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INNOVATIVE FIXED POINT RESULTS ON ISTRATESCU TYPE CONTRACTIONS WITH SIMULATION FUNCTION IN $b-$METRIC SPACES AND APPLICATIONS

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Abstract. In the present study, we extend the utilization of contractions to $b-$metric spaces by utilizing simulation functions. We will conduct research into the Istratescu type contractions, a type of contractions that have simulation functions produced in them, and we will demonstrate new fixed point solutions for this type. By doing this, we want to provide further light on the characteristics of $b-$metric spaces and contribute to the continued improvement of fixed point theory in this field. The results of this study are significant because they are supported by examples that are presented, giving concrete proof of their practical applications in the solution of exploring nonlinear integral equations of Caputo-Type and nonlinear fractional differential equations. This use case emphasizes the ability to change and the significance of our approach in solving complex mathematical problems in a variety of fields.

Keywords: b-metric space; simulation function; fixed point; integral equation; Caputo class derivative.

2020 AMS Subject Classification: 47H09, 47H10.

1. Introduction

Fixed point theory (FPT) finds widespread utility across various scientific and engineering fields, spanning physics, economics, and computer science. In physics, it aids in analyzing the dynamics of systems and modeling physical phenomena. In economics, FPT contributes to understanding market dynamics and devising strategies in game theory. In computer science,

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it facilitates the evaluation of program behavior and algorithmic validation. FPT's versatility extends to integral and differential equations, making it indispensable for solving intricate problems in modern research. Its broad applicability underscores its importance as a fundamental tool in scientific and engineering endeavors. such as [3, 4, 5, 14, 21, 22, 23, 24, 25, 26, 27, 28].

The $b-$metric spaces ($b-$MS) defined Bakhtin [6] and Czerwik [9] are a generalization of metric spaces(MS) that allow the distance function to take values from a partially ordered set rather than non-negative real integers. This generalization can be useful when modeling problems with qualitative or insufficient information. The distance function in a $b-$MS might represent a notion of "closeness" that is not always an accurate numerical distance which is explained with [1, 2, 16, 20, 32].

The simulation functions declared by Khojasteh et al. [15] are mappings utilized for modeling and approximating complex systems. They can be used to replicate the behavior of a system under various settings and provide significant insights into the system's behavior. In the framework of fixed point theory, simulation functions are a valuable tool for extending the concept of contractions to more general spaces, such as $b-$MS. This is accomplished by incorporating the simulation function into the contraction mapping, resulting in a new type of mapping known as a simulation contraction. Simulation contractions can be used to demonstrate novel fixed point results in $b-$MS. One example of this use is traffic flow analysis. A $b-$MS can be used to describe traffic flow, where the distance between two points indicates the trip time between those two points. To generate a simulation contraction, a simulation function is inserted within a contraction mapping and utilized to reproduce the traffic flow. This simulation contraction's fixed point corresponds to a steady-state traffic flow.

In this research, we extend the concept of contractions to $b-$MS with simulation functions and prove new fixed point results for a class of contractions called Istratescu type contractions embedded with simulation function.

2. Preliminaries

Let's begin with foundational concepts that are of significance.

Definition 2.1. *([6],[9]) Consider a nonempty set Λ and a real number $t \geq 1$. A function $d_b : \Lambda \times \Lambda \to [0,\infty)$ is defined to be a $b-$MS on Λ if the following conditions are satisfied for any $o,\omega,z \in \Lambda$:*

(1) $d_b(o,\omega) = 0$ *if and only if* $o = \omega$,

(2) $d_b(o,\omega) = d_b(\omega,o)$,

(3) $d_b(o,z) \leq t[d_b(o,\omega) + d_b(\omega,z)]$.

The triplet (Λ, d_b, t) is called a $b-$MS.

A $b-$MS is a generalization of a MS, which replaces the triangle inequality with a more flexible inequality that includes a parameter t. This parameter t is a real number between 0 and 1, and it determines the strength of the triangle inequality. When t=1, the b-MS is identical to an MS, and all of the typical MS qualities, such as completeness and compactness, apply. While $b-$MS may have features that do not hold in MS for t values between 0 and 1. In a b-MS, for example, the distance between two unique spots maybe 0. In an MS, this is not the case. As a result, $b-$MS allows for more general results in mathematics than MS. Researchers can gain an improved knowledge of fixed points and contractions in more general spaces by studying $b-$MS. Furthermore, $b-$MS are utilized to simulate real-world systems that are unsuitable under the typical MS framework. As a result, the study of $b-$MS is an effective tool for mathematical modeling and analysis in a variety of contexts.

Example 2.1. *Suppose we have the set $\Lambda = [0,1]$ with the function $d_b : \Lambda \times \Lambda \to [0,\infty)$ given by $d_b(o,\omega) = |o-\omega|^2$ for all $o,\omega \in \Lambda$. It's evident that $(\Lambda, d, 3)$ qualifies as a $b-$MS, yet it does not meet the criteria for being a MS.*

Example 2.2. *Suppose we have the $b-$MS defined by $d_b(o,\omega) = (\frac{|o-\omega|}{1+|o-\omega|})^t$ for all o,ω in Λ, where Λ denotes the set of all non-negative real numbers. This $b-$MS on Λ, with t ranging from 0 to 1, serves as a crucial tool in the analysis of sequence convergence.*

In each of these examples, the $b-$metric is a generalization of the standard metric on Λ, with the triangle inequality replaced with a more flexible inequality involving the parameter t. This flexibility allows for a broader range of possible features and behaviors in the space Λ, as well as the study of fixed points and contractions in more general spaces.

Consider (Λ, d_b, s) as a $b-$MS. The following concepts naturally stem from their metric equivalents.

(i) Convergence: $o_n \subseteq \Lambda$ converges to $o \in \Lambda$ if $\lim_{n\to\infty} d_b(o_n, o) = 0$.

(ii) Cauchy Sequence: o_n is a Cauchy sequence if, for every given $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that $d_b(o_n, o_m) < 0$ for every $m, n \geq n(\varepsilon)$.

(iii) Completeness: A $b-$MS (Λ, d, s) is considered complete if each Cauchy sequence converges to a certain $o \in \Lambda$.

Lemma 2.1. [10] *Let (Λ, d_b, s) be a $b-MS$ with $s \geq 1$, and let o_n and ω_n be sequences $b-$converging to o, ω, respectively. Then we have*

$$\frac{1}{s} d_b(o, \omega) \quad \leq \lim_{n\to\infty} \inf d_b(o_n, \omega_n) \leq \lim_{n\to\infty} \sup d_b(o_n, \omega_n) \leq s^2 d_b(o, \omega),$$

particularly, if $o = \omega$, then we have $\lim_{n\to\infty} d_b(o_n, \omega_n) = 0$. *Furthermore, for every $z \in \Lambda$, we obtain,*

$$\frac{1}{s} d_b(o, \omega) \leq \lim_{n\to\infty} \inf d_b(o_n, \omega) \leq \lim_{n\to\infty} \sup d_b(o_n, \omega) \leq s^2 d_b(o, \omega).$$

Lemma 2.2. [10] *Let (Λ, d_b, s) be a $b-MS$ with $s \geq 1$, and suppose that $\{o_n\}$ is a sequence in Λ such that*

$$\lim_{n\to\infty} d_b(o_n, o_{n+1}) = 0.$$

If $\{o_n\}$ is not a $b-$Cauchy sequence, then for $\varepsilon > 0$ and two sequences of positive integers, $\{o_{m(k)}\}$ and $\{o_{n(k)}\}$, there exist four sequences satisfying:

$$\begin{aligned}
\varepsilon \leq & \quad \lim_{n\to\infty} \inf d_b(o_{m(k)}, o_{n(k)}) \leq \lim_{n\to\infty} \sup d_b(o_{m(k)}, o_{n(k)}) && \leq s\varepsilon \\
\frac{\varepsilon}{s} \leq & \quad \lim_{n\to\infty} \inf d_b(o_{m(k)}, o_{n(k)+1}) \leq \lim_{n\to\infty} \sup d_b(o_{m(k)}, o_{n(k)+1}) && \leq s^2\varepsilon \\
\frac{\varepsilon}{s} \leq & \quad \lim_{n\to\infty} \inf d_b(o_{m(k)+1}, o_{n(k)}) \leq \lim_{n\to\infty} \sup d_b(o_{m(k)+1}, o_{n(k)}) && \leq s^2\varepsilon \\
\frac{\varepsilon}{s^2} \leq & \lim_{n\to\infty} \inf d_b(o_{m(k)+1}, o_{n(k)+1}) \leq \lim_{n\to\infty} \sup d_b(o_{m(k)+1}, o_{n(k)+1}) && \leq s^3\varepsilon
\end{aligned}$$

Definition 2.2. [15] *A function $\Xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is considered a simulation function (SF) if it satisfies the following conditions:*

(1) $\Xi(0,0) = 0$;

(2) $\Xi(t,s) < s - t$ *for all* $t,s > 0$*;*

(3) For sequences $\{t_n\}$ *and* $\{s_n\}$ *in* $(0,\infty)$ *converging to* $\ell \in (0,\infty)$*, we have:*

$$\lim_{n\to\infty} \sup \Xi(t_n, s_n) < 0.$$

Example 2.3. [15] *Let* $\phi_i : [0,\infty) \to [0,\infty)$ *be continuous functions for* $i = 1,2,3$ *satisfying* $\phi_i(t) = 0$ *if and only if* $t = 0$*. Define the functions* $\Xi_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *as follows:*

(1) For every $t,s \in [0,\infty)$*, set* $\Xi_1(t,s) = \phi_1(s) - \phi_2(t)$*, where* $\phi_1(t) < t \le \phi_2(t)$ *for each* $t > 0$*.*

(2) Define $\Xi_2(t,s) = s - \frac{f(t,s)}{g(t,s)}$ *for all* $t,s \in [0,\infty)$*, where* $f,g : [0,\infty)^2 \to (0,\infty)$ *are continuous functions such that* $f(t,s) > g(t,s)$ *for all* $t,s > 0$*.*

(3) Let $\Xi_3(t,s) = s - \phi_3(s) - t$ *for all* $t,s \in [0,\infty)$*.*

Lemma 2.3. [32]. *A sequence* $\{o_n\}$ *with elements from a b-metric space* (Λ, d_b, s) *is Cauchy if there exists* $c \in [0,1)$ *such that* $d_b(o_n, o_{n+1}) \le c \cdot d_b(o_n, o_{n-1})$ *for every* $n \in \mathbb{N}$*.*

3. Main Results

Our essential result Istratescu type contractions in the setting of $b-$MS embedded with SF is defined as

Definition 3.1. *Assuming that* Λ *is a nonempty set, let* $d_b : \Lambda \times \Lambda \to \mathbb{R}$ *be a function with parameters* $s \ge 1$ *and* $k \in [0,1]$ *for all* $o, \omega, z \in \Lambda$*, and let* $\alpha : \Lambda \times \Lambda \to [0,+\infty)$ *be a function. Suppose* $T : \Lambda \to \Lambda$ *is an* α*-admissible Istratescu* Ξ*-contraction mapping as*

$$o, \omega \in \Lambda, \Xi(\alpha(o,\omega) d_b(T^2 o, T^2 \omega), k.M(o,\omega)) \ge 0, \tag{1}$$

where

$$M(o,\omega) = d_b(To, T\omega) + |d_b(To, T^2 o) - d_b(T\omega, T^2 \omega)|$$

Theorem 3.1. *Given a* $b-$*MS with coefficient* s *and a mapping* $T : \Lambda \to \Lambda$*. The requirements that follows must be met for* T *to be considered an Istratescu type* $\Xi-$*contraction mapping:*

(1) There exits a SF Ξ*.*

(2) The mapping T *is an* $\alpha-$*orbital admissible and there exists* $o_0 \in \Lambda$ *such that* $\alpha(o_0, To_0) \ge 1$*.*

(3) The mapping T is continuous; or

(4) The mapping T^2 is continuous and $\alpha(To,o) \geq 1$ for any $o \in \Lambda$.

Then T has a fixed point.

Proof. By assumption, consider a sequence o_n in Λ such that $o_{n+1} = To_n$ and $o_{n+2} = T^2 o_n$ for all $n \in \mathbb{N} \cup 0$. If $o_n = o_{n+1}$ for all $n \in \mathbb{N} \cup 0$, then S has a fixed point and the proof is complete. Therefore, we suppose that $o_n \neq o_{n+1}$ for some $n \in \mathbb{N} \cup 0$; that is, $d_b(o_n, o_{n+1}) \neq 0$ for all $n \in \mathbb{N} \cup 0$.

$$\Xi(\alpha(o,\omega)d_b(T^2 o_n, T^2 o_{n+1}), k.M(o_n, o_{n+1})) \geq 0,$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M(o_n, o_{n+1}) &= d_b(To_n, To_{n+1}) + |d_b(To_n, T^2 o_n) - d_b(To_{n+1}, T^2 o_{n+1})| \\ &= d_b(o_{n+1}, o_{n+2}) + |d_b(o_{n+1}, o_{n+2}) - d_b(o_{n+2}, o_{n+3})|. \end{aligned}$$

It follows that

$$\Xi\bigg(d_b(o_{n+2}, o_{n+3}), k.(d_b(o_{n+1}, o_{n+2}) + |d_b(o_{n+1}, o_{n+2}) - d_b(o_{n+2}, o_{n+3})|)\bigg) \geq 0.$$

According to condition (2) of definition 2.2, we infer that

$$\begin{aligned} 0 &\leq \Xi\bigg(d_b(o_{n+2}, o_{n+3}), k(d_b(o_{n+1}, o_{n+2}) + |d_b(o_{n+1}, o_{n+2}) - d_b(o_{n+2}, o_{n+3})|)\bigg) \\ &< k(d_b(o_{n+1}, o_{n+2}) + |d_b(o_{n+1}, o_{n+2}) - d_b(o_{n+2}, o_{n+3})|) - d_b(o_{n+2}, o_{n+3}). \end{aligned}$$

Thus, we get

$$d_b(o_{n+2}, o_{n+3}) < k(d_b(o_{n+1}, o_{n+2}) + |d_b(o_{n+1}, o_{n+2}) - d_b(o_{n+2}, o_{n+3})|),$$

for all $n \geq 1$.

If $d_b(o_{n+1}, o_{n+2}) \leq d_b(o_{n+2}, o_{n+3})$, then we have

$$\begin{aligned} d_b(o_{n+2}, o_{n+3}) &< k(d_b(o_{n+1}, o_{n+2}) + d_b(o_{n+2}, o_{n+3}) - d_b(o_{n+1}, o_{n+2})), \\ &= k.d_b(o_{n+2}, o_{n+3}) < d_b(o_{n+2}, o_{n+3}), \end{aligned}$$

which is a contradiction, thus $d_b(o_{n+1},o_{n+2}) > d_b(o_{n+2},o_{n+3})$, then

$$\begin{aligned} d_b(o_{n+2},o_{n+3}) &< k(d_b(o_{n+1},o_{n+2})+d_b(o_{n+1},o_{n+2})-d_b(o_{n+2},o_{n+3})), \\ &= 2k(d_b(o_{n+1},o_{n+2}))+k.d_b(o_{n+2},o_{n+3}) \\ &\Leftrightarrow d_b(o_{n+2},o_{n+3}) < \frac{2k}{1-k}d_b(o_{n+1},o_{n+2}) \end{aligned}$$

By continuing in the same approach,

$$d_b(o_n,o_{n+1}) = d_b(T^n o_0, T^{n+1}o_0) \ < \ \frac{2k}{1-k}d_b(o_{n-1},o_n) = \frac{2k}{1-k}d_b(T^{n-1}o_0,T^n o_0,$$

then

$$\begin{aligned} d_b(T^n o_0, T^{n+1}o_0) &< \frac{2k}{1-k}d_b(T^{n-1}o_0,T^n o_0, \\ &\leq (\frac{2k}{1-k})^{n-1}d_b(To_0,T^2 o_0, \\ &\to 0 \ as \ n \to 0 \ since \ \frac{2k}{1-k} < 1. \end{aligned}$$

The last inequality implies that $d_b(o_n,o_{n+1}) < \frac{2k}{1-k}d_b(o_{n-1},o_n)$ and with $W = \frac{2k}{1-k} < 1$ which it is coincided with Lemma 2.3, then we get that the sequence $\{o_n\}$ creates a Cauchy sequence on a complete $b-$MS. Also, it is concurrently convergent. Then, there exists $v \in \Lambda$ such that

$$\lim_{n\to\infty} d_b(o_n,v) = 0.$$

Given that the mapping T is continuous, it follows that $\lim_{n\to\infty} d_b(To_{n-1},Tv) = \lim_{n\to\infty} d_b(o_n,Tv) = 0$, leading to the conclusion that $Tv = v$, indicating that v serves as a fixed point for T.

Similarly, considering the continuity of the mapping T^2, we observe

$$\lim_{n\to\infty} d_b(T^2 o_{n-2},T^2 v) = \lim_{n\to\infty} d_b(o_n,T^2 v) = 0.$$

Since each sequence in $b-$MS has a unique limit, we deduce that $T^2 v = v$, implying that v serves as a fixed point for T^2. If $Tv \neq v$, then we have

$$\begin{aligned} 0 &\leq \Xi\Big(\alpha(Tv,v)d_b(T^2(Tv),T^2v), k.(d_b(Tv,T^2v)+|d_b(Tv,T^2v)-d_b(T^2v,T^3v)|)\Big) \\ &< k.(d_b(Tv,T^2v)+|d_b(Tv,T^2v)-d_b(T^2v,T^3v)|)-d_b(T^2(Tv),T^2v). \end{aligned}$$

Then, we get

$$\begin{aligned}
0 < d_b(Tv,v) &= d_b(T^2(Tv),T^2v)\\
&\leq \alpha(Tv,v)d_b(T^2(Tv),T^2v)\\
&\leq k.(d_b(Tv,T^2v)+|d_b(Tv,T^2v)-d_b(T^2v,T^3v)|)\\
&= k.(d_b(Tv,v)+|d_b(Tv,v)-d_b(v,Tv)|)\\
&= k.(d_b(Tv,v))\\
&< d_b(Tv,v).
\end{aligned}$$

Thus, $Tv=v$. At last, we also need to prove the uniqueness of fixed point results by using the approach of argument to absurdity, so we consider that $\iota,\kappa\in\Lambda$ are two different fixed points, we get

$$\begin{aligned}
0 &\leq \Xi\bigg(\alpha(\iota,\kappa)d_b(T^2\iota,T^2\kappa),k.(d_b(T\iota,T\kappa)+|d_b(T\iota,T^2\iota)-d_b(T\kappa,T^2\kappa)|)\bigg)\\
&< k.(d_b(T\iota,T\kappa)+|d_b(T\iota,T^2\iota)-d_b(T\kappa,T^2\kappa)|)-d_b(T^2\iota,T^2\kappa).
\end{aligned}$$

Then, we get

$$\begin{aligned}
0 < d_b(\iota,\kappa) &= d_b(T^2\iota,T^2\kappa)\\
&\leq \alpha(\iota,\kappa)d_b(T^2\iota,T^2\kappa)\\
&\leq k.(d_b(T\iota,T\kappa)+|d_b(T\iota,T^2\iota)-d_b(T\kappa,T^2\kappa)|)\\
&= k.(d_b(\iota,\kappa)+|d_b(\iota,\iota)-d_b(\kappa,\kappa)|)\\
&= k.(d_b(\iota,\kappa))\\
&< d_b(\iota,\kappa),
\end{aligned}$$

which is a contradiction. Thus, $\iota=\kappa$, which means there is only one fixed point of T. □

Corollary 3.1. *Let (Λ,d_b) be a $b-MS$ with coefficient s, and let $T:\Lambda\to\Lambda$ be a mapping. Then T is said to be an Istratescu type $\Xi-$contraction mapping if it satisfies the following conditions:*

(1) There exists a SF Ξ.

(2) The mappings T and T^2 are continuous.

Under these conditions, T possesses a fixed point.

Proof. By adopting the mapping $\alpha(To,T\omega)=1$ for each $o,\omega\in\Lambda$, it follows from Theorem 3.1. □

Corollary 3.2. *Consider (Λ,d_b) as a $b-MS$ with coefficient s, and let $T:\Lambda\to\Lambda$ be a mapping. If T is defined as an Istratescu type $\Xi-$contraction mapping, it must satisfy the following conditions:*

(1) There exists a SF Ξ such that

$$o,\omega\in\Lambda,\Xi(\alpha(o,\omega)d_b(T^2o,T^2\omega),k.d_b(To,T\omega))\geq 0, \tag{2}$$

(2) The mappings T and T^2 are continuous.

Under these conditions, T must possess a fixed point.

Proof. This follows from Theorem 3.1 with the assumption that $\alpha(To,T\omega)=1$ and $M(o,\omega)=d_b(To,T\omega)$ for all $o,\omega\in\Lambda$. □

Corollary 3.3. *The mapping T is termed an Istratescu type $\Xi-$contraction mapping if it meets the following criteria under (Λ,d_b):*

(1) There exits a SF Ξ such that

$$o,\omega\in\Lambda,\Xi(\alpha(o,\omega)d_b(T^2o,T^2\omega),k.d_b(o,\omega))\geq 0, \tag{3}$$

(2) Both T and T^2 are continuous.

Under these conditions, T possesses a fixed point.

Proof. From Theorem 3.1, the assumption that $\alpha(To,T\omega)=1$ and $M(o,\omega)=d_b(o,\omega)$ for any $o,\omega\in\Lambda$ leads to this conclusion. □

Corollary 3.4. *Let (Λ,d_b) denote a $b-MS$ with coefficient s, and let $T:\Lambda\to\Lambda$ be a mapping. T is considered an Istratescu type $\Xi-$contraction mapping if it satisfies the following conditions:*

(1) There exits a SF Ξ such that

$$o,\omega\in\Lambda,\Xi(\alpha(o,\omega)d_b(To,T\omega),k.d_b(o,\omega))\geq 0, \tag{4}$$

(2) The mappings T is continuous.

Then T possesses a fixed point.

Proof. This follows directly from Theorem 3.1 by considering $\alpha(To, T\omega) = 1$, $M(o, \omega) = d_b(o, \omega)$, and $d_b(T^2o, T^2\omega) = d_b(To, T\omega)$ for any $o, \omega \in \Lambda$ □

Corollary 3.5. *Let (Λ, d_b) be a $b-MS$ with coefficient s, and $T : \Lambda \to \Lambda$ be a mapping. Then T is considered an Istratescu type $\Xi-$contraction mapping if either T or T^2 is continuous. In such cases, T has a fixed point.*

Proof. It follows directly from Theorem 3.1 by setting $\alpha(To, T\omega) = 1$ for any $o, \omega \in \Lambda$.

$$d_b(o_{n+2}, o_{n+3}) \quad < \quad k(d_b(o_{n+1}, o_{n+2}) + |d_b(o_{n+1}, o_{n+2}) - d_b(o_{n+2}, o_{n+3})|),$$

for all $n \geq 1$ and continue by supposing the SF Ξ is unavailable. □

Example 3.1. *Let $\Lambda = [0, \infty)$ with $b-MS$ with coefficient $s = 2$. Define $d_b : \Lambda \times \Lambda \to \mathbb{R}$ by $d_b(o, \omega) = (o - \omega)^2$. Additionally, let's define the mapping $T : \Lambda \to \Lambda$ and the SF $\Xi : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ as follows:*

$$To = \begin{cases} o^3 & \text{if } o \in [0, 1) \\ 1 & \text{if } o \in [1, 3) \\ \frac{2o^2+o+1}{o^2+o+1} & \text{if } o \in [3, \infty), \end{cases}$$

$$\alpha(o, \omega) = \begin{cases} 2 & \text{if } o, \omega \in [1, \infty) \\ 1 & \text{otherwise} \end{cases}$$

and

$$\Xi(t, s) = s - t \ \forall t, s \in \Lambda$$

Proof. It is evident that (Λ, d_b, s) constitutes a completed $b-$MS and Ξ qualifies as a SF. We observe that while T exhibits discontinuity at $x = 3$, T^2 maintains continuity across Λ owing to

$$T^2o = \begin{cases} o^9 & \text{if } o \in [0, 1) \\ 1 & \text{if } o \in [1, \infty) \end{cases}$$

It is simple to see that T is an Istratescu type $\Xi-$contraction mapping. Without a doubt, due to definition α, we see the as it were curiously case is for $o,\omega\in[1,\infty)$, as $k\in(0,1]$ we have

$$\begin{aligned}
&\Xi\big(\alpha(o,\omega)d_b(T^2o,T^2\omega),\ k\cdot d_b(To,T\omega)+\big|d_b(To,T^2o)-d_b(T\omega,T^2\omega)\big|\big)\\
&\quad=k\cdot d_b(To,T\omega)+\big|d_b(To,T^2o)-d_b(T\omega,T^2\omega)\big|-\alpha(o,\omega)d_b(T^2o,T^2\omega)\\
&\quad=k\cdot d_b(To,T\omega)+|d_b(To,1)-d_b(T\omega,1)|-2\cdot d_b(1,1)\\
&\quad=k\cdot d_b(To,T\omega)+|d_b(To,1)-d_b(T\omega,1)|\\
&\quad\geq 0.
\end{aligned}$$

Then T is generalized $\Xi-$contraction and meets all of the requirements of Theorem 3.1. Thus, T has a fixed point. □

4. An Application

The theory of nonlinear integral equations (NIEs) is a vast subject that is used in numerous applications across many fields of mathematics today. Given that the growth of the fractional calculus (FC) is perturbation theory and that possesses characteristics related to memory effects, FC and the theory of nonlinear fractional differential equations (NFDEs) are crucial for the investigation of natural problems. In many fields, including physical sciences, economics, chaos theory, and dynamic programming, the theory of NFDEs can therefore be effectively utilised. We suggest [8, 12, 13, 17, 28, 29, 30, 31] and any references therein for additional information.

4.1. The nonlinear fractional differential equations.

Let's begin by revisiting some fundamental definitions from fractional calculus [34]. The Caputo class derivative of a continuous function $g:[0,\infty)\to\mathbb{R}$ with order $\hbar>0$ is categorized as follows:

$$^{C}D^{\hbar}_{0_+}(f(t))=\frac{1}{\Gamma(n-\hbar)}\int_0^t(t-s)^{\hbar}g^{(n)}(s)ds,n-1<\hbar<n,n=[\hbar]+1,\tag{5}$$

where $[\hbar]$ represents the integer part of the positive real number $\hbar$, and Γ denotes the gamma function.

Let the NFDE of the Caputo class represented as:

$$ {}^{C}D^{\hbar}_{0_+}(o(t)) = f(t, o(t)), \tag{6} $$

with boundary conditions: $o(0) = 0$, $o(1) = \int_0^{\eta} o(s)ds$, where $1 < \hbar \leq 2$, $0 < \eta < 1$ and $o \in C[0,1]$,

Since $f : [0,1]\mathbb{R} \to \mathbb{R}$ is a given continuous function, then the equation (6) is flipped as the:

$$ \begin{aligned} o(t) &= \frac{1}{\Gamma(\hbar)} \int_0^t (t-s)^{\hbar-1} f(s, o(s))ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^1 (1-s)^{\hbar-1} f(s, o(s))ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^{\eta} (\int_0^s (s-m)^{\hbar-1} f(m, o(m))dm)ds. \end{aligned} \tag{7} $$

Next, we present the ensuing existence theorem.

Theorem 4.1. *Consider the NFDEs (6), which satisfy the following conditions:*

(1) There exits a SF $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *such that*

$$ \Xi(\|Fo(t) - F\omega(t)\|_{\infty}, (\|o-\omega\|_{\infty})^k \geq 0, $$

such with some contraction

$$ | Fo(t) - F\omega(t) | \leq \frac{\Gamma(\hbar+1)}{5}(| o - \omega |)^k, $$

holds for all $t \in [0,1]$, $o, \omega \in C[0,1]$, $k \in [0, \frac{1}{s})$ *and* $s \in [0,1]$

(2) There exists $o_0 \in C[0,1]$ *where the function* $F : C[0,1]) \to C[0,1]$ *is typified by*

$$ \begin{aligned} fo(t) &= \frac{1}{\Gamma(\hbar)} \int_0^t (t-s)^{\hbar-1} f(s, o(s))ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^1 (1-s)^{\hbar-1} f(s, o(s))ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^{\eta} (\int_0^s (s-m)^{\hbar-1} f(m, o(m))dm)ds. \end{aligned} $$

(3) if $\{o_n\}$ *is a sequence in* $C[0,1]$ *such that* $o_n \to o$ *in* $C[0,1]$ *and*

$$\begin{aligned} o_n(t) &= \frac{1}{\Gamma(\hbar)} \int_0^t (t-s)^{\hbar-1} f(s, o_{n-1}(s)) ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^1 (1-s)^{\hbar-1} f(s, o_{n-1}(s)) ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^\eta \Big(\int_0^s (s-m)^{\hbar-1} f(m, o_{n-1}(m)) dm \Big) ds. \end{aligned}$$

Then, the NFDE problem (6) possesses at least one solution.

In that case, f possesses a fixed point.

Proof. It is commonly acknowledged that if $o \in C[0,1]$ is a solution of equation (6) then $o \in C[0,1]$ is also a solution of equation (7) and the opposite is accurate. Now, let $o, \omega \in C[0,1]$ and by using condition (1) and Definition (2.2), we have

$$\Xi(\|Fo(t) - F\omega(t)\|_\infty, (\|o-\omega\|_\infty)^k) \geq 0,$$

such with some contraction $| fo(t) - f\omega(t) | \leq \frac{\Gamma(\hbar+1)}{5} | (o-\omega |)^k$, holds for all $t \in [0,1]$ and $o, \omega \in C[0,1]$, then $(\|o-\omega\|_\infty)^k - \|Fo(t) - F\omega(t)\|_\infty \geq 0$, then we want to prove that

$$\|Fo(t) - F\omega(t)\|_\infty \leq (\|o-\omega\|_\infty)^k.$$

Now,

$$\begin{aligned} To(t) - T\omega(t) &= \Big| \frac{1}{\Gamma(\hbar)} \int_0^t (t-s)^{\hbar-1} f(s, o(s)) ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^1 (1-s)^{\hbar-1} f(s, o(s)) ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^\eta \Big(\int_0^s (s-m)^{\hbar-1} f(m, o(m)) dm \Big) ds \\ &\quad - \frac{1}{\Gamma(\hbar)} \int_0^t (t-s)^{\hbar-1} f(s, o(s)) ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^1 (1-s)^{\hbar-1} f(s, \omega(s)) ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^\eta \Big(\int_0^s (s-m)^{\hbar-1} f(m, o(m)) dm \Big) ds \Big| \\ &\leq \frac{1}{\Gamma(\hbar)} \int_0^t | t-s |^{\hbar-1} | f(s, o(s)) - f(s, \omega(s)) | ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^1 | 1-s |^{\hbar-1} | f(s, o(s)) - f(s, \omega(s)) | ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{2t}{(2-\eta^2)\Gamma(\hbar)}\int_0^{\eta}(\int_0^{s} \mid s-m\mid^{\hbar-1}\mid f(m,\omega(m))-f(m,o(m))dm)\mid ds\\
\leq\ & \frac{1}{\Gamma(\hbar)}\int_0^{t}\mid t-s\mid^{\hbar-1}\ \frac{\Gamma(\hbar+1)}{5}(\mid o(s)-\omega(s)\mid)ds\\
&+\frac{2t}{(2-\eta^2)\Gamma(\hbar)}\int_0^{1}\mid 1-s\mid^{\hbar-1}\ \frac{\Gamma(\hbar+1)}{5}(\mid o(s)-\omega(s)\mid)ds\\
&+\frac{2t}{(2-\eta^2)\Gamma(\hbar)}\int_0^{\eta}(\int_0^{s}\mid s-m\mid^{\hbar-1}\ \frac{\Gamma(\hbar+1)}{5}(\mid \omega(m)-o(m)\mid)dm)ds\\
\leq\ & \frac{\Gamma(\hbar+1)}{5}(\|x-y\|_\infty)\sup(\int_0^{t}\mid t-s\mid^{\hbar-1}\ ds\\
&+\frac{2t}{(2-\eta^2)\Gamma(\hbar)}\int_0^{1}\mid 1-s\mid^{\hbar-1}\ ds\\
&+\frac{2t}{(2-\eta^2)\Gamma(\hbar)}\int_0^{\eta}(\int_0^{s}\mid s-m\mid^{\hbar-1}\ dm)ds)\\
\leq\ & \frac{\Gamma(\hbar+1)}{5}(\|o-\omega\|_\infty)\\
\leq\ & (\|o-\omega\|_\infty).
\end{aligned}
$$

Thus, for each $o,\omega\in C[0,1]$, $t\in[0,1]$, $1<\hbar\leq 2$, $0<\eta<1$ $k\in[0,\frac{1}{s})$ and $s\in[0,1]$, we have

$$\|To(t)-T\omega(t)\|_\infty\leq(\|o-\omega\|_\infty).$$

Aware of the supposing

$$\Xi(t,s)=s-t,\ \forall t,s\in\Lambda$$

and

$$\alpha(o,\omega)=\begin{cases}1, & \Xi(o(t),\omega(t))>0, t\in[a,b],\\ 0, & \text{otherwise},\end{cases}$$

We get

$$\Xi(\|To(t)-T\omega(t)\|_\infty,(\|o-\omega\|_\infty))=(\|o-\omega\|_\infty)-\|To(t)-T\omega(t)\|_\infty>0$$

which satisfy Corollary (3.4). Consequently, we infer the existence of $x\in C[0,1]$ such that $To=x$. □

4.2. The nonlinear integral equations.

In this subsection, we demonstrate both the existence and uniqueness of a solution for the NLIEs using our major study results from the previous section

Theorem 4.2. *let* $\Lambda = C([a,b],\mathbb{R})$ *include all continuously defined functions on* $[a,b]$. *Effectuate* $d_b(o,\omega) = \sup\{\mid o(t) - \omega(t) \mid\}$ *for all* $t \in [a,b]$. (Λ, d) *is a* $b-MS$ *such that*

$$o(t) = \int_a^b \hbar(t,s)T(s,o(s))ds, \ t \in [a,b], \tag{8}$$

where the functions $\hbar : [a,b] \times [a,b] \to [0,\infty)$ *and* $T : [a,b] \times C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ *and* $\eta(p,q) < \frac{1}{b-a}$.

Take into consideration the non-empty subsets Λ *such that* $o(t), \omega(t) \in \Lambda$, *where* $t \in [a,b]$. *Consider the case when there is* $0 < \alpha \leq 1$ *such that for all* $t,s,o,\omega \in [a,b]$, *we utilize*

$$\mid To(t) - T\omega(t)) \mid \leq \alpha \mid o - \omega \mid . \tag{9}$$

Then the reference equation (8) possesses a unique solution.

Proof. Let $\mathbb{T} : \Lambda^2 \to \Lambda$ and $g : \Lambda \to \Lambda$ as

$$\mathbb{T}o(t) = \int_a^b \hbar(s,t)T(s,o(s))ds.$$

Assume that$o, \omega \in \Lambda$ such that the simulation function $\Xi(o(t), \omega(t)) \geq 0$ for all $t \in [a,b]$ and by 9, we get

$$\begin{aligned}
|\mathbb{T}o(t) - \mathbb{T}\omega(t)| &= \int_a^b \hbar(s,t)T(s,o(s))ds - \int_a^b \hbar(s,t)T(s,\omega(s))ds \\
&= \int_a^b \hbar(s,t)[T(s,o(s)) - T(s,\omega(s))]ds \\
&\leq (\int_a^b \hbar(s,t)ds)(\int_a^b \mid T(s,o(s)) - T(s,\omega(s)) \mid ds) \\
&\leq \sup_{t\in[a,b]} (\int_a^b \hbar(s,t)ds)(\int_a^b \sup_{s\in[a,b]} \mid T(s,o(s)) - T(s,\omega(s)) \mid ds) \\
&\leq \sup_{t\in[a,b]} (\int_a^b \hbar(s,t)ds)(d_b(o,\omega)).
\end{aligned}$$

Let $\sup_{t\in[a,b]}(\int_a^b \hbar(s,t)ds) = \frac{1}{b-a}$ for all $a < b$ then we get

$$\mid To(t) - T\omega(t)) \mid \leq \frac{1}{b-a} d_b(o,\omega) \tag{10}$$

,

which it is equivalent with 9. Aware of the supposing

$$\Xi(t,s) = s - t,\ \forall t,s \in \Lambda$$

and

$$\alpha(o,\omega) = \begin{cases} 1, & \Xi(o(t),\omega(t)) > 0, t \in [a,b], \\ 0, & \text{otherwise}, \end{cases}$$

So, for all $a < b$ we get

$$\begin{aligned} \Xi(\alpha(o,\omega)d_b(To,T\omega), \frac{1}{b-a}.d_b(o,\omega)) &= \frac{1}{b-a}.d_b(o,\omega) - \alpha(o,\omega)d_b(To,T\omega), \\ &\geq \frac{1}{b-a}.d_b(o,\omega) - d_b(To,T\omega), \end{aligned}$$

but from 10, we get

$$\Xi(\alpha(o,\omega)d_b(To,T\omega), \frac{1}{b-a}.d_b(o,\omega)) \geq 0,$$

which it is equivalent with Corollary 3.4. Hence, $\mathbb{T}$ offers an unique solution in Λ. □

5. Conclusion

This study presents novel findings on the existence and uniqueness of fixed points in $b-$MS when coupled with SF. The results suggest that specific conditions can lead to the existence and uniqueness of fixed points in such spaces. Moreover, the paper explores the utilization of fixed point methods to establish the existence and uniqueness of solutions to NFDEs with periodic boundary conditions. Additionally, the authors propose extending the study's results to other generalized $b-$MS or alternative forms of contractions, which could offer fresh insights and outcomes. Overall, this research contributes to the ongoing advancement of FPT and its applications across diverse domains.

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CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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