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SUPER MULTIPLICATIVE METRIC SPACES AND ITS APPLICATIONS

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Abstract. In this paper, we introduce the general form of multiplicative metric spaces known as super multiplicative metric spaces. First we prove an analogous of Banach contraction principle in the setting of super multiplicative metric spaces and prove various results related to expansive mappings, ϕ weak contraction, weak compatible maps, property (E.A) and any kind weakly compatible maps.

Keywords: super multiplicative metric space; expansive mappings; weak compatible maps.
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1. PRELIMINARIES

In fixed point theory literature, a large number of results either coincides or deduced from existing results in literature. To overcome these difficulties, Karapinar et al. [8] introduced the notion of super metric spaces with the help of b-metric spaces [4] and generalized metric spaces [17].

In 1989, Bakhtin [4] introduced the notion of b-metric as follows:

Let X be a non-empty set. A function $b : X \times X \longrightarrow [0,\infty)$ satisfies the following conditions:

(1) for every $x, y \in X$, $b(x,y) \geq 0$ and $b(x,y) = 0$ iff $x = y$;

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(2) for every $x,y \in X$, $b(x,y) = b(y,x)$;

(3) there exits $s \geq 1$ such that $b(x,y) \leq s[b(x,z)+b(z,y)]$ for every $x,y,z \in X$.

Then (X,b,s) is called a $b-$metric space.

In 1993, Czerwik [7] proved Banach contraction principle in the setting of $b-$metric space.

A decade ago, Jleli et al.[17] introduced the notion of generalized metric as follows: Let $X \neq \phi$ be a non-empty set. A generalized metric D on $X \times X \to [0,\infty)$ satisfies the following conditions:

(1) for every $x,y \in X$, $D(x,y) \geq 0$ and $D(x,y) = 0$ iff $x = y$;

(2) for every $x,y \in X$, we have $D(x,y) = D(y,x)$;

(3) there exists $C > 0$ such that if $x,y \in X$, $\langle x_n \rangle \in C(D,X,x)$, then
$D(x,y) \leq C \lim_{n\to\infty} sup D(x_n,y)$, where $C(D,X,x) = \{\{x_n\} \in X : \lim_{n\to\infty} D(x_n,x) = 0\}$.

Then (X,D) is called a generalized metric space.

Recently, Karapinar et al. [8] introduced the notion super-metric spaces as follows:

Definition 1.1. Let X be a non-empty set. A function $m : X \times X \to [0,\infty)$ is a super-metric or super metric if

(1) $m(x,y) \geq 0$, if $m(x,y) = 0$, then $x = y$ *for all* $x,y \in X$

(2) $m(x,y) = m(y,x)$, *for all* $x,y \in X$

(3) there exists $s \geq 1$ such that for all $y \in X$ there exist distinct sequences $(x_n),(y_n) \subset X$, with $m(x_n,y_n) \longrightarrow 0$ as $n \longrightarrow \infty$ such that

$$\lim_{n\to\infty} \sup m(y_n,y) \leq s \lim_{n\to\infty} \sup m(x_n,y).$$

In 2008, Bashirov et al. [5] introduced the concept of multiplicative metric spaces as follows:

Definition 1.2. Let X be a non-empty set. A multiplicative metric is a mapping $d^* : X \times X \longrightarrow [1,\infty)$ satisfying conditions:

(1) $d^*(x,y) \geq 1$ for all $x,y \in X$ and $d^*(x,y) = 1$ if and only if $x = y$;

(2) $d^*(x,y) = d^*(y,x)$ for all $x,y \in X$;

(3) $d^*(x,y) \leq d^*(x,z).d^*(z,y)$ for all $x,y,z \in X$. Then (X,d^*) is known as multiplicative metric spaces.

Motivated by the notion of super metric and multiplicative metric spaces we introduce the general notion of multiplicative metric spaces known as super multiplicative metric spaces.

Definition 1.3. Let X be a non-empty set. A function $m^*: X \times X \to [1,\infty)$ is a super multiplicative metric if

(1) $m^*(x,y) \geq 1$, for all $x,y \in X$ and $m^*(x,y) = 1$ then $x = y$ *for all* $x,y \in X$;
(2) $m^*(x,y) = m^*(y,x)$ *for all* $x,y \in X$;
(3) there exists $s \geq 1$ such that for all $y \in X$ there exist distinct sequences $(x_n),(y_n) \subset X$, with $m^*(x_n,y_n) \longrightarrow 1$ as $n \longrightarrow \infty$ and

$$\lim_{n\to\infty} \sup m^*(y_n,y) \leq \lim_{n\to\infty} \sup m^{*s}(x_n,y).$$

Then (X,m^*) is called a super multiplicative metric space.

Definition 1.4. Let (X,m^*) be a super multiplicative metric space, $x \in X$ and $\varepsilon > 1$. Define a set

$$B_\varepsilon(x) = \{y \in X \,|\, m^*(x,y) < \varepsilon\},$$

which is called a super multiplicative open ball of radius ε with centre x. Similarly, one can describe super multiplicative closed ball as

$$\bar{B}_\varepsilon(x) = \{y \in X \,|\, m^*(x,y) \leq \varepsilon\}.$$

Definition 1.5. Let (X,m^*) be a super multiplicative metric space. A sequence $\langle x_n \rangle$ in X is said to be

(1) convergent to a point x, if $\lim_{n\to\infty} m^*(x_n,x) = 1$.
(2) Cauchy sequence, if $\lim_{n,m\to\infty} \sup m^*(x_n,x_m) = 1$. In other words, for every $\varepsilon > 0$ there exists n_0 such that $m^*(x_n,x_m) < \varepsilon$ for all $n,m \geq n_0$.

Example 1.6. Let $X = [1,\infty)$ and define
$m^*: X \times X \longrightarrow [0,\infty)$ by

$$m^*(x,y) = \begin{cases} e^{\frac{|x-y|+|x|+|y|}{2}} & if\, x \neq y \neq 0 \\ 1 & if\, x = y = 0. \end{cases}$$

Then (X,m^*) is a super multiplicative metric space.

Proof. Clearly $m^*(x,y) \geq 1$

If $m^*(x,y) = 1$, then $e^{\frac{|x-y|+|x|+|y|}{2}} = 1$, implies $x = y$.

But if $x = y \neq 0$, then $m^*(x,x) = e^{\frac{0+|x|+|x|}{2}} = e^{|x|} \neq 1$

Clearly $m^*(x,y) = m^*(y,x)$

Suppose for all $y \in X$ and $\langle x_n \rangle$ and $\langle y_n \rangle$ be two distinct sequences in X such that $m^*(x_n, y_n) \to 1\, as\, n \to \infty$, i.e,

$$\lim_{n\to\infty} e^{\frac{|x_n-y_n|+|x_n|+|y_n|}{2}} \to 1,$$

so $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.

Now there exists $n_0 > 0$ such that for $n \geq n_0$, we have

$$\begin{aligned}
\limsup_{n\to\infty} m^*(y_n,y) &= \limsup_{n\to\infty} e^{\frac{|y_n-y|+|y_n|+|y|}{2}} \\
&= e^{|y|} \\
&\leq e^{s|y|} \\
&= \limsup_{n\to\infty} e^{s\frac{|x_n-y|+|x_n|+|y|}{2}} \\
&= \limsup_{n\to\infty} m^{*s}(x_n,y)
\end{aligned} \tag{1}$$

In case $y = 0$, the proof is straight forward. Hence, (X, m^*) is a super multiplicative metric space. It is worth-mentioning that it is not a multiplicative metric space as $m^*(x,y) \neq 1$ when $x = y \neq 0$. □

2. Relationship Between Super Metric Spaces and Super Multiplicative Spaces

We establish the relation between super metric spaces and super multiplicative metric spaces as follows:

Let (X, m^*) be a super multiplicative metric space. Define $m : X \times X \longrightarrow [0,\infty)$ by

$$m(x,y) = ln(m^*(x,y))$$

Then (X,m) is a super metric space. It follows from the properties of logarithms.

Moreover, if we have a super metric space (X,m) then the corresponding super multiplicative metric space (X,m^*) is given by

$$m^*(x,y) = e^{m(x,y)}.$$

It follows from the properties of exponential functions. One can interchange and prove many applications of super multiplicative spaces to super metric spaces and vice versa.
One can note that if (X,m^*) is a complete super multiplicative metric space, then corresponding super metric space (m,X) is also a complete super metric space i.e.,

$$m(fx_1,fx_2) = ln(m^*(fx_1,fx_2)) \leq \lambda\, m(x_1,x_2),$$

$$m^*(fx_1,fx_2) = e^{m(fx_1,fx_2)} \leq \lambda\, m^*(x_1,x_2).$$

3. Fixed Point Theorems for Various Contractions in Super Multiplicative Metric Spaces

We prove an analogue of the Banach contraction principle in the setting of super-multiplicative metric spaces. The Banach contraction priciple in metric space states:
Let T be a self map on a complete metric space $X \neq \phi$ satisfying the following:

$$d(Tx,Ty) \leq \alpha d(x,y),\ 0 \leq \alpha < 1.$$

Then T has a unique fixed point. Before proving our main results, we need the following lemma.

Lemma 3.1. *Let T be a self mapping on a complete super multiplicative metric space X satisfying*

$$m^*(Tx,Ty) \leq m^{*\alpha}(x,y),\ 0 \leq \alpha < 1. \tag{2}$$

Then $\langle x_n \rangle$ is a Cauchy sequence in X.

Proof. Let $x_0 \in X$ and set $x_1 = Tx_0$. For this x_1 , there exists x_2 such that $x_2 = Tx_1$. Continuing this way, one can define in general,

$$x_{n+1} = Tx_n\ \forall n = 0,1,2,\dots\ . \tag{3}$$

From (2), we have

$$
\begin{aligned}
m^*(x_{n+1},x_n) &= m^*(Tx_n,Tx_{n-1}) \leq m^{*\alpha}(x_n,x_{n-1}) \\
&\leq m^{*\alpha^2}(x_{n-1},x_{n-2}) \\
&\vdots \\
&\leq m^{*\alpha^n}(x_1,x_0).
\end{aligned} \tag{4}
$$

Proceeding limit as $n \to \infty$, we have $\lim_{n\to\infty} m^*(x_n,x_{n+1}) = 1$, since $0 \leq \alpha < 1$.
Now by definition of super multiplicative metric spaces, for $s \geq 1$ and for all $x_{n+2} \in X$ there exist distinct sequences $\langle x_n\rangle, \langle x_{n+1}\rangle$ with $\lim_{n\to\infty} m^*(x_n,x_{n+1}) \to 1$ then

$$\lim_{n\to\infty} \sup m^*(x_n,x_{n+2}) \leq \lim_{n\to\infty} \sup m^{*s}(x_{n+1},x_{n+2}).$$

Since $\lim_{n\to\infty} m^*(x_n,x_{n+1}) = 1$, therefore, $\lim_{n\to\infty} \sup m^*(x_n,x_{n+2}) = 1$. Continuing in this way, for $s \geq 1$ and for all $x_{n+3} \in X$ there exist distinct sequences $\langle x_n\rangle, \langle x_{n+2}\rangle$ with $\lim_{n\to\infty} m^*(x_n,x_{n+2}) \to 1$ such that

$$\lim_{n\to\infty} \sup m^*(x_n,x_{n+3}) \leq \lim_{n\to\infty} \sup m^{*s}(x_{n+2},x_{n+3}).$$

i.e., $\lim_{n\to\infty} \sup m^*(x_n,x_{n+3}) = 1$.
Inductively, one can conclude

$$\lim_{n\to\infty} \sup m^*(x_n,x_m) = 1, \textit{ for all} m,n \in \mathbb{N}\ m > n.$$

Thus $\langle x_n\rangle$ is Cauchy sequence in X. □

Now we prove analogue of Banach contraction principle in the setting of super multiplicative metric spaces.

Definition 3.2. Let (X,m^*) be a super multiplicative metric space. A map T is said to be contraction if $m^*(Tx,Ty) \leq m^{*\alpha}(x,y)$ for all $x,y \in X$, $0 \leq \alpha < 1$.

Theorem 3.3. *Let (X,m^*) be a complete super multiplicative metric space and T be a contraction mapping on (X,m^*) i.e.,*

$$m^*(Tx,Ty) \leq m^{*\alpha}(x,y) \textit{ for all } x,y \in X,\ 0 \leq \alpha < 1. \tag{5}$$

Then T has a unique fixed point in X.

Proof. By Lemma 3.1, the sequence $\langle x_n \rangle$ defined by (3) is a Cauchy sequence in X. Since (X,m^*) is a complete super multiplicative metric space, therefore, the sequence $\langle x_n \rangle$ converges to a point say $z \in X$.

We claim that z be a fixed point of T. Now,

$$m^*(x_{n+1},Tz) = m^*(Tx_n,Tz) \leq m^{*\alpha}(x_n,z) \textit{ for all } n = 0,1,2,3.... \ .$$

Proceeding limit $n \to \infty$, we get $Tz = z$.

This implies z is a fixed point of T.

Uniqueness:

Let $w(\neq z)$ be another fixed point of T.

$m^*(w,z) = m^*(Tw,Tz) \leq m^{*\alpha}(w,z)$, a contradiction, since $0 \leq \alpha < 1$.

This implies $z = w$.

Hence T has a unique fixed point. □

In 1968, Kannan [10] gave this following contraction in metric space:

Let (X,d) be a complete metric space and let T a self map such that there exists $k < 1/2$ satisfying

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)]$$

for all $x,y \in X$. Then T has a unique fixed point.

Now we prove this in the setting of super multiplicative metric space.

Theorem 3.4. *Let (X,m^*) be a complete super multiplicative metric space and T be a self map on X satisfying Kannan contraction:*

$$m^*(Tx,Ty) \leq \{m^*(x,Tx).m^*(y,Ty)\}^{\alpha} \tag{6}$$

for all $x,y \in X$ and $0 < \alpha < 1/2$. Then T has a fixed point in X.

Proof. Let $x_0 \in X$ and consider the iterate of sequence $x_{n+1} = Tx_n$.

From (6), we have

$$m^*(x_1,x_2) = m^*(Tx_0,Tx_1) \leq \{m^*(x_0,Tx_0).m^*(x_1,Tx_1)\}^{\alpha}$$

$$\leq \{m^*(x_0,x_1).m^*(x_1,x_2)\}^{\alpha}$$

$$m^*(x_1,x_2) \leq m^{*\frac{\alpha}{1-\alpha}}(x_0,x_1)$$

$$i.e, m^*(x_1,x_2) \leq m^{*\beta}(x_0,x_1) \tag{7}$$

$$where\ \beta = \frac{\alpha}{1-\alpha} < 1$$

Continuing, this way we get

$$m^*(x_n,x_{n+1}) \leq m^{*\beta^n}(x_0,x_1)$$

Proceeding limit as $n \to \infty$, we have $\lim_{n\to\infty} m^*(x_n,x_{n+1}) = 1$ as $0 \leq \alpha < 1$.

Now by definition of super metric space, for $s \geq 1$ and for all $x_{n+2} \in X$, there exist distinct sequences $\langle x_n \rangle, \langle x_{n+1} \rangle$ with $m^*(x_n,x_{n+1}) \to 1$ such that

$$\lim_{n\to\infty} \sup m^*(x_n,x_{n+2}) \leq \lim_{n\to\infty} \sup m^{*s}(x_{n+1},x_{n+2}).$$

Since $\lim_{n\to\infty} m^*(x_n,x_{n+1}) = 1$, therefore $\lim_{n\to\infty} \sup m^*(x_n,x_{n+2}) = 1$. Further,for $s \geq 1$ and for all $x_{n+3} \in X$ there exist distinct sequences $\langle x_n \rangle, \langle x_{n+2} \rangle$ with $m^*(x_n,x_{n+2}) \to 1$ such that

$$\lim_{n\to\infty} \sup m^*(x_n,x_{n+3}) \leq \lim_{n\longrightarrow\infty} \sup m^{*s}(x_{n+2},x_{n+3}).$$

This implies $\lim_{n\to\infty} \sup m^*(x_n,x_{n+3}) = 1$.

Inductively, one can conclude that

$$\lim_{n\to\infty} \sup m^*(x_n,x_m) = 1\ , m > n\, and\, m,n \in \mathbb{N}.$$

Thus $\langle x_n \rangle$ is Cauchy sequence in X. Since (X,m^*) is a complete super metric space, therefore, the sequence $\langle x_n \rangle$ converges to a point, say $z \in X$.

We claim that z be a fixed point of T.

From (6), we have

$$m^*(x_{n+1},Tz) = m^*(Tx_n,Tz) \leq \{m^*(x_n,Tx_n).m^*(z,Tz)\}^{\alpha}$$

Proceeding as $n \to \infty$

$$m^*(Tz,z) \leq \{m^*(z,Tz).m^*(z,Tz)\}^{\alpha}$$

$$= m^{*2\alpha}(z,Tz)$$

$$m^{*(1-2\alpha)}(z,Tz) \leq 1 \;\; since\; 0 \leq \alpha < 1/2 \tag{8}$$

i.e, $Tz = z$.

Therefore z is a fixed point for T. □

Reich [22] in 1971 gave the following contraction for complete metric space:

Let (X,d) be a complete metric space and T be a self map with property:

$$d(Tx,Ty) \leq ad(x,Tx) + bd(y,Ty) + cd(x,y),$$

for all $x,y \in X$, where a,b,c are non-negative and satisfy $a+b+c < 1$. Then T has a unique fixed point. Next we prove Reich type contraction [22] in setting of super multiplicative metric spaces.

Theorem 3.5. *Let (X,m^*) be a complete super multiplicative metric space and T be a Reich type contraction map on X ,and there exits non-negative numbers a,b,c with $a+b+c < 1$ such that*

$$m^*(Tx,Ty) \leq m^{*a}(x,y).m^{*b}(x,T(x)).m^{*c}(y,T(y)). \tag{9}$$

Then T has a fixed point.

Proof. Let $x_0 \in X$ and consider the iterate of sequence $x_{n+1} = Tx_n$.

From (9),

$$m^*(x_1,x_2) = m^*(Tx_0,Tx_1) \leq m^{*a}(x_0,x_1).m^{*b}(x_0,Tx_0).m^{*c}(x_1,Tx_1)$$

$$= m^{*a}(x_0,x_1).m^{*b}(x_0,x_1).m^{*c}(x_1,x_2)$$

$$m^{*(1-c)}(x_1,x_2) \leq m^{*(a+b)}(x_0,x_1)$$

$$m^*(x_1,x_2) \leq m^{*\frac{a+b}{1-c}}(x_0,x_1)$$

$$m^*(x_1,x_2) \leq m^{*\beta}(x_0,x_1),\; where\; \beta = \frac{a+b}{1-c}.$$

Continuing this, way we get

$$m^*(x_n,x_{n+1}) \leq m^{*\beta^n}(x_0,x_1),\;\; where\; \beta < 1$$

Proceeding limit as $n \to \infty$, we have $\lim_{n\to\infty} m^*(x_n,x_{n+1}) = 1$ as $0 \le \alpha < 1$.
Now by definition of super metric space, for $s \ge 1$ and for all $x_{n+2} \in X$ there exist distinct sequences $\langle x_n\rangle, \langle x_{n+1}\rangle$ with $m^*(x_n,x_{n+1}) \to 1$ such that

$$\lim_{n\to\infty} \sup m^*(x_n,x_{n+2}) \le \lim_{n\to\infty} \sup m^{*s}(x_{n+1},x_{n+2}).$$

Since $\lim_{n\to\infty} m^*(x_n,x_{n+1}) = 1$. Therefore ,$\lim_{n\to\infty} \sup m^*(x_n,x_{n+2}) = 1$.Similarly we have for $s \ge 1$ and for all $x_{n+3} \in X$ there exist distinct sequences $\langle x_n\rangle, \langle x_{n+2}\rangle$ with $m^*(x_n,x_{n+2}) \to 1$ such that

$$\lim_{n\to\infty} \sup m^*(x_n,x_{n+3}) \le \lim_{n\longrightarrow\infty} \sup m^{*s}(x_{n+2},x_{n+3}).$$

i.e., $\lim_{n\to\infty} \sup m^*(x_n,x_{n+3}) = 1$.
Inductively, one can conclude

$$\lim_{n\to\infty} \sup m^*(x_n,x_m) = 1\,, m > n\, and\, m,n \in \mathbb{N}.$$

Thus $\langle x_n\rangle$ is Cauchy sequence in X. Since (X,m^*) is a complete super metric space, therefore, the sequence $\langle x_n\rangle$ converges to a point say $z \in X$.
We claim that z be a fixed point of T.
From(9) , we have,

$$m^*(x_{n+1},Tz) = m^*(Tx_n,Tz) \le m^{*a}(x_n,z).m^{*b}(x_n,Tx_n).m^{*c}(z,Tz),$$

letting as $n \to \infty$ *, we have*

$$m^*(z,Tz) \le m^{*b}(z,Tz).m^{*c}(z,Tz) \tag{10}$$

$$m^*(z,Tz) \le m^{*(b+c)}(z,Tz)$$

$$m^{*(1-(b+c))}(z,Tz) \le 1$$

since, $b+c<1$ we get ,$Tz = z$.
So we have z is a fixed point for T. □

In 1977, Jaggi [12] proved the following contraction for metric space:
Let T be a continous self map defined on a complete metric space (X,d). Suppose that T satisfies the following contraction condition:

$$d(Tx,Ty) \le \alpha \left[\frac{d(x,Tx)d(y,Ty)}{d(x,y)}\right] + \beta d(x,y)$$

for all $x, y \in X$, $x \neq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. Then T has a unique fixed point in X.

Now, we prove this contraction in the setting of super multiplicative metric space as follows:

Theorem 3.6. *Let f be a continuous self map defined on a complete super-multiplicative metric space (X, m^*). Suppose that f satisfies the following contractive condition:*

$$m^*(fx, fy) \leq \frac{m^{*\alpha}(x, fx) m^{*\alpha}(y, fy)}{m^{*\alpha}(x,y)} m^{*\beta}(x,y) \tag{11}$$

for all $x, y \in X$, $x \neq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta \leq 1$. Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ and consider the iterate of sequence $fx_n = x_{n+1}$.

From (11), we have

$$\begin{aligned} m^*(x_1,x_2) = m^*(fx_0, fx_1) &\leq \frac{m^{*\alpha}(x_0, fx_0) m^{*\alpha}(x_1, fx_1)}{m^{*\alpha}(x_0,x_1)} m^{*\beta}(x_0,x_1) \\ &= \frac{m^{*\alpha}(x_0,x_1) m^{*\alpha}(x_1,x_2)}{m^{*\alpha}(x_0,x_1)} m^{*\beta}(x_0,x_1) \\ m^{*(1-\alpha)}(x_1,x_2) &\leq m^{*\beta}(x_0,x_1) \\ m^*(x_1,x_2) &\leq m^{*(\frac{\beta}{1-\alpha})}(x_0,x_1) \end{aligned} \tag{12}$$

Continuing n times we have

$$m^*(x_n, x_{n+1}) \leq m^{*(\frac{\beta}{1-\alpha})^n}(x_0,x_1)$$

Taking limit as $n \to \infty$ we get $m^*(x_n, x_{n+1}) = 1$

Now by definition of super multiplicative metric space, for $s \geq 1$ and for all $x_{n+2} \in X$, there exist distinct sequences $\langle x_n \rangle$, $\langle x_{n+1} \rangle$ with $m^*(x_n, x_{n+1}) \to 1$ such that

$$\lim_{n\to\infty} sup\, m^*(x_{2n-1}, x_{2n+1}) \leq \lim_{n\to\infty} sup\, m^{*s}(x_{2n}, x_{2n-1}).$$

Since $\lim_{n\to\infty} m^*(x_n, x_{n+1}) = 1$, therefore $\lim_{n\to\infty} \sup m^*(x_n, x_{n+2}) = 1$. Further,for $s \geq 1$ and for all $x_{n+3} \in X$ there exist distinct sequences $\langle x_n \rangle, \langle x_{n+2} \rangle$ with $m^*(x_n, x_{n+2}) \to 1$ such that

$$\lim_{n\to\infty} \sup m^*(x_n, x_{n+3}) \leq \lim_{n\longrightarrow\infty} \sup m^{*s}(x_{n+2}, x_{n+3}).$$

This implies $\lim_{n\to\infty}\sup m^*(x_n,x_{n+3}) = 1$.

Inductively, one can conclude that

$$\lim_{n\to\infty}\sup m^*(x_n,x_m) = 1\ , m > n\, and\, m,n \in \mathbb{N}.$$

Thus $\langle x_n\rangle$ is Cauchy sequence in X. Since (X,m^*) is a complete super metric space, therefore, the sequence $\langle x_n\rangle$ converges to a point, say $z \in X$.

We claim that z be a fixed point of f.

From (11), we have

$$m^*(x_{n+1}, fz) \leq \frac{m^{*\alpha}(x_n, fx_n)m^{*\alpha}(z, fz)}{m^{*\alpha}(x_n, z)} m^{*\beta}(x_n, z)$$

Proceeding limit as $n \to \infty$

$$m^*(z, fz) \leq m^{*\alpha}(z, fz)$$

$$m^{*(1-\alpha}(z, fz) \leq 1$$

implies $fz = z$ as $\alpha < 1$. Hence f has a fixed point in X. □

4. Fixed Point Theorems for Pair of Mappings

G. Jungck [13] proved an interesting result for commutative mapping in metric spaces as follows:

Let (X,d) be a complete metric space and f be a continuous self mapping of (X,d), if there exists a mapping $g : X \to X$ and a constant $0 \leq \alpha < 1$ such that

(1) $fgx = gfx$ for every $x \in X$.

(2) $g(X) \subset f(X)$

(3) $d(gx, gy) \leq \alpha d(fx, fy)$ for every $x,y \in X$.

Then f and g have a unique common fixed point.

Now we prove the Jungck fixed point theorem in the setting of super-multiplicative metric spaces as follows:

Theorem 4.1. *Let (X,m^*) be a complete super multiplicative metric spaces and f be a continuous self mapping of super multiplicative metric space (X,m^*). If there exists a mapping $g : X \to X$ and a constant $0 \leq \alpha < 1$ such that*

(1) $fgx = gfx$ *for every* $x \in X$,

(2) $g(X) \subset f(X)$,

(3) $m^*(gx, gy) \leq m^{*\alpha}(fx, fy)$ *for every* $x, y \in X$.

Then f *and* g *have a unique common fixed point.*

Proof. Let $x_0 \in X$ and $x_1 \in X$ be such that $gx_0 = fx_1$. Then, by induction, a sequence $\langle y_n \rangle$ in X as follows:

$$fx_n = gx_{n-1} = y_n(say), \; n = 0, 1, 2, 3, ... \tag{13}$$

From (3) , we have

$$\begin{aligned} m^*(y_{n+1}, y_n) = m^*(fx_{n+1}, fx_n) = m^*(gx_n, g_{n-1}) &\leq m^{*\alpha}(fx_n, fx_{n-1}) = m^{*\alpha}(y_n, y_{n-1}) \\ &\leq m^{*\alpha^2}(y_{n-1}, y_{n-2}) \\ &\vdots \\ &\leq m^{*\alpha^n}(y_1, y_0). \end{aligned} \tag{14}$$

Proceeding from the limit as $n \to \infty$, we have $\lim_{n\to\infty} m^*(y_{n+1}, y_n) = 1$, since $0 \leq \alpha < 1$. Now by Lemma 3.1 we get $\langle y_n \rangle$, i.e., $\langle fx_n \rangle$ is a Cauchy sequence in X. But X is complete so $\lim_{n\to\infty} fx_n = t$. Now from (13) $\lim_{n\to\infty} gx_n = t$. Since f is continuous, so by (3), f and g both are continuous. Therefore $fgx_n \to ft$ and $gfx_n \to gt$. But f and g commute for all $x \in X$, so $gfx_n = fgx_n \; \forall n$, i.e., $gt = ft$. Consequently $f(ft) = f(gt) = g(gt)$.

$$m^*(gt, ggt) \leq m^{*\alpha}(ft, fgt) = m^{*\alpha}(gt, ggt)$$

$$m^{*(1-\alpha)}(gt, ggt) \leq 1.$$

This implies $gt = ggt$, since $0 \leq \alpha < 1$. Therefore, $gt = g(gt) = f(gt)$, i.e., gt is common fixed point of f and g.

Uniqueness

Let $x = fx = gx$ and $y = gy = fy$

Now $m^*(x, y) = m^*(gx, gy) \leq m^{*\alpha}(fx, fy) = m^{*\alpha}(x, y)$, implies $x = y$, as $0 \leq \alpha < 1$. □

In 1998, Jungck and Rhoades [16] introduced the notion of weakly compatible as follows:

Definition 4.2. Two maps f and g are said to be weakly compatible if the maps commute at their coincidence points i.e., $fx = gx$ implies $fgx = gfx$ for $x \in X$.

Example 4.3. Let $X = [0,3]$. Define self maps f and g on X by $fx = \frac{x}{2}$ and $gx = x$, then $f(0) = g(0)$ and $fg(0) = gf(0)$. Hence f and g are weakly compatible maps.

Now we prove a fixed point theorem in the setting of super multiplicative metric space for a pair of weakly compatible maps.

Theorem 4.4. *Let (X, m^*) be a complete super multiplicative-metric space and f and g be self maps of X satisfying conditions:*

$$m^*(fx, fy) \leq m^{*\alpha}(gx, gy), \tag{15}$$

where $0 \leq \alpha < 1$ and $f(X) \subseteq g(X)$. If one of the subspaces $f(X)$ or $g(X)$ is a complete subspace in X, then f and g have a unique common fixed point, provided f and g are weakly compatible maps.

Proof. Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n+1} = fx_n, \;\; n = 0, 1, 2, ..., \;\; where \;\; x_0 \in X.$$

Therefore, from (15),we have

$$\begin{aligned} m^*(y_n, y_{n+1}) = m^*(fx_n, fx_{n+1}) &\leq m^{*\alpha}(gx_n, gx_{n+1}) \\ &= m^{*\alpha}(y_{n-1}, y_n) \\ &\vdots \\ &\leq m^{*\alpha^n}(y_0, y_1). \end{aligned} \tag{16}$$

Taking limit as $n \to \infty$, we have

$$m^*(y_n, y_{n+1}) = 1.$$

From Lemma 3.1, sequence $\langle y_n \rangle$ is Cauchy sequence in X and (X, m^*) is complete super multiplicative metric space, so it converges to a point say z in X i.e.,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z.$$

Without loss of generality, one can assume, $g(X)$ be complete subspace of X, so there exists $p \in X$ such that $gp = z$.

From (15), we have

$$m^*(fp, fx_n) \leq m^{*\alpha}(gp, gx_n) = m^{*\alpha}(z, y_{n-1}).$$

Taking limit as $n \to \infty$, we get $fp = z$.

Since f and g are weakly compatible, so $fgp = gfp$, implies $fz = gz$. Now it remains to show that z is a common fixed point of f and g. Now from (15), we have,

$$m^*(fz, fx_n) \leq m^{*\alpha}(gz, gx_n)$$

$$m^*(fz, z) \leq m^{*\alpha}(fz, z),$$

implies, $fz = z$ as $\alpha < 1$. Therefore, $fz = gz = z$. Hence z is a common fixed point of f and g. Uniqueness follows easily. □

Now we prove Theorem 4.4 with minor modifications as follows:

Theorem 4.5. *Let (X, m^*) be a complete super multiplicative metric space and f and g be self maps of X such that*

$$m^*(fx, fy) \leq m^{*\alpha}(gx, gy) \; x, y \in X \;, 0 \leq \alpha < 1, \tag{17}$$

and $f(X) \subseteq g(X)$, such that one of $f(X)$ and $g(X)$ is closed subset of X. Then f and g have a unique common fixed point in X, provided f and g are weakly compatible maps.

Proof. Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n+1} = fx_n, \;\; n = 0, 1, 2, ..., \;\; where \;\; x_0 \in X.$$

From the proof of Theorem 4.4, we conclude that $\langle y_n \rangle$ is a Cauchy sequence in X. Since either $f(X)$ or $g(X)$ is closed, for definiteness assume that $g(X)$ is closed subset of X, so it has a limit point in $g(X)$, call it z. Therefore for some $p \in X$ we have $gp = z$. Now we show that $fp = z$. From (17), we have,

$$m^*(fx_n, fp) \leq m^{*\alpha}(gx_n, gp)$$

$$m^*(fx_n, fp) \leq m^{*\alpha}(y_{n-1}, z).$$

Letting limit $n \to \infty$, we get $fp = z$. Rest part of the proof follows from Theorem 4.4 . □

Theorem 4.6. *Let (X, m^*) be a complete super multiplicative metric space and f and g be self maps of X. Assume there exists a right continuous function $\phi : [1,\infty) \longrightarrow [1,\infty)$ such that $\phi(t) < t$, if $t > 1$ and $\phi(t) = 1$ if and only if $t = 1$ satisfying following conditions:*

$$m^*(fx, fy) \leq \phi(m^*(gx, gy)) \text{ for all } x, y \in X. \tag{18}$$

$f(X) \subseteq g(X)$ and one of $f(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique common fixed point in X, provide f and g are weakly compatible maps.

Proof. Let $x_0 \in X$. Since $f(X) \subseteq g(X)$, therefore choose $x_1 \in X$ such that $gx_1 = fx_0$. In general, define x_{n+1} such that,

$$fx_n = gx_{n+1} = y_n \text{ (say)},\ n = 0, 1, 2, \dots .$$

Put $\alpha_n = m^*(y_n, y_{n-1})$.
From (18),we have

$$m^*(y_n, y_{n-1}) = m^*(fx_n, fx_{n-1}) \leq \phi(m^*(gx_n, gx_{n-1})) = \phi(m^*(y_{n-1}, y_{n-2})), \tag{19}$$

i.e., $\alpha_n \leq \phi(\alpha_{n-1}) < \alpha_{n-1}$. Thus $\langle \alpha_n \rangle$ is a decreasing sequence of reals , so converges in $\mathbb{R}^+$, say to a limit L. i.e., $\lim_{n\to\infty} m^*(y_n, y_{n-1}) = L$. We claim that $L = 1$ as if $L \neq 1$, $L \leq \phi(L) < L$, a contradiction, therefore, $L = 1$. Hence $\lim_{n\to\infty} m^*(y_n, y_{n-1}) = 1$. From Lemma 3.1, $\langle y_n \rangle$ is Cauchy sequence. Since (X, m^*) is a complete super multiplicative metric space, so it converges to a point say $z \in X$. Then $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$.
Without loss of generality, assume $g(X)$ be complete subspace of X, therefore, there exists $p \in X$ such that $gp = z$.
Using (18), we have

$$m^*(fp, fx_n) \leq \phi(m^*(gp, gx_n)) = \phi(m^*(z, y_{n-1})).$$

Proceeding limit $n \to \infty$,we have $fp = z$. Therefore $fp = gp = z$.
Since f and g are weakly compatible we have $fgp = gfp$, which implies $fz = gz$.
Now again using (18), we have

$$m^*(fz, fx_n) \leq \phi(m^*(gz, gx_n)) = \phi(m^*(fz, gx_n)).$$

Taking limit $n \to \infty$,

$$m^*(fz,z) \leq \phi(m^*(fz,z)),$$

we get $fz = z$. So $fz = gz = z$.

Hence z is a common fixed point of f and g.

Uniqueness:

Let $w(\neq z)$ be another common fixed point of f and g.

Using (18), we have ,

$$m^*(z,w) = m^*(fz,fw) \leq \phi(m^*(gz,gw)) = \phi(m^*(z,w)),$$

a contradiction, thus $w = z$. Hence f and g have a unique common fixed point. □

Theorem 4.7. *Let (X,m^*) be a complete super multiplicative metric space and f,g be self mapping on X, satisfying,*

$$m^*(fx,gy) \leq m^{*\alpha}(x,y).[m^*(x,fx).m^*(y,gy)]^{\beta} \tag{20}$$

with $\alpha,\beta > 0$, $\alpha+2\beta < 1$, and for all $x,y \in X$. Then f and g have a fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Consider sequence of iterate,

$x_{2n-1} = fx_{2n-2}$, $x_n = gx_{2n-1}$.

From (20) , on putting $x = x_{2n-2}$ and $y = x_{2n-1}$, we have on simplification

$$m^*(x_{2n-1},x_{2n}) = m^*(fx_{2n-2},gx_{2n-1}) \leq m^{*\alpha}(x_{2n-2},x_{2n-1}).[m^*(x_{2n-2},fx_{2n-2}).m^*(x_{2n-1},gx_{2n-1})]^{\beta}.$$

or

$$m^*(x_{2n-1},x_{2n}) \leq m^{*\alpha}(x_{2n-2},x_{2n-1}).[m^*(x_{2n-2},x_{2n-1}).m^*(x_{2n-1},x_{2n})]^{\beta}$$

$$m^{*(1-\beta)}(x_{2n-1},x_{2n}) \leq m^{*(\alpha+\beta)}(x_{2n-2},x_{2n-1})$$

$$m^*(x_{2n-1},x_{2n}) \leq m^{*\left(\frac{\alpha+\beta}{1-\beta}\right)}(x_{2n-2},x_{2n-1})$$

$$\text{i.e, } m^*(x_{2n-1},x_{2n}) \leq m^{*\gamma}(x_{2n-2},x_{2n}), \text{ where } \gamma = \frac{\alpha+\beta}{1-\beta} \tag{21}$$

continuing in the same way

$$m^*(x_{2n-1},x_{2n}) \leq m^{*\gamma^{2n}}(x_0,x_1)$$

proceeding $lim n \to \infty$, we get $\lim\limits_{n\to\infty} m^*(x_{2n-1}, x_{2n}) = 1$.
By definition of super metric space , for $\langle x_{2n-1}\rangle$,$\langle x_{2n}\rangle$ in X there exists $x_{2n+1} \in X$ such that

$$\lim_{n\to\infty} sup\, m^*(x_{2n-1}, x_{2n+1}) \leq \lim_{n\to\infty} sup\, m^{*s}(x_{2n}, x_{2n-1})$$

then by definition of super multiplicative metric, we have $\lim\limits_{n\to\infty} sup\, m^*(x_{2n-1}, x_{2n+1}) = 1$.
Again for $\langle x_{2n+1}\rangle$ and $\langle x_{2n-1}\rangle$ there exists $x_{2n+2} \in X$ such that

$$\lim_{n\to\infty} sup\, m^*(x_{2n-1}, x_{2n+2}) \leq \lim_{n\to\infty} sup\, m^{*s}(x_{2n+2}, x_{2n+1})$$

i.e., $\lim\limits_{n\to\infty} sup\, m^*(x_{2n-1}, x_{2n+2}) = 1$.
Proceeding in this way, we get
$\lim\limits_{n\to\infty} sup\, m^*(x_n, x_m) = 1$ $m > n$ and $m, n \in \mathbf{N}$. Therefore $\{x_n\}$ is Cauchy. Since (X, m^*) is complete, the sequence $\langle x_n\rangle$ converges to a point say $u \in X$.
On putting $x = x_{2n-2}$ and $y = u$ in (20), we have
$m^*(x_{2n-1}, gu) = m^*(fx_{2n-2}, gu) \leq m^{*\alpha}(x_{2n-1}, u).[m^*(x_{2n-2}, fx_{2n-2}).m^*(u, gu)]^{\beta}$
$m^*(x_{2n-1}, gu) \leq m^{*\alpha}(x_{2n-1}, u).[m^*(x_{2n-2}, fx_{2n-2}).m^*(u, gu)]^{\beta}$.
Proceeding $\lim n \to \infty$ we get
$m^*(u, gu) \leq m^{*\beta}(u, gu)\ m^{*(1-\beta)}(u, gu) \leq 0$ if $1-\beta \leq 0$ then $\beta \geq 1$ but $\alpha + 2\beta \leq 1$ and $\alpha > 0$ hence $m^*(u, gu) = 1$ i.e. $gu = u$ similarly we can show that $fu = u$.
Hence f and g have a fixed point in X. □

5. Expansive Mappings

In 1984, Wang, Li, Gao and Iséki [25] and Rhoades [20] proved some fixed point theorems for expansive mappings that corresponds to some contractive mappings in metric spaces. Now we introduce expansive mappings in the setting of super multiplicative metric spaces that corresponds to some contractive mappings in metric spaces and multiplicative metric spaces.
Let f be a mapping of a super multiplicative metric space (X, m^*) into itself. Then f is said to be expansive mapping if there exists a constant $\alpha > 1$ such that for all $x, y \in X$, we have

$$m^*(fx, fy) \geq m^{*\alpha}(x, y).$$

Now we prove a result related to expansive mapping in super multiplicative metric spaces.

Theorem 5.1. *Let (X, m^*) be a complete super multiplicative metric space and $T : X \longrightarrow X$ be a surjective mapping. Suppose that $\alpha > 1$ such that*

$$m^*(Tx, Ty) \geq m^{*\alpha}(x, y), \text{ for all } x, y \in X. \tag{22}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Since T is surjective then $\exists$ an element x_1 such that $x_1 \in T^{-1}x_0$. Continuing this way, we get a sequence $\langle x_n \rangle$ such that $x_n \in T^{-1}x_{n-1}$. If $x_n = x_{n-1}$ for some n, then x_n is a fixed point of T. Assume that $x_n \neq x_{n-1}$ for every $n \in \mathbf{N}$.

Then from inequality (22), we have

$$\begin{aligned} m^*(x_{n+1}, x_n) = m^*(Tx_{n+2}, Tx_{n+1}) &\geq m^{*\alpha}(x_{n+2}, x_{n+1}) \\ m^{*\frac{1}{\alpha}}(Tx_{n+2}, Tx_{n+1}) &\geq m^*(x_{n+2}, x_{n+1}) \\ m^{*\frac{1}{\alpha}}(x_{n+1}, x_n) &\geq m^*(x_{n+2}, x_{n+1}) \\ m^*(x_{n+2}, x_{n+1}) &\leq m^{*\frac{1}{\alpha}}(x_{n+1}, x_n) \\ &\leq m^{*\frac{1}{\alpha^2}}(x_n, x_{n-1}) \\ &\leq m^{*\frac{1}{\alpha^3}}(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq m^{*\frac{1}{\alpha^{n+1}}}(x_1, x_0), \ \ \alpha > 1. \end{aligned} \tag{23}$$

By Lemma 3.1, the sequence $\langle x_n \rangle$ is Cauchy sequence. Since (X, m^*) is a complete super multiplicative metric space, therefore , the sequence $\langle x_n \rangle$ converges to a point say $z \in X$. We show that z is a fixed point of T.

Now we show that $Tz = z$. Let $p \in T^{-1}z$, for n such that $x_n \neq z$, we have

$$m^*(x_n, z) = m^*(Tx_{n+1}, Tp) \geq m^*(x_{n+1}, p).$$

Taking limit as $n \to \infty$ $m^*(z, p) \leq 1$ i.e., $z = p$ i.e., $Tz = z$.

Hence z is the fixed point of T.

Uniqueness:

Let $w(\neq z)$ be another fixed point for T.

Since

$$m^*(w,z) \leq m^{*\frac{1}{\alpha}}(Tw,Tz) = m^{*\frac{1}{\alpha}}(w,z)$$

$$m^{*(1-\frac{1}{\alpha})}(w,z) \leq 1 \tag{24}$$

a contradiction since $\alpha > 1$.

This implies $z = w$,

Hence T has a unique fixed point in X. □

We generalize Theorem 5.1 for a pair of maps.

Theorem 5.2. *Let f be a continuous self map of a complete super multiplicative metric space (X,m^*). If there exists a mapping $g : X \to X$ and constant $0 \leq \alpha < 1$ such that*

(1) $fgx = gfx$ for every $x \in X$,

(2) $g(X) \subset f(X)$,

(3) $m^*(fx, fy) \geq m^{*\alpha}(gx, gy)$ for every $x, y \in X$.

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $g(X) \subseteq f(X)$, choose $x_1 \in X$ such that $fx_1 = gx_0$. In general choose x_{n+1} such that

$$fx_{n+1} = gx_n = y_n \text{ (say)}$$

Then from (3),

$$m^*(y_n, y_{n+1}) = m^*(gx_n, g_{n+1}) \leq m^{*\frac{1}{\alpha}}(fx_n, fx_{n+1}) = m^{*\frac{1}{\alpha}}(y_{n-1}, y_n). \tag{25}$$

From Lemma 3.1, the sequence $\langle y_n \rangle$ is a Cauchy sequence. Since (X,m^*) is complete, so $\langle y_n \rangle$ converges to a point, say $z \in X$.

Then $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} fx_{n+1} = \lim\limits_{n\to\infty} gx_n = z$.

Since f is continuous so by (3), f and g both are continuous.

Therefore $fgx_n \to gt$ and $gfx_n \to ft$. But f and g commute for all $x \in X$, so $gfx_n = fgx_n \ \forall n$.

So $gt = ft$. Consequently $f(ft) = f(gt) = g(gt)$

$$m^*(gt, ggt) = m^*(ft, ggt) \leq m^{*\frac{1}{\alpha}}(ft, fgt) = m^{*\frac{1}{\alpha}}(gt, ggt)$$

$$m^{*(1-\frac{1}{\alpha})}(gt,ggt) \leq 1.$$

This implies $gt = ggt$, since $0 \leq \alpha < 1$. $gt = g(gt) = f(gt)$ i.e., gt is a common fixed point of f and g.

Uniqueness:

$x = fx = gx$ and $y = gy = fy$

Now $m^*(x,y) = m^*(gx,gy) \leq m^{*\frac{1}{\alpha}}(fx,fy) = m^{*\frac{1}{\alpha}}(x,y)$, implies $x = y$. □

6. Φ−Weak Contraction

In 1997, Alber and Guerre-Delabriere [24] presented the notion of $\phi-$weak contraction in Hilbert spaces. In 2001 Rhoades [19], extended the notion of $\phi-$weak contraction in setting of complete metric spaces as follows: there exists a function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$ such that

$$d(fx,fy) \leq d(x,y) - \phi(x,y).$$

Analogously, we define the concept of $\phi-$weak contraction in the setting of super multiplicative metric spaces as follows:

Definition 6.1. Let $\phi : [1,\infty) \longrightarrow [1,\infty)$ with $\phi(t) > 1$ for $t > 1$ and $\phi(1) = 1$ such that

$$m^*((fx,fy) \leq \frac{m^*(x,y)}{\phi(m^*(x,y))}.$$

Now we prove the following theorem in super-multiplicative metric spaces.

Theorem 6.2. *Let (X,m^*) be a complete super-multiplicative metric space and $T : X \longrightarrow X$ be a mapping such that*

$$m^*(Tx,Ty) \leq \frac{m^*(x,y)}{\phi(m^*(x,y))} \tag{26}$$

for all $x,y \in X$, where $\phi : [1,\infty) \longrightarrow [1,\infty)$ is continuous function with $\phi(t) = 1$ if and only if $t = 1$ and $\phi(t) > 1$ for all $t > 1$. Then T has a unique fixed point.

Proof. Let us define a sequence $\langle x_n \rangle$ in X by

$$x_n = Tx_{n+1},\ \textit{for}\ n = 0,1,2...\ \textit{where}\ x_0 \in X.$$

If $x_0 = x_1$, then x_1 is a fixed point of T and the proof is completed. Suppose $x_0 \neq x_1$, thus $m^*(x_0,x_1) > 1$. Without loss of generality, one can assume, $x_n \neq x_{n+1}$, so $m^*((x_n,x_{n+1}) > 1$ for all $n = 0,1,2...$.

Put $\alpha_n = m^*(x_n,x_{n+1})$.

Therefore, from (26), we have

$$m^*(x_n,x_{n+1}) = m^*(Tx_{n-1},Tx_n) \leq \frac{m^*(x_{n-1},x_n)}{\phi(m^*(x_{n-1},x_n))}. \tag{27}$$

On simplification, we have $m^*(x_n,x_{n+1}) \leq m^*(x_{n-1},x_n),\ i.e., \alpha_n \leq \alpha_{n-1}$.

The sequence α_n is non-increasing sequence of reals, so it converges in $\mathbb{R}^+$, consequently, there exists $L \geq 1$ such that $\lim_{n\to\infty} \alpha_n = L$, i.e., $\lim_{n\to\infty} m^*(x_n,x_{n+1}) = L$. We claim that $L = 1$, if $L \neq 1$, then $L > 1$ and by using continuity of ϕ and inequality (27), we get,

$$L \leq \frac{L}{\phi(L)},\ since,\ \phi(t) > 1\ for\ all\ t > 1,$$

a contradiction, so , $\phi(L) = 1$, i.e., $L = 1$. Hence $\lim_{n\to\infty} m^*(x_n,x_{n+1}) = 1$. By Lemma 3.1, $\langle x_n \rangle$ is Cauchy sequence. Since (X,m^*) is a complete super multiplicative metric space so $\langle x_n \rangle$ converges to a point say $z \in X$. We show that z is fixed point of T.

From (26), we have,

$$m^*(Tz,x_{n+1}) = m^*(Tz,Tx_n) \leq \frac{m^*(z,x_n)}{\phi(m^*(z,x_n))},$$

Proceeding limit $n \to \infty$, implies we have $m^*(Tz,z) \leq 1$, implies $Tz = z$. Hence z is a fixed point of T.

Uniqueness:

Let $w(\neq z)$ be another fixed point of T.

$$m^*(z,w) = m^*(Tz,Tw) \leq \frac{m^*(z,w)}{\phi(m^*(z,w))},$$

implies, $w = z$, as $\phi(t) > 1$ for all $t > 1$. □

Now we generalize Theorem 6.2 for a pair of weakly compatible mappings in setting of super multiplicative metric spaces.

Theorem 6.3. *Let (X,m^*) be a complete super multiplicative metric space. Let f and g be self-mappings satisfying the following:*

$$m^*(fx,fy) \leq \frac{m^*(gx,gy)}{\phi(m^*(gx,gy))} \tag{28}$$

where $\phi:[1,\infty)\longrightarrow[1,\infty)$ is a continuous function with $\phi(t)>1$ for all $t>1$ and $\phi(t)=1$ if and only if $t=1$. Further, $f(X)\subseteq g(X)$ and $g(X)$ or $f(X)$ are complete subspace of X. Then f and g have a unique common fixed point, provided f and g are weakly compatible maps.

Proof. Let $x_0\in X$. Since $f(X)\subseteq g(X)$. Choose $x_0\in X$ such that $fx_1=gx_0$. In general, choose x_{n+1} such that

$$fx_n = gx_{n+1} = y_n \text{ (say) } , n=0,1,2\dots .$$

Put $\alpha_n = m^*(y_n,y_{n+1})$

From (28), we have,

$$\begin{aligned} m^*(y_n,y_{n+1}) = m^*(fx_n,fx_{n+1}) &\leq \frac{m^*(gx_n,gx_{n+1})}{\phi(m^*(gx_n,gx_{n+1}))} \\ &= \frac{m^*(y_{n-1},y_n)}{\phi((m^*(y_{n-1},y_n))}. \end{aligned} \tag{29}$$

On simplification, we have $\alpha_n \leq \frac{\alpha_{n-1}}{\phi(\alpha_{n-1})}$, implies α_n is non-increasing sequence, so it converges in $\mathbb{R}^+$.

Consequently, there exists $L\geq 1$ such that $\lim\limits_{n\to\infty} m^*(y_n,y_{n+1})=L$. We claim that $L=1$. If $L\geq 1$, then by using definition of ϕ and inequality (28), we get,

$$L \leq \frac{L}{\phi(L)} \text{ since, } \phi(t)>1 \text{ for all } t>1,$$

implies $L=1$. Hence $\lim\limits_{n\to\infty} m^*(y_n,y_{n+1})=1$. By Lemma 3.1, $\langle y_n\rangle$ is a Cauchy sequence in X and (X,m^*) is a complete super multiplicative metric space, so it converges to a point say $z\in X$.

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z.$$

Since $g(X)$ is complete subspace of X, there exists $p\in g(X)$ such that $gp=z$.

From (28), we have,

$$m^*(fp,fx_n) \leq \frac{m^*(gp,gx_n)}{\phi(m^*(gp,gx_n))} = \frac{m^*(z,y_{n-1})}{\phi(m^*(z,y_{n-1})}.$$

Proceeding limit $n \to \infty$, we get

$\lim_{n\to\infty} m^*(fp, z) = 1$, i.e., $fp = z$.

Thus $fp = gp$. Since f and g are weakly compatible, so $fgp = gfp$, which implies $fz = gz$.

From (28), we have

$$m^*(fz, fx_n) \leq \frac{m^*(gz, gx_n)}{\phi(m^*(gz, gx_n)} = \frac{m^*(fz, gx_n)}{\phi(m^*(fz, gx_n)}.$$

Taking limit $n \to \infty$, we have,

$$m^*(fz, z) \leq \frac{m^*(fz, z)}{\phi(m^*(fz, z))},$$

implies $fz = gz = z$. Hence z is the common fixed point of f and g.

Uniqueness:

Let $w(\neq z)$ be another fixed point of f and g. From (28), we have,

$$m^*(z, w) = m^*(fz, fw) \leq \frac{m^*(gz, gw)}{\phi(m^*(gz, gw))} = \frac{m^*(z, w)}{\phi(m^*(z, w))} \leq 1.$$

Thus $w = z$.

Hence z is the unique common fixed point of f and g. □

7. WEAKLY COMPATIBLE MAPS AND PROPERTY(E.A)

In 2002, Amari and Moutawakil [1] introduced the notion of property (E.A) as follows:

Definition 7.1. Let f and g be two self-maps of a metric space (X, d). The pair (f,g) is said to satisfy property (E.A), if there exists a sequence $\langle x_n \rangle$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

The notion of property (E.A) is used in a multiplicative space in the same way as it is used in metric spaces.

Example 7.2. Let $X = [0, 1]$. Define $f, g : X \longrightarrow X$ by $fx = \frac{x}{2}$ and $gx = \frac{3x}{4}$ for all $x \in X$. Consider sequence $x_n = \frac{1}{n}$. Clearly $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$. Then f and g satisfy property (E.A).

The importance of property (E.A) is that it relaxes the continuity requirement of maps completely, weakens the completeness requirement of spaces and property (E.A) buys containment of range spaces without any continuity requirement.

Now we prove a theorem for a pair of weakly compatible maps along property (E.A) in the setting of super multiplicative metric spaces.

Theorem 7.3. *Let (X, m^*) be a super multiplicative metric space and f and g be self maps of X satisfying conditions:*

$$m^*(gx, gy) \leq m^{*\alpha}(fx, fy), \tag{30}$$

where $0 \leq \alpha < 1$ and f and g satisfying property (E.A) and $f(X)$ is a closed subspace of X. Then f and g have a unique common fixed point, provided f and g are weakly compatible.

Proof. Since f and g satisfy property (E.A), there exists a sequence $\langle x_n \rangle$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$=$z \in X$. Since $f(X)$ is a closed subspace of X, so $z = fp \in f(X)$ for some $p \in X$. Now we show that $fp = gp$. From (30), we have

$$m^*(gp, gx_n) \leq m^{*\alpha}(fp, fx_n)$$

$$m^*(gp, gx_n) \leq m^{*\alpha}(z, y_n).$$

Proceeding limit $n \to \infty$, we have, $fp = gp = z$.

Rest of the proof follows from Theorem 4.4 . □

8. Any Kind of Weakly Compatible Maps

In 2010, Murthy et al. [11] introduced the notion of any kind of weakly compatible maps in fuzzy metric spaces as follows:

Definition 8.1. Let $X \neq 0$ be a non-empty set. A pair of self mappings (f, g) on non-empty set X is said to be any kind of weakly compatible maps if and only if there is a sequence $\langle x_n \rangle$ in X satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$ and $fgt = gft$.

Example 8.2. Define $f, g : [0,2] \longrightarrow [0,2]$ by $fx = 2$ if $x \in [0,1]$ and $fx = \frac{x}{2}$ if $x \in (1,2]$ and $gx = 2$ if $x \in [0,1]$ and $\frac{x+3}{5}$ if $x \in (1,2]$. Consider the sequence $\langle x_n \rangle = (2 - \frac{1}{2n})$. Clearly

$$fx_n = (1 - \frac{1}{4n}), gx_n = (1 - \frac{1}{10n}).$$

Proceeding limit $n \longrightarrow \infty$

$$fx_n \to 1, gx_n \to 1.$$

Moreover, $fg(1) = gf(1) = 1$. Therefore, we can say f and g are any kind of weakly compatible maps.

Now we prove a theorem for weakly compatible maps along with notion of any kind of weakly compatible maps defined by Murthy et al. [11].

Theorem 8.3. *Let (X, m^*) be a complete super multiplicative metric space and f and g be self maps of X satisfying conditions:*

$$m^*(fx, fy) \le m^{*\alpha}(gx, gy), \tag{31}$$

where $0 \le \alpha < 1$ and $f(X) \subset g(X)$, one of the spaces $f(X)$ or $g(X)$ is a closed subset of X. Further f and g are any kind of weakly compatible maps. Then maps f and g have a unique common fixed point, provided f and g are weakly compatible maps.

Proof. Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n+1} = fx_n, \;\; n = 0, 1, 2, ..., \; where \;\; x_0 \in X.$$

Since f and g are any kind of weakly compatible maps, therefore, there exists a sequence $\langle x_n \rangle \in X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X$. Let $g(X)$ be a closet subset of X, then for sequence $\langle x_n \rangle$ in $g(X)$, there is a limit in $g(X)$. Call it be z such that $z = gp$. Therefore, $\lim_{n\to\infty} fx_n = z = gp = \lim_{n\to\infty} gx_n$. This implies $z = gp \in g(X)$. Now we have to show $z = fp = gp$. Using (31), we have,

$$m^*(fp, fx_n) \le m^{*\alpha}(gp, gx_n) = m^*(z, y_{n-1})$$

Proceeding limit $n \to \infty$, gives $fp = z$. Rest of the proof follows from Theorem 4.4.

□

Conflict of Interests

The authors declare that there is no conflict of interests.

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