_{p_r+1},a_{p_r+1})+\rho\sigma_s(a_{q_r},a_{q_r+1})^2)\\ &+ (1+\rho\sigma_s(a_{p_r},a_{p_r+1}))\sigma_s(a_{q_r},a_{p_r}).\end{aligned}$$

Therefore letting $r\to\infty$ in the above inequalities (8) and (11), we show that

$$\lim_{r\to\infty}\sigma_s(a_{p_r+1},a_{q_r+1})=\varepsilon. \tag{12}$$

Since, $\mathfrak{F}$ is triangular α-orbital admissible, we have

$$\alpha(a_{p_r},a_{q_r})\geq 1, \tag{13}$$

for all $p,q\in\mathbb{N}$.

Now, by equations (3) and (13), we have

$$\begin{aligned}0 &\leq \xi(\alpha(a_{p_r},a_{q_r})\sigma_s(\mathfrak{F}a_{p_r},\mathfrak{F}a_{q_r}),\sigma_s(a_{p_r},a_{q_r})),\\ &< \sigma_s(a_{p_r},a_{q_r})-\alpha(a_{p_r},a_{q_r})\sigma_s(\mathfrak{F}a_{p_r},\mathfrak{F}a_{q_r}),\end{aligned}$$

$$\begin{aligned} \alpha(a_{p_r},a_{q_r})\sigma_s(\mathfrak{F}a_{p_r},\mathfrak{F}a_{q_r}) &< \sigma_s(a_{p_r},a_{q_r}), \\ \sigma_s(a_{p_r+1},a_{q_r+1}) &\leq \alpha(a_{p_r},a_{q_r})\sigma_s(a_{p_r+1},a_{q_r+1}) < \sigma_s(a_{p_r},a_{q_r}). \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \alpha(a_{p_r},a_{q_r})\sigma_s(a_{p_r+1},a_{q_r+1}) = \varepsilon. \tag{14}$$

Choose $a_n = \alpha(a_{p_r},a_{q_r})\sigma_s(a_{p_r+1},a_{q_r+1})$ and $b_n = \sigma_s(a_{p_r},a_{q_r})$, by the condition (3) of Definition 1, we have

$$o \leq \limsup_{n\to\infty} \xi(\alpha(a_{p_r},a_{q_r})\sigma_s(a_{p_r+1},a_{q_r+1}),\sigma_s(a_{p_r},a_{q_r})) < 0,$$

a contradiction.

So, $\varepsilon = 0$.

Hence, $\{a_n\}$ is a Cauchy sequence. Considering the fact that $\mathfrak{M}$ is a complete s. ms. there exists $a \in \mathfrak{M}$ such that

$$\lim_{n\to\infty} \sigma_s(a_n,a) = 0. \tag{15}$$

By the hypothesis, $\mathfrak{F}$ is continuous, so

$$\lim_{n\to\infty} \sigma_s(\mathfrak{F}a_n,\mathfrak{F}a) = \lim_{n\to\infty} \sigma_s(a_{n+1},\mathfrak{F}a) = 0. \tag{16}$$

Uniqueness of limit from equations (15) and (16), we have

$\mathfrak{F}a = a.$ □

Theorem 5. *If a,b are fix. pt. of $\mathfrak{F}$ as in Theorem 8, such that $\alpha(a,b) \geq 1$, then the fix. pt. is unique.*

Proof. To prove uniqueness if possible suppose that a,b are two distinct fix. pt. of $\mathfrak{F}$, then equation (4) implies that

$$\sigma_s(a,b) = \sigma_s(\mathfrak{F}a,\mathfrak{F}b) \leq \alpha(a,b)\sigma_s(\mathfrak{F}a,\mathfrak{F}b) < \sigma_s(a,b), \tag{17}$$

which holds when $a = b$. □

Example 2. *Suppose* $\mathfrak{M} = [0,1] \cup \{2\}$ *and define* $\sigma_s : \mathfrak{M} \times \mathfrak{M} \to [0,\infty)$ *as* $\sigma_s(a,b) = |a-b|(|a-b| + \frac{1}{5})$.

By Example 1, $(\mathfrak{M}, \sigma_s)$ *is a complete s. ms.*

Define, $\mathfrak{F} : \mathfrak{M} \to \mathfrak{M}$ *as*

$$\mathfrak{F}a = \left\{ \begin{array}{ll} \frac{a}{2}, & \text{if } a \in [0,1] \\ 5, & \text{if } a = 2 \end{array} \right\}$$

which is a continuous map. And

$$\alpha(a,b) = \left\{ \begin{array}{ll} 1, & \text{if } a,b \in [0,1] \\ 0, & \text{if otherwise} \end{array} \right\}$$

Suppose $\xi(a,b) = \frac{2}{3}b - a$.

Taking $a = 1, b = 2$, $\mathfrak{F}a = \frac{1}{2}, \mathfrak{F}b = 5$.

Now,

$\sigma_s(\mathfrak{F}a, \mathfrak{F}b) = \sigma_s(\frac{1}{2}, 5) = |\frac{1}{2} - 5|(|\frac{1}{2} - 5| + \frac{1}{5}) = \frac{423}{20}$,

and

$\sigma_s(a,b) = |1-2|(|1-2| + \frac{1}{5}) = \frac{6}{5}$.

There does not exists any $k \in [0,1)$ *such that*

$\frac{423}{20} \leq k.\frac{6}{5}$.

So, $\mathfrak{F}$ *is not a contraction.*

Now, we prove the inequality (3).

Suppose either a or $b = 2$, *then* $\alpha(a,b) = 0$.

Then, $\xi(\alpha(a,b)\sigma_s(\mathfrak{F}a, \mathfrak{F}b), E(a,b)) = \frac{2}{3}E(a,b) - \alpha(a,b)\sigma_s(\mathfrak{F}a, \mathfrak{F}b) = E(a,b) \geq 0$.

So, inequality (3) holds.

The remaining case is $a,b \in [0,1]$, *then*

$$\begin{array}{rcl} \xi(\alpha(a,b)\sigma_s(\mathfrak{F}a,\mathfrak{F}b), E(a,b)) & = & \frac{2}{3}.E(a,b) - \alpha(a,b)\sigma_s(\mathfrak{F}a,\mathfrak{F}b), \\ & = & \frac{2}{3}(|a-b|(|a-b| + \frac{1}{5}) + |\sigma_s(a,\mathfrak{F}a) - \sigma_s(b,\mathfrak{F}b)|), \\ & - & \frac{1}{2}(|a-b|)(\frac{1}{2}|a-b| + \frac{1}{5}), \\ \frac{2}{3}(|a-b|(|a-b| + \frac{1}{5}) + |\sigma_s(a,\mathfrak{F}a) - \sigma_s(b,\mathfrak{F}b)|) & > & \frac{1}{2}(|a-b|)(\frac{1}{2}|a-b| + \frac{1}{5}), \\ \xi(\alpha(a,b)\sigma_s(\mathfrak{F}a,\mathfrak{F}b), E(a,b)) & \geq & 0. \end{array}$$

Hence, inequality (3) holds.

Thus, all the conditions of Theorems 8 and 9 are satisfied. So, $\mathfrak{F}$ has a unique fix. pts. Clearly, 0 is the unique fix. pt.

Now, we discuss about some consequences of our proved results of Theorems 8 and 9.

4. Consequences

Consequence 1. *Suppose $(\mathfrak{M}, \sigma_s)$ be a complete s. ms. and $\mathfrak{F}$ is an E-type $\mathscr{Z}$ contraction with respect to ξ, then $\mathfrak{F}$ has a unique fix. pt.*

Consequence 2. *Suppose $(\mathfrak{M}, \sigma_s)$ be a complete s. ms. and $\mathfrak{F}$ be a continuous triangular α-orbital admissible self map such that*

$$\phi(\alpha(a,b)\sigma_s(\mathfrak{F}a, \mathfrak{F}b)) \leq \psi(E(a,b)), \tag{18}$$

where
$E(a,b) = \sigma_s(a,b) + |\sigma_s(a, \mathfrak{F}a) - \sigma_s(b, \mathfrak{F}b)|$ *for all* $a, b \in \mathfrak{M}$, $\xi(a,b) = \xi_1(a,b)$ *and* $\phi, \psi : [0,\infty) \to [0,\infty)$ *are two continuous functions such that* $\psi(t) = \phi(t) = 0$ *iff* $t = 0$ *and* $\psi(t) < t \leq \phi(t)$ *for all* $t > 0$. *Then,* $\mathfrak{F}$ *has a unique fix. pt.*

Consequence 3. *Suppose $(\mathfrak{M}, \sigma_s)$ be a complete s. ms. and $\mathfrak{F}$ be a self map such that*

$$\phi(\sigma_s(\mathfrak{F}a, \mathfrak{F}b)) \leq \psi(E(a,b)), \tag{19}$$

where
$E(a,b) = \sigma_s(a,b) + |\sigma_s(a, \mathfrak{F}a) - \sigma_s(b, \mathfrak{F}b)|$ *for all* $a, b \in \mathfrak{M}$ *and* $\phi, \psi : [0,\infty) \to [0,\infty)$ *are two continuous functions such that* $\psi(t) = \phi(t) = 0$ *iff* $t = 0$ *and* $\psi(t) < t \leq \phi(t)$ *for all* $t > 0$. *Then,* $\mathfrak{F}$ *has a unique fix. pt.*

5. Application

In this section, we solve an integral equation as an application of proved result. Consider the integral equation

$$y(t) = x(t) + \lambda . \int_a^b H(t, \varpi) f(\varpi, y(\varpi)) d\varpi, \tag{20}$$

$t \in [a,b]$, where $\lambda \geq 0$ and $x : [a,b] \to \mathbb{R}, H : [a,b] \times [a,b] \to \mathbb{R}$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ are the given continuous functions.

Suppose Γ denotes the set of all continuous functions on $[a,b]$. Define the supra metric σ_s on $\mathfrak{M} = \Gamma$ as $\sigma_s(y_1(t), y_2(t)) = \sup_{t\in[a,b]} 2(e^{|y_1(t)-y_2(t)|} - 1)$.

(Γ, σ_s) is complete s. ms.(By Example 1).

Theorem 6. *The equation (20) has a unique solution on Gamma if*

(1) $\sup_{t\in[a,b]} \int_a^b H(t,\varpi)d\varpi \leq \log_e(\frac{1}{\pi.\lambda})$,

(2) $|f(\varpi, y_1(\varpi)) - f(\varpi, y_2(\varpi)| \leq |y_1(t) - y_2(t)| + ||y_1(t) - \mathfrak{F}y_1(t)| - |y_2(t) - \mathfrak{F}y_2(t)||$.

Proof. Define a self map

$\mathfrak{F} : \Gamma \to \Gamma$ as $\mathfrak{F}y(t) = x(t) + \lambda. \int_a^b H(t,\varpi)f(\varpi, y(\varpi))d\varpi$.

$$
\begin{aligned}
\sigma_s(\mathfrak{F}y_1(t), Ty_2(t)) &= \sup_{t\in[a,b]} 2(e^{|\mathfrak{F}y_1(t)-\mathfrak{F}y_2(t)|} - 1), \\
&= \sup_{t\in[a,b]} 2(e^{|x(t)+\lambda.\int_a^b H(t,\varpi)f(\varpi,y_1(\varpi))d\varpi-(x(t)+\lambda.\int_a^b H(t,\varpi)f(\varpi,y_2(\varpi))d\varpi}) | - 1), \\
&= \sup_{t\in[a,b]} 2(e^{|x(t)+\lambda.\int_a^b H(t,\varpi)f(\varpi,y_1(\varpi))d\varpi-(x(t)+\lambda.\int_a^b H(t,\varpi)f(\varpi,y_2(\varpi))d\varpi)|} - 1), \\
&\leq \sup_{t\in[a,b]} 2(e^{\lambda.\int_a^b H(t,\varpi)|f(\varpi,y_1(\varpi))-f(\varpi,y_2(\varpi)|d\varpi} - 1), \\
&\leq \frac{1}{\pi}\sup_{t\in[a,b]} 2(e^{(|y_1(t)-y_2(t)|+||y_1(t)-Ty_1(t)|-|y_2(t)-Ty_2(t)||)} - 1), \\
&= \frac{1}{\pi}\sigma_s(y_1(t), y_2(t)).
\end{aligned}
$$

which can be obtained by taking $\phi(t) = t$ and $\psi(t) = \frac{1}{\pi}t$ in the Consequence 3.

So, $\mathfrak{F}$ has a unique fix. pt. (by Consequence 3).

This implies that equation (20) has a unique solution. □

6. Conclusion

In this article, we have defined new notion of α-admissible mappings of type-E with respect to simulation function ξ and proved fix. pt. theorem for it. We have discussed an example to validate the results. Finally, we discuss some consequences as well as an application in the form of solution of an integral equation.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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