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FIXED POINT THEOREMS IN PARTIALLY ORDERED C^*-ALGEBRA VALUED METRIC SPACES AND CYCLIC CONTRACTIONS

ALSHAIMAA AWAD[1,*], SALEH OMRAN[1], MOHAMED ELMURSI[2]

[1]Department of Mathematics, Faculty of Science, South Valley University, Qena 83523, Egypt

[2]Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

Abstract. In this paper, we established partially ordered C^*-algebra valued metric spaces. In particular, we prove some Banach fixed point theorems and some fixed point theorems for cyclic contractions in partially ordered C^*-algebra valued metric spaces. We give some examples to illustrate our results.

Keywords: fixed point theorems; partially ordered metric space; partially ordered C^*-algebra valued metric space; cyclic contraction.

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1. INTRODUCTION

Fixed point theory (FPT) is a substantial concept in nonlinear analysis and branches of modern mathematics. This theory plays a central role in solving many important problems in pure and applied mathematics [1, 2]. Banach contraction principle is one of the most important result in FPT and approximation theory [2, 3, 4]. On the other side, FPT in partially ordered metric spaces has grown quickly. FP problems have also been considered in partially ordered cone metric spaces, in partially ordered G-metric spaces and in partially ordered probabilistic metric spaces [5].

*Corresponding author
E-mail address: shawad@sci.svu.edu.eg
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Wolk [6] and Monjardet [7] initiated the initial outcome in partially ordered sets. Ran and Reurings [8] studied the existence of fixed points for certain mappings in partially ordered metric spaces and used their results to the solution of matrix equations. The results of Ran and Reurings [8] were extended by Nieto et al. [9, 10] for non-decreasing mappings and got solutions of certain partial differential equations with periodic boundary conditions. While Agarwal et al. [11] have discussed some new results for a generalized contractions in partially ordered metric spaces. In order to obtain fixed points, common fixed point results for single valued and multivalued operators in various ordered spaces with topological features have been greatly improved and generalized [3, 12]. Seshagiri Rao et al. [12, 13, 14, 15, 16, 17, 18] have recently investigated several findings about FP, coincidence point, coupled FP and coupled common FP for mappings in partially ordered metric spaces and partially ordered b-metric spaces [3].

The study of fixed points of mappings satisfying cyclic contractive conditions has been at the center of vigorous research activity in the last years. In 2003, Kirk et al. [19] obtained some FP results for mapping satisfying cyclical contractive condition. In fact, in 2010, Pãcurar and Rus [20] proved fixed point results for cyclic ϕ-contractions. Karapinar [21] proved FP results for cyclic weak ϕ-contraction [22, 23].

In this paper, we will continue to study Banach FP theorems and some FP theorems for cyclic contractions in partially ordered C^*-algebra valued metric space.

Suppose that $\mathbb{A}$ is a unital algebra with the unit I. An involution on $\mathbb{A}$ is a conjugate linear map $x \to x^*$ on $\mathbb{A}$ such that $x^{**} = x$ and $(xy)^* = y^*x^*$ for all $x, y \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a *-algebra. A Banach *-algebra is a *-algebra $\mathbb{A}$ together with a complete submultiplicative norm, $\|xy\| \leq \|x\|\|y\|$ such that $\|x^*\| = \|x\|$. C^*-algebra is a *-Banach algebra such that $\|x^*x\| = \|x\|^2$, when $\mathbb{A}$ is a unital C^*-algebra, then for any $x \in \mathbb{A}_+$ we have $x \preceq I \Leftrightarrow \|x\| \leq 1$ (see[2, 24, 25]).

2. PRELIMINARIES

The following definitions are frequently used in results.

Definition 2.1. [25] Let X be a nonempty set. Suppose that mapping $d : X \times X \to \mathbb{A}$ satisfies:

(i) $\theta \preccurlyeq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta \Leftrightarrow x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,

(iii) $d(x,y) \preccurlyeq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a C^*-algebra valued metric on X and $(X,\mathbb{A},d)$ is called a C^*-algebra valued metric space.

Definition 2.2. [12] The triple $(X,d,\leq)$ is called partially ordered metric space, if $(X,\leq)$ is a partially ordered set together with (X,d) is a metric space.

Definition 2.3. [12] If (X,d) is a complete metric space, then the triple $(X,d,\leq)$ is called complete partially ordered metric space.

Definition 2.4. [12] A partially ordered metric spaces $(X,d,\preccurlyeq)$ is called ordered complete if for each convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X$, the following conditions hold:

(i) If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \longrightarrow x$ implies $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$ that is $x = \sup\{x_n\}$

(ii) If $\{x_n\}$ is a non-increasing sequence in X such that $x_n \longrightarrow x$ implies $x \preccurlyeq x_n$ for all $n \in \mathbb{N}$ that is $x = \inf\{x_n\}$

Definition 2.5. [12] Let $(X,\preccurlyeq)$ be a partially ordered set and let $T : X \longrightarrow X$ be a mapping. Then

(1) elements $x,y \in X$ are comparable, if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds,

(2) a non empty set X is called well ordered set, if every two elements of it are comparable,

(3) T is said to be monotone non-decreasing with respect to (w.r.t.)$\preccurlyeq$, if for all $x,y \in X$, $x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,

(4) T is said to be monotone non-increasing with respect to (w.r.t.)$\preccurlyeq$, if for all $x,y \in X$, $x \preccurlyeq y$ implies $Tx \succcurlyeq Ty$.

Theorem 2.6. [10] Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that there exists $\alpha \in [1,0)$ with

$$d(Tx,Ty) \leq \alpha\, d(x,y),$$

for all $x \geq y$. If there exists $x_0 \in X$ with $x_0 \leq T(x_0)$, then T has a FP.

Theorem 2.7. [27] Let $(X,\leq)$ be a partially ordered set such that every pair $x,y\in X$ has a lower bound and an upper bound. Furthermore, let d be a metric in X such that (X,d) is a complete metric space. Let $T:X\longrightarrow X$ be a continuous and monotone map such that there exists $\alpha\in[1,0)$ with

$$d(Tx,Ty)\leq\alpha\ d(x,y),$$

for all $x\geq y$. Then T has a unique FP.

3. Main Results

In this section, we mention some basic definitions, examples and we study some theorems in partially ordered C^*-algebra valued metric spaces.

Definition 3.1. The quadruple $(X,\mathbb{A},d_P,\preccurlyeq)$ is called partially ordered C^*-algebra valued metric space, if $(X,d_P,\preccurlyeq)$ is a partially ordered metric space together with $(X,\mathbb{A},d_P)$ is a C^*-algebra valued metric space.

The following example support the definition (3.1).

Example 3.2. Let $X=\mathbb{R}$ and $A=M_n(\mathbb{R})$ where $A,\ B\in M_n(\mathbb{R})$. Define

$$d_P(A,B)=(|a_{ij}-b_{ij}|)\in M_n(\mathbb{R}),$$

where $A=(a_{ij}),\ B=(b_{ij})$ and $i,j=1,2,\dots$ such that $A\preccurlyeq B$ if and only if $a_{ij}\leq b_{ij}$. Then $(X,M_n(\mathbb{R}),d_P,\preccurlyeq)$ is a partially ordered C^*-algebra valued metric space.

Theorem 3.3. Let $(X,\mathbb{A},d_P,\preccurlyeq)$ be complete partially ordered C^*-algebra valued metric space and $x,y\in X$ has upper and lower bounded and $T:X\to X$ is continuous and monotone,

$$d_P(Tx,Ty)\preccurlyeq a^*\ d_P(x,y)\ a,\ x\leq y, \text{ and } \|a\|<I_{\mathbb{A}}.$$

Then T has a unique FP.

Proof. Choose $x_0\in X$ and $x_0\leq T(x_0)$, $x_0\leq x_1\leq x_2\leq\cdots\leq x_n\leq x_{n+1}$,

$$\begin{aligned} d_P(x_{n+1},x_n)=d_P(Tx_n,Tx_{n-1}) &\preccurlyeq a^*\ d_P(x_n,x_{n-1})\ a \\ &= a^*\ d_P(Tx_{n-1},Tx_{n-2})\ a \end{aligned}$$

$$\begin{aligned} &\preccurlyeq a^*a^*\, d_P(x_{n-1},x_{n-2})\, a\, a \\ &= (a^*)^2\, (d_P(x_{n-1},x_{n-2}))\, (a)^2 \\ &\vdots \\ &\preccurlyeq (a^*)^n\, d_P(x_1,x_0)\, (a)^n. \end{aligned}$$

We show that $\{x_n\}$ is a Cauchy sequence and let $n,m \in N$, $n \geq m$, $x_{n+1} \geq x_m$,

$$\begin{aligned} d_P(x_{n+1},x_m) &\preccurlyeq d_P(x_{n+1},x_n)+d_P(x_n,x_{n-1})+\cdots+d_P(x_{m+1},x_m) \\ &\preccurlyeq (a^*)^n\, d_P(x_1,x_0)\, (a)^n+(a^*)^{n-1}\, d_P(x_1,x_0)\, (a)^{n-1}+\ldots \\ &\quad +(a^*)^m\, d_P(x_1,x_0)\, (a)^m \\ &= \sum_{k=m}^{n} (a^*)^k\, d_P(x_1,x_0)\, (a)^k \\ &= \sum_{k=m}^{n} (d_P(x_1,x_0)^{\frac{1}{2}}(a)^k)^*(d_P(x_1,x_0)^{\frac{1}{2}}(a)^k) \\ &= \sum_{k=m}^{n} |(d_P(x_1,x_0))^{\frac{1}{2}}\, (a)^k|^2 \\ &\preccurlyeq \|\sum_{k=m}^{n} |(d_P(x_1,x_0))^{\frac{1}{2}}\, (a)^k|^2\|.\, I_{\mathbb{A}} \\ &\preccurlyeq \sum_{k=m}^{n} \|(d_P(x_1,x_0))^{\frac{1}{2}}\|^2\, \|(a)\|^{2k}.\, I_{\mathbb{A}} \to \theta \text{ as } m,n \to \theta. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence.

Since T is continuous, there exists $x \in X$ shut that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = Tx = x$.

Thus, x is a FP of the mapping T.

To show uniqueness, suppose that $y \neq x$ is another FP of T.

Case 1. if x is comparable to y, then $T^n x = x$ is comparable $T^n y = y$ and x,y are two FP.

$$\begin{aligned} d_P(x,y) = d_P(Tx_n,y_n) &\preccurlyeq a^* d_P(x_n,y_n)\, a \\ &\preccurlyeq (a^*)^n d_P(x,y)\, (a)^n \\ &= |(d_P(x,y))^{\frac{1}{2}}\, (a)^n|^2 \\ &\preccurlyeq \||(d_P(x,y))^{\frac{1}{2}}\, (a)^n|^2\|.\, I_{\mathbb{A}} \end{aligned}$$

$$\preccurlyeq \quad \|(d_P(x_1,x_0))^{\frac{1}{2}}\|^2 \ \|(a)\|^{2n}.\ I_{\mathbb{A}} \to \theta \ \textit{as take}\ n \to \infty;$$

then $x = y$.

Case 2. if x is not comparable to y then there exists either an upper or a lower bound of x and y that is, there exists $z \in X$ comparable to x and y. Monotonicity implies that $T^n z$ is comparable to $T^n x = x$ and $T^n y = y$ for all $n = 1, 2, \ldots$

$$\begin{aligned}
d_P(x,y) &= d_P(T^n x, T^n y) \\
&\preccurlyeq d_P(T^n x, T^n z) + d_P(T^n z, T^n y) \\
&\preccurlyeq (a^*)^n \ d_P(x,z)\ (a)^n + (a^*)^n \ d_P(z,y)\ (a)^n \\
&= (d_P(x,z)^{\frac{1}{2}}\ (a)^n)^*(d_P(x,z)^{\frac{1}{2}}\ (a)^n) + (d_P(z,y)^{\frac{1}{2}}\ (a)^n)^*(d_P(z,y)^{\frac{1}{2}}\ (a)^n) \\
&= |(d_P(x,z))^{\frac{1}{2}}\ (a)^n + (d_P(z,y))^{\frac{1}{2}}(a)^n|^2 \\
&\preccurlyeq \||(d_P(x,z))^{\frac{1}{2}}\ (a)^n + (d_P(z,y))^{\frac{1}{2}}\ (a)^n|^2\|.\ I_{\mathbb{A}} \\
&\preccurlyeq \|(a)\|^{2n}\|(d_P(x,z))^{\frac{1}{2}}\|^2 + \|(a)\|^{2n}\|(d_P(z,y))^{\frac{1}{2}}\|^2.\ I_{\mathbb{A}} \to \theta\ (as\ n \longrightarrow \infty).
\end{aligned}$$

So $d_P(x,y) = \theta$ and $x = y$. □

Lemma 3.4. [30] let ψ is mapping from $\psi : \mathbb{A}_+ \to \mathbb{A}_+$ with conditions

(i) $\psi(a^*) = (\psi(a))^*$

(ii) $\psi(ab) = \psi(a)\psi(b)$

(iii) $\psi(a+b) = \psi(a) + \psi(b)$

(iv) $\lim_{n\to\infty} \psi^n(a) = \theta$ for all $a \succ \theta$ where $\psi^n(a) = \psi^{n-1} \circ \psi(a)$

(v) $\psi(a) = \theta$ if $a = \theta$.

Theorem 3.5. Let $(X, \mathbb{A}, d_P, \preccurlyeq)$ be partially ordered C^*-algebra valued metric space, X be a partially order set $x, y \in X$ has upper and lower bounded furthermore d_P is a complete C^*-algebra valued metric space and $T : X \to X$ is continuous and monotone and $\psi : \mathbb{A}_+ \to \mathbb{A}_+$

$$d_P(Tx, Ty) \preccurlyeq \psi(a^*\ d_P(x,y)\ a),$$

then T has a unique FP.

Proof. Choose $x_0 \in X$ and $x \leq T(x_0)$, $x_0 \leq T(x_0) \leq T^2(x_0) \leq \cdots \leq T^n(x_0) \leq T^{n+1}(x_0)$, i.e., $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1}$,

$$\begin{aligned}
d_P(x_{n+1},x_n) = d_P(Tx_n,Tx_{n-1}) &\preccurlyeq \psi(a^*\, d_P(x_n,x_{n-1})\, a)\\
&= \psi(a^*)\ \psi\big(d_P(x_n,x_{n-1})\big)\ \psi(a)\\
&\preccurlyeq \psi(a^*)\psi\Bigg(\psi\big(a^*\, d_P(x_{n-1},x_{n-2})\, a\big)\Bigg)\psi(a)\\
&= \psi(a^*)\ \psi\Bigg(\psi(a^*)\ \psi\big(d_P(x_{n-1},x_{n-2})\big)\ \psi(a)\Bigg)\ \psi(a)\\
&= \psi^3(a^*)\ \psi^2(d_P(x_{n-1},x_{n-2}))\ \psi^3(a)\\
&\preccurlyeq \psi^6(a^*)\ \psi^3(d_P(x_{n-2},x_{n-3}))\ \psi^6(a)\\
&\ \vdots\\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*)\ \psi^n(d_P(x_1,x_0))\ \psi^{\frac{n(n+1)}{2}}(a).
\end{aligned}$$

We show that $\{x_n\}$ is a Cauchy sequence.

Let $n,m \in N$, $n \geq m$, $x_{n+1} \geq x_m$,

$$\begin{aligned}
d_P(x_{n+1},x_m) &\preccurlyeq d_P(x_{n+1},x_n) + d_P(x_n,x_{n-1}) + \ldots + d_P(x_{m+1},x_m)\\
&\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a)^*\ \psi^n(d_P(x_1,x_0))\ \psi^{\frac{n(n+1)}{2}}(a) + \psi^{\frac{n(n-1)}{2}}(a)^*\ \psi^{n-1}(d_P(x_1,x_0))\ \psi^{\frac{n(n-1)}{2}}(a)\\
&+ \cdots + \psi^{\frac{m(m+1)}{2}}(a)^*\ \psi^m(d_P(x_1,x_0))\ \psi^{\frac{m(m+1)}{2}}(a)\\
&= \sum_{k=m}^{n} \psi^{\frac{k(k+1)}{2}}(a^*)\ \psi^k(d_P(x_1,x_0))\ \psi^{\frac{k(k+1)}{2}}(a)\\
&= \sum_{k=m}^{n} (\psi^{\frac{k}{2}}(d_P(x_1,x_0))\ \psi^{\frac{k(k+1)}{2}}(a))^*(\psi^{\frac{k}{2}}(d_P(x_1,x_0))\psi^{\frac{k(k+1)}{2}}(a))\\
&= \sum_{k=m}^{n} |\psi^{\frac{k}{2}}(d_P(x_1,x_0))\ \psi^{\frac{k(k+1)}{2}}(a)|^2\\
&\preccurlyeq \|\sum_{k=m}^{n} |\psi^{\frac{k}{2}}(d_P(x_1,x_0))\ \psi^{\frac{k(k+1)}{2}}(a)|^2\|.\ I_{\mathbb{A}}\\
&\preccurlyeq \sum_{k=m}^{n} \|\psi^{\frac{k(k+1)}{2}}(a)\|^2\|\psi^{\frac{k}{2}}(d_P(x_1,x_0))\|^2.\ I_{\mathbb{A}} \to \theta.
\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence.

Since T is continuous, there exists $x \in X$ shut that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = Tx = x,$.

Thus, x is a FP of the mapping T.

To show uniqueness, suppose that $y \neq x$ is another fixed point of T.

Case 1. if x is comparable to y, then $T^n x = x$ is comparable $T^n y = y$.

$$d_P(x,y) = d_P(T^n x, T^n y) \preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*)\psi^n(d_P(x,y))\psi^{\frac{n(n+1)}{2}}(a) \to \theta \text{ as take } n \to \infty,$$

then $x = y$.

Case 2. if x is not comparable to y then there exists either an upper or a lower bound of x and y that is, there exists $z \in X$ comparable to x and y. Monotonicity implies that $T^n z$ is comparable to $T^n x = x$ and $T^n y = y$ for all $n = 1, 2, \ldots$

$$\begin{aligned} d_P(x,y) &= d_P(T^n x, T^n y) \\ &\preccurlyeq d_P(T^n x, T^n z) + d_P(T^n z, T^n y) \\ &\preccurlyeq \psi^{\frac{n(n+1)}{2}}(a^*)\ \psi^n(d_P(x,z))\ \psi^{\frac{n(n+1)}{2}}(a) \\ &+ \psi^{\frac{n(n+1)}{2}}(a^*)\ \psi^n(d_P(z,y))\ \psi^n(a) \to \theta.\ (as\ n \to \infty) \end{aligned}$$

So $d_P(x,y) = \theta$ and $x = y$. □

4. Cyclic Contractions

In this section, we introduce FPT for cyclic contractions in partially ordered C^*-algebra valued metric space and we give some basic definitions and examples.

Definition 4.1. [31] Let A and B be non empty subsets of a metric space (X,d) and $T : A \cup B \longrightarrow A \cup B$. Then T is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$.

Definition 4.2. [22] Let X be a non-empty set and $f : X \longrightarrow X$ an operator. By definition, $X = \cup_{i=1}^m x_i$ is a cyclic representation of X with respect to f if

(i) X_i, $i = 1, \ldots, m$ are non-empty sets,

(ii) $f(X_1) \subset X_2, \ldots, f(X_{m-1}) \subset X_m, f(X_m) \subset X_1$.

Example 4.3. Let $X = \mathbb{R}^+$. Let $X_1 = [0, \frac{\pi}{2}]$, $X_2 = [1,2]$, $X_3 = [1,3]$ and $X = \cup_{i=1}^3 X_i$. Define $f : X \longrightarrow X$ by $f(x) = \sin(x) + 1$, for all $x \in X$. Clearly, $X = \cup_{i=1}^3 X_i$ is a cyclic representation of X with respect to f.

Theorem 4.4. [28] Let A and B be nonempty closed subset of a complete C^*-algebra valued b-metric space $(X,\mathbb{A},d)$. Assume that $T : A\cup B \longrightarrow A\cup B$ is a cyclic mapping that satisfies $d(Tx,Ty) \leq a^*\ d(x,y)\ a$, for all $x\in A$, $y\in B$, where $a\in A$ with $\|a\| < \dfrac{1}{\|b\|}$. Then, T has a unique FP in $A\cup B$.

Theorem 4.5. [29] Let A,B be two nonempty subsets of a metric space (X,d), let A be complete, and let $\leq$ be partially ordered relation on A. Let $T : A\cup B \longrightarrow A\cup B$ be cyclic mapping, T be continuous on A and T^2 be nondecreasing on A and $d(Tx^`,T^2x) \leq \alpha\ d(x^`,Tx)$ for some $\alpha\in[0,1)$ and for all $(x,x^`)\in A\times A$ with $x\leq x^`$. If there exists $x_0\leq A$ with $x_0\leq T^2x_0$, then $A\cap B\neq\phi$, hence T has a FP in $A\cap B$.

Theorem 4.6. [19] Let A and B be nonempty closed subsets of a complete metric space (X,d). Suppose that $T : A\cup B \longrightarrow A\cup B$ is cyclic map such that

$$d(Tx,Ty)\leq\alpha\ d(x,y)\ \forall x\in A,\ \forall\ y\in B.$$

If $\alpha\in[0,1)$, then T has a unique FP in $A\cap B$.

Theorem 4.7. Let A,B be two nonempty closed subsets of a complete partially ordered C^*-algebra valued metric space $(X,\mathbb{A},d_P,\preccurlyeq)$. Let $T : A\cup B \longrightarrow A\cup B$ be cyclic mapping, T be continuous on A and T^2 be nondecreasing on A that satisfies

$$d_P(Tx,T^2y)\preccurlyeq a^*\ d_P(x,Ty)\ a,$$

where for all $(x,y)\in A\times A$ with $x\preccurlyeq y$. Then T has a FP in $A\cap B$.

Proof. Let $x_0\in A$ with $x_0\leq T^2x_0$ and since T^2 is nondecreasing we get
$x_0\leq T^2(x_0)\leq\cdots\leq T^{2n}(x_0)\leq\ldots,$
for all $n\in\mathbb{N}$,

$$\begin{aligned}
d_P(T^nx_0,T^{n+1}x_0)=d_P(T^nx_0,TT^nx_0) &\preccurlyeq a^*\ d_P(T^{n-1}x_0,TT^{n-1}x_0)\ a\\
&\preccurlyeq (a^*)^2\ d_P(T^{n-2}x_0,TT^{n-2}x_0)\ (a)^2\\
&\ \ \vdots\\
&\preccurlyeq (a^*)^n\ d_P(x_0,Tx_0)\ (a)^n.
\end{aligned}$$

We show that $\{T^n x_0\}$ is a Cauchy sequence, let $n,m \in \mathbb{N}$ and $n \geq m$,

$$\begin{aligned}
d_P(T^m x_0, T^{n+1}x_0) &\preccurlyeq d_P(T^m x_0, T^{m+1}x_0) + d_P(T^{m+1}x_0 + T^{n+1}x_0) \\
&\preccurlyeq d_P(T^m x_0, T^{m+1}x_0) + d_P(T^{m+1}x_0, T^{m+2}x_0) + ... + d_P(T^n x_0, T^{n+1}x_0) \\
&\preccurlyeq (a^*)^m\ d_P(x_0, Tx_0)\ (a)^m + (a^*)^{m+1}\ d_P(x_0, Tx_0)\ (a)^{m+1} + \dots \\
&+ (a^*)^n\ d_P(x_0, Tx_0)\ (a)^n \\
&= \sum_{k=m}^{n} (a^*)^k d_P(x_0, Tx_0)(a)^k \\
&= \sum_{k=m}^{n} ((a)^k\ d_P(x_0, Tx_0)^{\frac{1}{2}})^* ((a)^k\ d_P(x_0, Tx_0)^{\frac{1}{2}}) \\
&= \sum_{k=m}^{n} |(a)^k\ (d_P(x_1, x_0))^{\frac{1}{2}}|^2 \\
&\preccurlyeq \| \sum_{k=m}^{n} |(a)^k\ (d_P(x_0, Tx_0))^{\frac{1}{2}}|^2 \|.\ I_{\mathbb{A}} \\
&\preccurlyeq \sum_{k=m}^{n} \|(a)\|^{2k} \|(d_P(x_0, Tx_0))^{\frac{1}{2}}\|^2.\ I_{\mathbb{A}} \to \theta.\ (as\ n,m \to \theta)
\end{aligned}$$

Therefore $\{T^n x_0\}$ is a Cauchy sequence in $\mathbb{A}$.

Since T is continuous, there exists $x \in X$ shut that $\lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^{n-1}x = Tx = x$, x is a FP of the mapping T.

We have $\{T^{2n}x_0\}$ is sequence in A converge to x and $\{T^{2n-1}x_0\}$ is sequence in B converge x, we get both sequences converge to the same limit x. i.e., $x \in A \bigcap B$. □

The following example support theorem 4.7

Example 4.8. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow M_2(\mathbb{R})$, define

$$d_P((x,y),(x,y)) = diag(|x_1 - y_1| + \alpha_1|x_2 - y_2|,\ |x_1 - y_1| + \alpha_2|x_2 - y_2|),$$

where $\alpha_1, \alpha_2 > 0$ such that $(x_1, y_1) \preccurlyeq (x_2, y_2)$ if and only if $x_1 \prec x_2,\ y_1 \prec y_2$.

Let

$$A = \{(x,0) : 0 \leq x \leq 2\},\ B = \{(0,y) : 0 \leq y \leq 10\}.$$

We define $T : A \cup B \longrightarrow A \cup B$ by $T(x,0) = (0,2x)$ for $0 \leq x \leq 2$, and $T(0,y) = (\frac{1}{y},0)$ if $y > 0$, and $T(0,y) = (y,0)$ for $0 \leq y \leq 10$.

Then T is a cyclic mapping and has a FP.

Conflict of Interests

The authors declare that there is no conflict of interests.

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