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ON ENHANCED k-FOLD AVERAGED MAP OF WEAK ENRICHED $\mathscr{F}$-CONTRACTION WITH APPLICATION TO BOUNDARY LAYER MODEL

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Abstract. Recently, two separate generalizations of enriched contraction maps, namely, weak enriched contraction and weak enriched $\mathscr{F}$-contractions, were introduced to approximate fixed points using the higher order Kirk iteration. In this article, we introduce an enhanced k-fold averaged iterative procedure that can approximate fixed points of operators that may not meet the hypotheses of the previous k-fold averaged iterative scheme for each k. Our first attempt is to prove the strong convergence and stability of the enhanced k-fold averaged iteration associated with the weak enriched $\mathscr{F}$-contraction in Banach spaces. Also, we justify the equivalence of the enhanced k-fold Kirk iteration with other comparable iterative schemes using the weak enriched $\mathscr{F}$-contractive map. Furthermore, we show the validation and versatility of the enhanced map with some numerical examples. The results indicate that the improved k-fold averaged iteration (a) has a better convergent rate than others and (b) exhibits contracting behavior when others fail for some enriching constants. As an application, the enhanced k-fold map is employed to solve a boundary layer model.

Keywords: k-fold averaged map; weak enriched $\mathscr{F}$-contraction; asymptotically regular; boundary layer model.

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1. INTRODUCTION

Let $(\mathscr{X},\|\cdot\|)$ be a nonempty normed space and $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ be a self-map. A point $p\in\mathscr{X}$ is a fixed point of $\mathscr{T}$ if $\mathscr{T}p=p$. We denote $Fix(\mathscr{T})$ as the set of all fixed points of $\mathscr{T}$.
In [4], Banach examined the existence of fixed point of an operator $\mathscr{T}$ that satisfies

$$\|\mathscr{T}x-\mathscr{T}y\|\leq\beta\|x-y\|, \tag{1}$$

for $\beta\in(0,1)$ and $x,y\in\mathscr{X}$, using the Picard sequence $\{x_n\}_{n=0}^{\infty}$, for $x_0\in\mathscr{X}$,

$$x_n=\mathscr{T}x_{n-1},\ \forall\, n=1,2,\ldots \tag{2}$$

Later, it turns out that the Picard sequence is not suitable for approximating fixed point of $\mathscr{T}$ if $\beta=1$. Therefore, if the condition (1) is slightly weaken and $Fix(\mathscr{T})\neq\emptyset$, the Picard iteration does not converge to the fixed point of $\mathscr{T}$. This leads to the need for more robust and elaborate iterative procedures to approximate the fixed point.
One of the first such procedures was developed by Krasnoselskii [11] in a uniformly convex Banach space $\mathscr{X}$. Krasnoselskii demonstrated that if $\mathscr{T}$ is a nonexpansive mapping on a closed convex subset $\mathscr{C}$ of $\mathscr{X}$, then the Krasnoselskii iterative process, for $x_0\in\mathscr{C}$,

$$x_n=\mathscr{T}_\lambda x_{n-1}=(1-\lambda)x_{n-1}+\lambda\mathscr{T}x_{n-1} \tag{3}$$

converges to the fixed point of $\mathscr{T}$. Indeed, the iterative process (3) generalized the Picard iteration (2) with $\lambda=1$. Kirk [12] defined a more general procedure which is a convex combination of the higher order of $\mathscr{T}$ as follows:

$$x_n=Sx_{n-1}=\alpha_0x_{n-1}+\alpha_1\mathscr{T}x_{n-1}+\alpha_2\mathscr{T}^2x_{n-1}+\ldots+\alpha_k\mathscr{T}^kx_{n-1}, \tag{4}$$

where $k\geq1$ is a fixed integer, $\alpha_i\geq0$ for $i=0,1,2,...,k$, $\alpha_1>0$, and $\sum_{i=0}^{k}\alpha_i=1$. Kirk deduced that if $S:\mathscr{C}\to\mathscr{C}$ whenever $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ is non-expansive, then $Fix(S)=Fix(\mathscr{T})$. Obviously, the Kirk iteration reduces to Picard and Krasnoselskii iterations for $k=0$ and $k=1$, respectively. For several generalizations of the Kirk iteration, see [3, 10, 14, 15, 20].
A new fundamental class of mappings called enriched contraction [6] was introduced to study the existence and approximation of fixed points through Krasnoselskii iteration.

Definition 1.1. [6] Let $\mathscr{C}$ be a convex subset of a normed space $(\mathscr{X}, \|\cdot\|)$. A self-mapping $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ is called an enriched contraction if there exist non-negative real numbers b and $\mu\in[0,b+1)$ such that

$$\|b(x-y)+\mathscr{T}x-\mathscr{T}y\|\leq\mu\|x-y\| \tag{5}$$

for every $x,y\in\mathscr{C}$.

It was asserted, among other notable observations, that the fixed point of $\mathscr{T}_\lambda$ associated with enriched contraction is a fixed point of $\mathscr{T}$. Since then, this class of enriched contraction mappings has drawn the attention of researchers in the fixed point theory. For some recent papers in this regard, see [2, 7, 8, 9, 13, 16, 18, 19].

In [13], Nithiarayaphaks and SinTunavarat introduced a notion of weakly enriched contraction mappings as follows:

Definition 1.2. [13] Let $\mathscr{C}$ be a convex subset of a normed space $(\mathscr{X}, \|\cdot\|)$. A self-mapping $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ is called a weakly enriched mapping if there exist non-negative real numbers b_1, b_2 and $\omega\in[0,b_1+b_2+1)$ such that

$$\left\|b_1(x-y)+\mathscr{T}x-\mathscr{T}y+b_2(\mathscr{T}^2x-\mathscr{T}^2y)\right\|\leq\omega\|x-y\| \tag{6}$$

for every $x,y\in\mathscr{C}$.

The authors also defined a relaxed 2-fold averaged mapping to achieve their results as presented below:

Theorem 1.3. [13] *Let $\mathscr{C}$ be a closed convex subset of a Banach space $(\mathscr{X}, \|.\|)$, and let $\mathscr{T}$ be a weak enriched contraction self mapping on $\mathscr{C}$, Then there are $\alpha_1>0$ and $\alpha_2\geq 0$ with $\alpha_1+\alpha_2\in(0,1]$ such that the following statements hold:*

I. $|Fix(\mathscr{T}_{\alpha_1\alpha_2})|=1$.

II. *for any $x_0\in\mathscr{C}$, the iteration scheme $\{x_n\}\subset\mathscr{C}$ defined as*

$$x_n=\mathscr{T}_{\alpha_1\alpha_2}x_{n-1}=(1-\alpha_1-\alpha_2)x_{n-1}+\alpha_1\mathscr{T}x_{n-1}+\alpha_2\mathscr{T}^2x_{n-1},\ \textit{for } n\in\mathbb{N} \tag{7}$$

converges to a unique fixed point of $\mathscr{T}_{\alpha_1\alpha_2}$.

Besides, the authors gave some sufficient conditions for the realization of $Fix(\mathscr{T}_{\alpha_1\alpha_2}) = Fix(\mathscr{T})$. Zhou et al. [21] unified the approximation of fixed points of weak enriched contractions using a class of mappings $\mathscr{F}$ as follows: Let $\mathscr{F}$ be the family of all mappings $f:\mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying the following conditions:

$\mathscr{F}_1$: f is continuous in each argument.

$\mathscr{F}_2$: there exists $\lambda \in [0,1)$ such that if $r < f(s,r,s)$ or $r < f(r,s,s)$, then $r \leq \lambda s$.

$\mathscr{F}_3$: for $\lambda > 0$ and for all $r,s,t \in \mathbb{R}_+$, $\lambda f(r,s,t) \leq f(\lambda r, \lambda s, \lambda t)$.

Below are some family of mappings $f:\mathbb{R}_+^3 \to \mathbb{R}_+$ belonging to the class $\mathscr{F}$.

Example 1.4. i. $f(r,s,t) = \beta(r+s)$, where $\beta \in [0,\frac{1}{2})$.

ii. $f(r,s,t) = \beta r$, where $\beta \in [0,1)$.

iii. $f(r,s,t) = ar+bs+ct$, where $0 \leq a,b,c < 1$ and $a+b+c = 1$.

iv $f(r,s,t) = \beta \max\{s,t\}$, where $\beta \in [0,1)$.

v. $f(r,s,t) = \beta(s+t)$, where $\beta \in [0,\frac{1}{2})$.

Using the notion of class $\mathscr{F}$, Zhou et al. defined a weak enriched $\mathscr{F}$-contraction map as follows:

$$\begin{aligned} \|b_1(x-y)+\mathscr{T}x-\mathscr{T}y+\sum_{i=2}^{k} b_i(\mathscr{T}^i x-\mathscr{T}^i y)\| \leq f\Bigg(&(\sum_{i=1}^{k} b_i+1)\,\|x-y\|, \\ &\left\|x-\mathscr{T}x+\sum_{i=2}^{k} b_i(x-\mathscr{T}^i x)\right\|, \left\|y-\mathscr{T}y+\sum_{i=2}^{k} b_i(y-\mathscr{T}^i y)\right\|\Bigg), \end{aligned} \tag{8}$$

where $f \in \mathscr{F}$, $b_i \in (0,\infty)$, $i = 1,2,...,k$, $k \geq 3$, $x,y \in \mathscr{X}$. With the imposition of (8) on the k-fold averaged iteration

$$\begin{aligned} x_n &= \overline{\mathscr{T}}x_{n-1} \\ &= (1-\alpha_1-\alpha_2-\cdots-\alpha_k)x_{n-1}+\alpha_1\mathscr{T}x_{n-1}+\alpha_2\mathscr{T}^2x_{n-1}+\cdots+\alpha_k\mathscr{T}^k x_{n-1}, \end{aligned} \tag{9}$$

for $n \in \mathbb{N}$, where $\alpha_1 > 0$ and $\alpha_i \geq 0,\ i = 2,3,\ldots,k$, the authors proved the existence of fixed points of the k-fold averaged mapping associated with weak enriched contraction. They also gave the conditions that guarantee the equality of the sets of fixed points for the weak enriched contractions. These results were unification and extension of the results in [6, 13]. Indeed, the enriched contraction and the weak enriched contraction are recovered by setting

$b_2 = b_3 = \cdots = b_k = 0$ and $b_3 = b_4 = \cdots = b_k = 0$, respectively.

However, the k-fold averaged map associated with the weak enriched $\mathscr{F}$-contraction may exhibit non-contracting behaviors for some constants b_i, $i = 1, 2, \ldots, k$, and consequently becomes unstable. Therefore, there is need for a robust map for approximating fixed points of weakly enriched map.

2. Enhanced k-Fold Averaged Iteration

Motivated by the above work done, we define an enhanced k-fold averaged iteration to control the deficiency of the k-fold average map as follows: For $u_0 \in \mathscr{X}$,

$$u_n = \overline{\mathscr{S}}u_{n-1} = (1-\alpha_1-\alpha_2-\cdots-\alpha_k)x_{n-1} + \alpha_1\mathscr{T}u_n + \alpha_2\mathscr{T}^2u_n + \cdots + \alpha_k\mathscr{T}^k u_n \tag{10}$$

where $\alpha_1 > 0$ and $\alpha_i \geq 0,\ i = 2, 3, \ldots, k$. By factoring α_1 in equation (10), we get

$$\frac{\overline{\mathscr{S}}u_{n-1}}{\alpha_1} = \frac{(1-\alpha_1-\alpha_2-\cdots-\alpha_k)}{\alpha_1}u_{n-1} + \mathscr{T}u_n + \frac{\alpha_2}{\alpha_1}\mathscr{T}^2u_n + \cdots + \frac{\alpha_k}{\alpha_1}\mathscr{T}^k u_n. \tag{11}$$

Also, by letting $b_1 = \frac{1-\alpha_1-\alpha_2-\cdots-\alpha_k}{\alpha_1}$, $b_i = \frac{\alpha_i}{\alpha_1}$ for each $i \geq 2$, the enhanced k-fold averaged iteration can be used for approximating fixed points of the weakly enriched $\mathscr{F}$-contraction. As shown in the next example, the improved k-fold averaged map exhibits contracting behavior when others are not.

Example 2.1. Let $\mathscr{X} = \mathbb{R}$ be endowed with the usual norm and $\mathscr{T} : \mathbb{R} \to \mathbb{R}$ be defined by $\mathscr{T}u = 1 - 2u$ for $u \in \mathbb{R}$ with $Fix(\mathscr{T}) = \{\frac{1}{3}\}$. Letting $b_1 = b = \frac{2}{5}$ and $b_2 = 3$, then the map

i. $\mathscr{T}$ (associated with $\mathscr{T}_\lambda$) is not an enriched contraction.
ii. $\mathscr{T}$ (associated with $\mathscr{T}_{\alpha_1,\alpha_2}$ for $k = 2$) is not a weak enriched contraction.
iii. $\mathscr{T}$ (associated with $\overline{\mathscr{S}}$ for $k = 1$ or $k = 2$) is an enriched/weak enriched contraction.

We justify these as follows:

i. For $u, v \in \mathbb{R}$ and $b = \frac{2}{5}$, we get

$$\begin{aligned}|b(u-v) + \mathscr{T}u - \mathscr{T}v| &= \left|\frac{2}{5}(u-v) + (1-2u) - (1-2v)\right| \\ &= \frac{8}{5}|u-v| \nleq \mu\,|u-v|.\end{aligned}$$

Since $\frac{8}{5} \notin [0, \frac{7}{5})$, then $\mathscr{T}$ is not an enriched contraction.

ii. Let $u,v\in\mathbb{R}$, $b=\frac{2}{5}$, and $b_2=3$, then $\omega\in[0,\frac{22}{5})$. By using (6), we have

$$\left|b_1(u-v)+\mathscr{T}u-\mathscr{T}v+b_2(\mathscr{T}^2u-\mathscr{T}^2v)\right| = \left|\frac{2}{5}(u-v)-2(u-v)+3[(4u-1)-(4v-1)]\right|$$
$$= \frac{52}{5}|u-v| \nleq \omega|u-v|.$$

This implies that $\mathscr{T}$ is not a weak enriched contraction.

iii. Also, for $u,v,w,z\in\mathbb{R}$ and $b=\frac{2}{5}$, then $\mu\in[0,\frac{22}{5})$.

$$|b(u-v)+\mathscr{T}w-\mathscr{T}z| = \left|\frac{2}{5}(u-v)+2[(1-\frac{8}{5}u)-(1-\frac{8}{5}v)]\right|$$
$$= \frac{18}{5}|u-v| \le \mu|u-v|.$$

Thus, $\mathscr{T}$ (associated with $\overline{\mathscr{S}}$) is an enriched contraction. It is easy to show that $\mathscr{T}$ is also a weakly enriched contraction for $k>1$.

The following definitions and lemma will be helpful to achieve the results in this paper.

Lemma 2.2. [5] *Let* $\{\upsilon_n\}_{n=0}^{\infty}$ *and* $\{\nu_n\}_{n=0}^{\infty}$ *be sequence of non-negative numbers and* $\delta\in[0,1)$ *such that*

$$\upsilon_{n+1}\le\delta\upsilon_n+\nu_n\forall n\ge 0.$$

i. *If* $\lim_{n\to\infty}\nu_n=0$, *then* $\lim_{n\to\infty}\upsilon_n=0$.
ii. *If* $\sum_{n=0}^{\infty}\nu_n<\infty$, *then* $\sum_{n=0}^{\infty}\upsilon_n<\infty$.

Definition 2.3. [5] Let $\{\upsilon_n\}_{n=0}^{\infty}$ and $\{\nu_n\}_{n=0}^{\infty}$ be two non-negative real sequences which converge to υ and ν respectively. Let

$$\iota=\lim_{n\to\infty}\frac{|\upsilon_n-\upsilon|}{|\nu_n-\nu|}$$

i. If $\iota=0$, then $\{\upsilon_n\}$ converges to υ faster than $\{\nu_n\}$ to ν.
ii. If $0<\iota<\infty$ then both $\{\upsilon_n\}$ and $\{\nu_n\}$ have the same convergence rate.
iii. If $\iota=\infty$, then $\{\nu_n\}$ converges to ν faster than $\{\upsilon_n\}$ to υ.

Definition 2.4. [14] Let $\mathscr{X}$ be a normed space and $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ be a self mapping. Suppose that $Fix(\mathscr{T})=\{p\in\mathscr{X}:\mathscr{T}p=p\}$ is the set of fixed points of $\mathscr{T}$. Let $\{x_n\}_{n=0}^{\infty}\subset\mathscr{X}$ be the sequence generated by an iterative procedure involving $\mathscr{T}$ which is defined by $x_{n+1}=f(\mathscr{T}x_n)$

where $x_0 \in \mathscr{X}$ is the initial approximation. Suppose that $\{x_n\}$ converges to a fixed point p of $\mathscr{T}$. Let $\{y_n\}_{n=0}^{\infty} \subset \mathscr{X}$ be arbitrary, then $x_{n+1} = f(\mathscr{T}x_n)$ is said to be $\mathscr{T}$-stable or stable with respect to $\mathscr{T}$ if and only if $\lim_{n\to\infty} y_n = p$

Definition 2.5. Let $\{\upsilon_n\}_{n=0}^{\infty}$ and $\{\nu_n\}_{n=0}^{\infty}$ be two real sequences in $(\mathscr{X}, \|\cdot\|)$. The difference $\{\upsilon_n - \nu_n\}_{n=0}^{\infty}$ is asymptotically regular if

$$\|\upsilon_n - \nu_n\| \to 0 \text{ as } n \to \infty.$$

3. Main Result

In this section, we prove the strong convergent and stability of a sequence defined by the mapping $\overline{\mathscr{S}}$ under the weak enriched $\mathscr{F}$-contraction. We also state the necessary and sufficient conditions for the equivalent of the sequence. Throughout, we define $f \in \mathscr{F}$ by $f(r,s,t) = ar + bs + ct$ with $a + b + c = 1$, where r,s,t are non-negative terms.

Theorem 3.1. *Let $\mathscr{C}$ be a nonempty closed subset of a Banach space $(\mathscr{X}, \|.\|)$. Let $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ be an operator with $Fix(\mathscr{T}) \neq \emptyset$ and the k-fold averaged mapping $\overline{\mathscr{S}}$ (associated with $\mathscr{T}$) is a weak enriched $\mathscr{F}$-contraction. For $u_0 \in \mathscr{C}$, the sequence $\{u_n\}$ defined by (12) converges strongly to a unique fixed point $p \in Fix(\mathscr{T})$.*

Proof. Let $u_0 \in \mathscr{C}$ and $Fix(\mathscr{T}) \neq \emptyset$. Let $\{u_n\}$ be the sequence defined by the k-fold averaged mapping $\overline{\mathscr{S}}$, by hypothesis

$$\begin{aligned}
\|u_n - p\| &= \left\| \left(1 - \sum_{i=1}^{k} \alpha_i\right) u_{n-1} + \alpha_1 \mathscr{T} u_n + \alpha_2 \mathscr{T}^2 u_n + \cdots + \alpha_k \mathscr{T}^k u_n - p \right\| \\
&= \left\| \left(1 - \sum_{i=1}^{k} \alpha_i\right) (u_{n-1} - p) + \sum_{i=1}^{k} \alpha_i (\mathscr{T}^i u_n - \mathscr{T}^i p) \right\| \\
&= \alpha_1 \left\| \frac{(1 - \sum_{i=2}^{k} \alpha_i)}{\alpha_1} (u_{n-1} - p) + \mathscr{T} u_n - p + \sum_{i=2}^{k} \frac{\alpha_i}{\alpha_1} (\mathscr{T}^i u_n - p) \right\| \\
&\leq \alpha_1 f \left(\frac{1}{\alpha_1} \|u_{n-1} - p\|, \left\| \frac{\sum_{i=1}^{k} \alpha_i}{\alpha_1} (u_n - \mathscr{T}^k u_n) \right\|, \left\| \frac{\sum_{i=1}^{k} \alpha_i}{\alpha_1} (p - \mathscr{T}^k p) \right\| \right)
\end{aligned}$$

By adopting the condition $\mathscr{F}_3$, we get

$$\|u_n - p\| \leq f \left(\|u_{n-1} - p\|, \left\| \sum_{i=1}^{k} \alpha_i (u_n - \mathscr{T}^k u_n) \right\|, 0 \right) \tag{12}$$

Since $f \in \mathscr{F}$, the last inequality becomes

$$\|u_n - p\| \leq a\,\|u_{n-1} - p\| + b\left\|\sum_{i=1}^{k} \alpha_i(u_n - \mathscr{T}^k u_n)\right\| \tag{13}$$

But,

$$\begin{aligned}\left\|\sum_{i=1}^{k} \alpha_i(u_n - \mathscr{T}^k u_n)\right\| &= \left\|(1 - \sum_{i=1}^{k} \alpha_i)(p - u_n) + (1 - \sum_{i=1}^{k} \alpha_i)(u_{n-1} - p)\right\| \\ &\leq (1 - \sum_{i=1}^{k} \alpha_i)\,\|p - u_n\| + (1 - \sum_{i=1}^{k} \alpha_i)\,\|u_{n-1} - p\| \qquad (14)\end{aligned}$$

Therefore,

$$\|u_n - p\| \leq \frac{a + b(1 - \sum_{i=1}^{k} \alpha_i)}{1 - (1 - \sum_{i=1}^{k} \alpha_i)b}\,\|u_{n-1} - p\| \equiv \theta\,\|u_{n-1} - p\| \tag{15}$$

Since $a + b + c = 1$ and $\alpha_1 > 0$, then $a + 2(1 - \sum_{i=1}^{k} \alpha_i)b < 1$ implies that $\theta < 1$. By the application of Lemma 2.2, we have

$$\lim_{n \to \infty} \|u_n - p\| = 0.$$

Therefore, the sequence $\{u_n\}$ converges strongly to $p \in Fix(\mathscr{T})$.

Moreover, let p and q be two convergent points of $\{u_n\}$ for which $p \neq q$, then

$$\begin{aligned}\|q - p\| &= \left\|\overline{\mathscr{S}}q - \overline{\mathscr{S}}p\right\| \\ &\leq f(\|q - p\|, 0, 0) \\ &= a\,\|q - p\|, \text{ where } a < 1.\end{aligned}$$

yields a contradiction. Hence, $\{u_n\}$ has a unique limit $p \in Fix(\mathscr{T})$. □

Next, we study the stability of map $\overline{\mathscr{S}}$ associated with the map $\mathscr{T}$ as follows:

Theorem 3.2. *Let $\mathscr{C} \neq \emptyset$ be a closed convex subset of a Banach space $(\mathscr{X}, \|.\|)$ and $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ be a weak enriched $\mathscr{F}$-contraction. The sequence $\{x_n\}_{n=0}^{\infty}$ defined by the map $\overline{\mathscr{S}}$ is stable.*

Proof. Let $y_n \subset \mathscr{C}$ be an arbitrary sequence and let $\gamma_n = \left\|y_n - \overline{\mathscr{S}}y_n\right\|$. Firstly, if $y_n \to p$ as $n \to \infty$ for $p \in Fix(\mathscr{T})$, then from inequality (15), we conclude that $\gamma_n \to 0$ as $n \to \infty$. On other hand, assume that $\lim_{n\to\infty} \gamma_n = 0$. By hypothesis,

$$\|y_n - p\| = \left\|y_n - \overline{\mathscr{S}}y_n + \overline{\mathscr{S}}y_n - \overline{\mathscr{S}}p\right\|$$

$$
\begin{aligned}
&\leq \left\|y_n - \overline{\mathscr{S}}y_n\right\| + \left\|\overline{\mathscr{S}}y_n - \overline{\mathscr{S}}p\right\| \\
&= \gamma_n + \left\|(1-\sum_{i=1}^{k}\alpha_i)(y_{n-1}-p) + \sum_{i=1}^{k}\alpha_i(\mathscr{T}^i y_n - \mathscr{T}^i p)\right\| \\
&= \gamma_n + \alpha_1\left\|\frac{(1-\sum_{i=2}^{k}\alpha_i)}{\alpha_1}(y_{n-1}-p) + \mathscr{T}y_n - p + \sum_{i=2}^{k}\frac{\alpha_i}{\alpha_1}(\mathscr{T}^i y_n - \mathscr{T}^i p)\right\| \\
&\leq \gamma_n + \alpha_1 f\left(\frac{1}{\alpha_1}\left\|y_{n-1}-p\right\|, \left\|y_n - \mathscr{T}y_n + \frac{\sum_{i=2}^{k}\alpha_i}{\alpha_1}(y_n - \mathscr{T}^k y_n)\right\|, 0\right) \\
&= \gamma_n + f\left(\left\|y_{n-1}-p\right\|, \gamma_n, 0\right) \qquad (16)
\end{aligned}
$$

Applying limit as $n \to \infty$ to the last inequality, we have

$$
\begin{aligned}
\lim_{n\to\infty}\left\|y_n - p\right\| &\leq f\left(\lim_{n\to\infty}\left\|y_{n-1}-p\right\|, 0, 0\right) \\
&= a\lim_{n\to\infty}\left\|y_n - p\right\|, \text{ where } a < 1.
\end{aligned}
$$

This gives

$$
\lim_{n\to\infty}\left\|y_n - p\right\| = 0
$$

Therefore, the iterative sequence $\{x_n\}$ is stable. □

Remark 3.3. Theorem 3.2 shows that the $Fix(\overline{\mathscr{S}})$ is well-posed.

Let $Fix(\overline{\mathscr{T}})$ and $Fix(\overline{\mathscr{S}})$ be the set of all fixed points of the k-fold averaged and the enhanced k-fold averaged mappings associated with a weak enriched $\mathscr{F}-$contraction mappings, respectively. In [21], the sequence $\{x_n\}$ defined by the the k-fold averaged map exists and converges to a unique fixed point in $Fix(\overline{\mathscr{T}})$. In what follows, we show the necessary condition for the convergent of the enhanced k-fold Kirk map associated with the weak enriched $\mathscr{F}-$contraction map.

Theorem 3.4. *Let $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ be a self-map with $Fix(\mathscr{T})\neq\emptyset$. For $u_0, v_0\in\mathscr{C}$, if the sequence $\{u_n\}$ defined by the k-fold averaged map $\overline{\mathscr{T}}$ converges to p, then the sequence $\{v_n\}$ defined by the enhanced k-fold averaged map $\overline{\mathscr{S}}$ also converges to p.*

Proof. Let $u_0 \in \mathscr{C}$ and $Fix(\mathscr{T}) \neq \emptyset$. Suppose that the sequence $\{u_n\}$ converges to p, we prove that $\{v_n\}$ converges to p. Let $v_0 \in \mathscr{C}$ be arbitrary and using both $\overline{\mathscr{T}}$ and $\overline{\mathscr{S}}$, we have

$$\|v_n - p\| \leq \|v_n - u_n\| + \|u_n - p\| \tag{17}$$

But then,

$$\begin{aligned}
\|v_n - u_n\| &= \left\| (1 - \sum_{i=1}^{k} \alpha_i)(v_{n-1} - u_{n-1}) + \sum_{i=1}^{k} \alpha_i (\mathscr{T}^i v_n - \mathscr{T}^i u_{n-1}) \right\| \\
&= \alpha_1 \left\| \frac{(1 - \sum_{i=2}^{k} \alpha_i)}{\alpha_1} (v_{n-1} - u_{n-1}) + \mathscr{T} v_n - \mathscr{T} u_{n-1} + \sum_{i=2}^{k} \frac{\alpha_i}{\alpha_1} (\mathscr{T}^i v_n - \mathscr{T}^i u_{n-1}) \right\| \\
&\leq \alpha_1 f \left(\frac{1}{\alpha_1} \|v_{n-1} - u_{n-1}\|, \left\| \frac{\sum_{i=1}^{k} \alpha_i}{\alpha_1} (v_n - \mathscr{T}^k v_n) \right\|, \right. \\
&\qquad \left. \left\| \frac{\sum_{i=1}^{k} \alpha_i}{\alpha_1} (u_{n-1} - \mathscr{T}^k u_{n-1}) \right\| \right)
\end{aligned}$$

By adopting the condition $\mathscr{F}_3$, we have

$$\begin{aligned}
\|v_n - u_n\| &\leq f \left(\|v_{n-1} - u_{n-1}\|, \left\| \sum_{i=1}^{k} \alpha_i (v_n - \mathscr{T}^i v_n) \right\|, \left\| \sum_{i=1}^{k} \alpha_i (u_{n-1} - \mathscr{T}^i u_{n-1}) \right\| \right) \\
&= f \left(\|v_{n-1} - u_{n-1}\|, (1 - \sum_{i=1}^{k} \alpha_i) \|v_{n-1} - v_n\|, \|u_{n-1} - u_n\| \right) \\
&= a \|v_{n-1} - u_{n-1}\| + b(1 - \sum_{i=1}^{k} \alpha_i) \|v_{n-1} - v_n\| + c \|u_{n-1} - u_n\|
\end{aligned}$$

Inserting $\|v_n - u_n\|$ into inequality (17), we get

$$\|v_n - p\| \leq a \|v_{n-1} - u_{n-1}\| + b(1 - \sum_{i=1}^{k} \alpha_i) \|v_{n-1} - v_n\| + c \|u_{n-1} - u_n\| + \|u_n - p\|$$

This further implies

$$\begin{aligned}
\|v_n - p\| &\leq \frac{a + b(1 - \sum_{i=1}^{k} \alpha_i)}{1 - b(1 - \sum_{i=1}^{k} \alpha_i)} \|v_{n-1} - p\| \\
&\qquad + \frac{1}{1 - b(1 - \sum_{i=1}^{k} \alpha_i)} [(a + c) \|u_{n-1} - p\| + (c + 1) \|u_n - p\|]
\end{aligned}$$

Obviously, $\frac{a+b(1-\sum_{i=1}^{k} \alpha_i)}{1-b(1-\sum_{i=1}^{k} \alpha_i)} < 1$ for $a+b+c = 1$. Since $\|u_n - p\| = 0$, by the application of Lemma 2.2, we have that $\lim_{n\to\infty} \|v_n - p\| = 0$. Therefore, the sequence $\{v_n\}$ converges to p. □

Remark 3.5. The converse of Theorem 3.4 may be false, that is, the convergent of the enhanced k-fold Kirk iteration does not imply the convergent of the k-fold Kirk iteration even though $Fix(\overline{\mathscr{T}}) \neq \emptyset$. As shown in the next example, both the averaged maps $\overline{\mathscr{T}}$ and $\overline{\mathscr{T}}$ under the weak enriched $\mathscr{F}$-contraction map are not equivalent.

Example 3.6. Let $\mathscr{X} = \mathbb{R}$ be a usual normed space and $\mathscr{T}$ be a self-map of $\mathscr{X}$ defined by a $\mathscr{T}x = 1 - 2x$ with $Fix(\mathscr{T}) = \{\frac{1}{3}\}$.

By letting $b_1 = \frac{1}{2}$, $b_2 = 4$, $b_3 = 3$, $f(r,s,t) = ar + bs + ct$ for $a + b + c = 1$. Then, the map $\overline{\mathscr{S}}$ of the enhanced k-fold averaged map is a weak enriched $\mathscr{F}$-contraction for $k = 1,2,3$. On the other hand, the Picard map (2), and k-fold averaged map (9) (for $k = 1,2,3$) does not satisfy the weak enriched $\mathscr{F}$-contraction.

To see this, we select three different initial data to verify the behavior of $\mathscr{T}$ for $k = 1,2,3$.

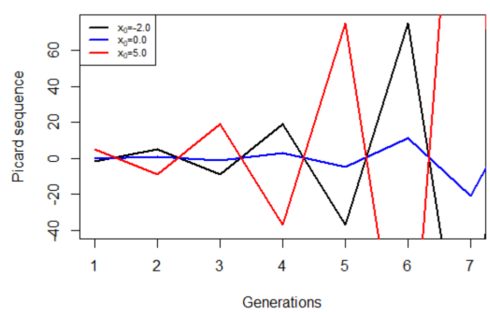

FIGURE 1. Picard sequence with three different initial seeds

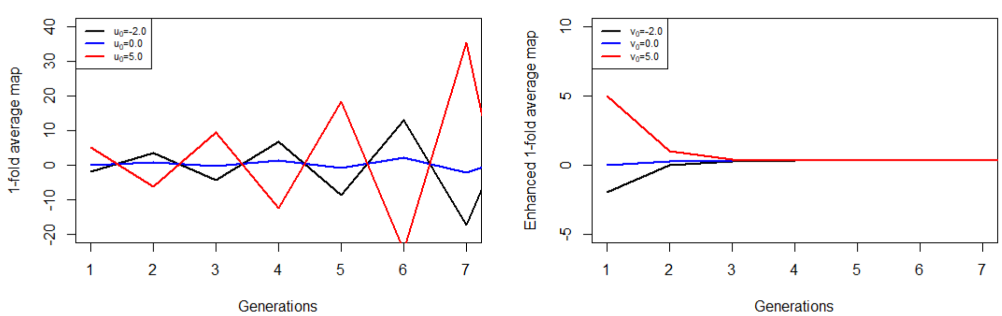

FIGURE 2. 1-fold and enhanced 1-fold averaged maps for $b_1 = \frac{1}{2}$.

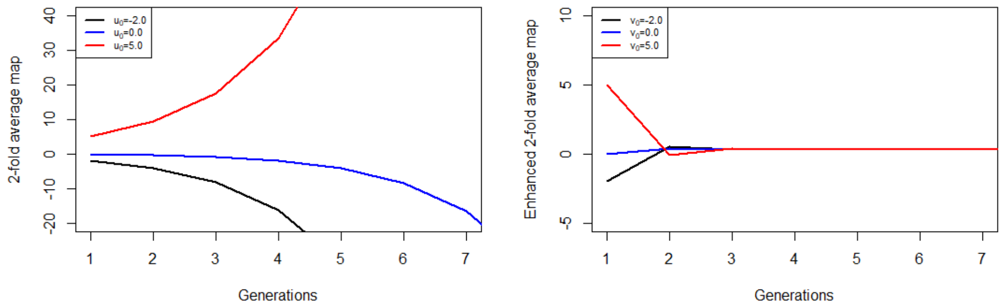

FIGURE 3. 2-fold and enhanced 2-fold averaged maps for $b_1 = \frac{1}{2}$, $b_2 = 4$.

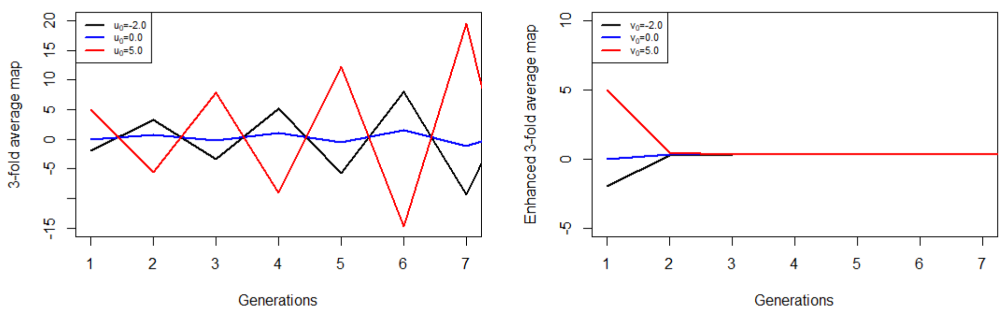

FIGURE 4. 3-fold and enhanced 3-fold averaged maps for $b_1 = \frac{1}{2}$, $b_2 = 4$, $b_3 = 3$.

In Figures 1, 2, 3, and 4, we show the behaviors of the Picard map and that of the k-fold averaged maps for $k = 1,2,3$; $b_1 = \frac{1}{2}$, $b_2 = 4$, $b_3 = 3$. By selecting three distinct initial data $-2,0,5 \in \mathbb{R}$, the sequence defined by the k-fold averaged map diverges for $k = 1,2,3$. On other hand, all the sequences defined by the enhanced map, with the same data points, are convergent. Also observe that for $u_0 = v_0$ and each k, the sequence $\{u_n - v_n\}$ becomes large as $n \to \infty$. This shows that u_n and v_n are not asymptotically equivalent. In what follows, we present a sufficient condition for the converse of Theorem 3.4.

Theorem 3.7. *Let $T : \mathscr{C} \to \mathscr{C}$ be a self-map with $Fix(\mathscr{T}) \neq \emptyset$. Assume that $\{u_n - v_n\}$ is asymptotically regular for $u_0, v_0 \in \mathscr{C}$, then the sequence $\{v_n\}$ defined by the map $\overline{\mathscr{S}}$ is convergent if and only if the sequence $\{u_n\}$ defined by the map $\overline{\mathscr{T}}$ is convergent.*

Proof. Let $v_0 \in \mathscr{C}$ and $Fix(\mathscr{T}) \neq \emptyset$. Suppose that the sequence $\{v_n\}$ converges to p and $\{u_n - v_n\}$ is asymptotically regular, for $u_0 \in \mathscr{C}$ we have that

$$\|u_n - p\| \leq \|u_n - v_n\| + \|v_n - p\|$$

Taking limit across the inequality as $n \to \infty$, this becomes

$$\lim_{n\to\infty} \|u_n - p\| = 0.$$

This completes the proof. $\square$

Remark 3.8. Theorem 3.7 demonstrates that the maps $\overline{\mathscr{T}}$ and $\overline{\mathscr{S}}$ are equivalent only if the sequence $\{u_n - v_n\}$ is asymptotically regular. However, equivalence of both $\overline{\mathscr{T}}$ and $\overline{\mathscr{S}}$ does not guarantee a similar convergent rate as illustrated in the next example.

Example 3.9. Let $\mathscr{T} : [0,1] \to [0,1]$ be defined by the map $\mathscr{T}x = \frac{x}{2}$ for $x \in [0,1]$ with $Fix(\mathscr{T}) = \{0\}$.

Let $x_0, u_0 \in (0,1]$ such that $x_0 \neq u_0$, the sequence $\{u_n - x_n\}$ is asymptotically regular for any $b_i \in (0,\infty)$, where $\{u_n\}$ and $\{x_n\}$ are defined by the maps $\overline{\mathscr{T}}$ and $\overline{\mathscr{S}}$, respectively. In what follows, we show that $\{x_n\}$ has better convergent rate than $\{u_n\}$. In particular, let $k = 1$ and $b_1 = \frac{\sqrt{3}}{2}$, then the 1-fold averaged map is obtained as follow:

$$\begin{aligned} u_n &= \frac{b_1}{b_1+1} u_{n-1} + \frac{1}{b_1+1} \mathscr{T} u_{n-1} \\ &= (1 - \frac{2}{\sqrt{3}+2}) u_{n-1} + \frac{1}{\sqrt{3}+2} u_{n-1} \\ &= \frac{\sqrt{3}+1}{\sqrt{3}+2} u_{n-1} \end{aligned}$$

Similarly, the enhanced 1-fold averaged map is obtained as:

$$x_n = \frac{\sqrt{3}}{\sqrt{3}+2} x_{n-1}$$

By using Definition 2.3, we have

$$\begin{aligned} \frac{|x_n - 0|}{|u_n - 0|} &= \frac{\sqrt{3}}{\sqrt{3}+2} \div \frac{\sqrt{3}+1}{\sqrt{3}+2} \times \frac{x_{n-1}}{u_{n-1}} \\ &= \prod_{j=1}^{n} \left(\frac{\sqrt{3}}{\sqrt{3}+2} \right)^j \end{aligned}$$

Applying limit as $n \to \infty$ to both sides, we have

$$\lim_{n\to\infty} \frac{|x_n - 0|}{|u_n - 0|} = 0.$$

Therefore, the sequence defined by the enhanced 1-fold averaged map has a better convergent rate than the 1-fold averaged map.

For $k = 2$, let $x_0, u_0 \in (0,1]$ such that $x_0 = u_0$, $b_1 = \frac{\sqrt{3}}{2}$, and $b_2 = 2$, then the sequences defined by 2-fold and enhanced 2-fold averaged maps are, respectively, obtained as:

$$u_n = \frac{\sqrt{3}+2}{\sqrt{3}+6} u_{n-1}$$

and

$$x_n = \frac{\sqrt{3}}{\sqrt{3}+4} x_{n-1}$$

Thus,

$$\begin{aligned}\lim_{n\to\infty} \frac{|x_n - 0|}{|u_n - 0|} &= \lim_{n\to\infty} \frac{\sqrt{3}}{\sqrt{3}+4} \cdot \frac{\sqrt{3}+6}{\sqrt{3}+2} \cdot \frac{x_{n-1}}{u_{n-1}} \\ &= \lim_{n\to\infty} \prod_{j=1}^{n} \left(\frac{2\sqrt{3}+1}{2\sqrt{3}+3} \right)^j = 0.\end{aligned}$$

Therefore, the sequence defined by the enhanced 2-fold averaged map converges faster than the 2-fold averaged map, and for $k > 2$.

4. APPLICATION TO BOUNDARY LAYER PROBLEM

The boundary layer problem is widely used in engineering to study fluid flow behavior near a solid boundary, such as a surface or an object moving through a fluid. It helps to understand the interaction between the fluids and surfaces which is essential in real-life applications. Here, we consider an equation of the boundary layer

$$\frac{dx(t)}{dt} = \frac{1}{\varepsilon}(x(t) - 1),\ x(0) = 0, \tag{18}$$

where ε is a small parameter. The equation (18) is Lipschitz continuous with a Lipschitz constant $\frac{1}{\varepsilon}$ but only has a boundary layer near $t = 0$ due to the nature of ε. The equation (18) is equivalent to integral model:

$$x(t)=\frac{1}{\varepsilon}\int_0^t (x(s)-1)ds, \tag{19}$$

Let $\mathscr{X}=C(I)$, $C(I)$ a space of all real-valued continuous functions defined on I. For $x(t)\in C(I)$, the function $\|\cdot\|:C(I)\to\mathbb{R}_+$ defines the norm

$$\|x\|=\sup_{t\in I}\{|x(t)|\}$$

and $(C(I),\|\cdot\|)$ is a Banach space. Let $\mathscr{T}$ be a self-map of $C(I)$ with the property that $x(t)\in C(I)$ whenever $\mathscr{T}x(t)\in C(I)$, then $\mathscr{T}$ is a weak enriched $\mathscr{F}$-contraction. In what follows, we determine the map that can effectively capture the behavior of the boundary layer for small and large step sizes. By setting $x_0=0$, $\varepsilon=0.1$, $b_1=\frac{1}{9}$, $k=1$, and step length 0.125, we compare the averaged iterations with the solution of (18) as follows:

TABLE 1. Comparison of the procedures (2), (9), and (10) using (18).

Part A: Small step-length				
$\mathscr{T}$	Exact values	Picard iteration	1-fold iteration (9)	1-fold iteration (10)
		Approx. (Error)		
0.0	0.0	0.0	0.0	0.0
0.025	-0.2840	-0.2500 (0.0340)	-0.2250 (0.0590)	-0.2840 ($1.8e^{-5}$)
0.050	-0.6487	-0.6250 (0.0237)	-0.5963 (0.0525)	-0.6488 ($4.5e^{-5}$)
0.075	-1.1170	-1.1016 (0.0590)	-0.9254 (0.1916)	-1.1190 ($1.9e^{-3}$)
0.100	-1.7183	-1.7083 ($9.98e^{-3}$)	-1.5744 (0.1438)	-1.7181 ($1.7e^{-4}$)
0.125	-2.4903	-2.4839 ($6.4e^{-3}$)	-2.3795 (0.1108)	-2.4886 ($1.6e^{-3}$)
0.150	-3.4817	-3.4747 ($6.99e^{-3}$)	-3.3950 ($8.6e^{-2}$)	-3.4808 ($9.0e^{-4}$)
0.175	-4.7546	-4.7429 (0.0117)	-4.6859 ($6.8e^{-2}$)	-4.7529 ($1.6e^{-3}$)
0.200	-6.3891	-6.3656 (0.0549)	-6.3342 ($5.3e^{-2}$)	-6.3915 ($2.4e^{-3}$)
Part B: Big step-length				
t	Exact values	Approx.		
1.0	-22025.4657	-10.0	-9.0	-22039.3882
2.0	-485165194.4098	-220.0	-181.8	-485227170.9249

In Table 4, we compare three iterative procedures (2), (9), and (10) with $k = 1$ to approximate the function $x(t_n)$ for each t_n with step length 0.125. As observed in the accuracy, the enhanced averaged iteration compares favorably with the solution of (18) while other procedures have lesser accuracy.

More so, by setting $x_0 = 0$, $\varepsilon = 0.1$, $b_1 = \frac{1}{9}$, and step length 1, there is a sharp transition in the solution. But then, only the enhanced averaged iteration nearly captures the solution while other procedures are struggling to approximate $x(t)$.

5. Conclusions

The study introduced enhanced k-fold averaged iteration for approximating fixed points of operators that do not meet the hypotheses of the k-fold averaged iteration under weak enriched $\mathscr{F}$-contraction. We proved the strong convergent and stability of the enhanced k-fold averaged iteration satisfying the weak enriched $\mathscr{F}$-contraction in Banach spaces. We gave the conditions for the equivalent of the improved k-fold averaged iteration with the k-fold averaged iteration. The validation of the improved map was investigated using some numerical examples. The results showed that the enhanced averaged iteration has a better convergent rate than other related iterations and its map exhibits contracting behavior when others fail for some enriching constants. The results were applied to solve a boundary layer problem before and after a sharp transition.

Conflict of Interests

The authors declare that there is no conflict of interests.

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