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ON A WEAKLY COMPATIBLE CONDITION OF MAPPING AND FIXED POINT RESULTS IN G-METRIC SPACE

PARDEEP KUMAR, NAWNEET HOODA*

Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal 131039, India

Abstract. In the present paper, we study the fixed point results in G-complete G-metric space and drive some well known results as corollaries of Mustafa et al. [7], Mustafa et al. [10], and Shatanawi [13].

Keywords: G-metric space; fixed point; contraction; weakly compatible mapping.

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1. INTRODUCTION

Banach fixed point theorem provides a constructive method to determine the existence and uniqueness of fixed point of certain self maps of metric spaces. It can be used to prove existence and uniqueness of solutions to integral and differential equations. The fixed point theory, initially stated in the metric space setting, has been extended in more general spaces even though most of them are metric like. Among of these general spaces, we will prove our results in the G-metric space, which has been discussed and explored by many authors.

Definition 1.1. [9]. Let $X \neq \phi$ be a set and $\mathrm{G} : X^3 \to [0,\infty)$ which satisfy the following properties:

*Corresponding author
E-mail address: nawneethooda@gmail.com
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(G_1) $G(r,s,t)=0$ if $r=s=t$ whenever $r,s,t \in X$;

(G_2) $G(r,r,s)>0$ whenever $r,s \in X with\ r \neq s$;

(G_3) $G(r,r,s) \leq G(r,s,t)$ whenever $r,s,t \in X with\ t \neq s$;

(G_4) $G(r,s,t)=G(r,t,s)=G(s,t,r)=\ldots,$

(G_5) $G(r,s,t) \leq G(r,a,a)+G(a,s,t)$ where $r,s,t,a \in X$ $(rectangle\ inequality)$.

Then (X,G) is known as a G-metric space.

Proposition 1.2. [9]. *In a* G-*metric space if*

$$G(r,s,t)=0,$$

then $r=s=t$ *where* $r,s,t \in X$.

Definition 1.3. [9]. In a G-metric space, a sequence $\{r_n\}$ converges to a point $r \in X$ if

$$\lim_{n,m \to \infty} G(r,r_n,r_m)=0,$$

that is, for any $\varepsilon>0$, there exists $N \in \mathbb{N}$ such that

$$G(r,r_n,r_m)<\varepsilon \quad for\ all\ \ n,m \geq N.$$

We call r the limit of the sequence and write $r_n \to r$ or $\lim_{n \to \infty} r_n = r$.

Proposition 1.4. [9]. *Let* (X,G) *be a* G-*metric space. A sequence* $\{r_n\} \subseteq X$*, has the following equivalent properties:*

(i) $\lim_{n \to \infty} r_n = r$.

(ii) $\lim_{n,m \to \infty} G(r,r_n,r_m)=0$.

(iii) $\lim_{n \to \infty} G(r,r_n,r_n)=0$.

(iv) $\lim_{n \to \infty} G(r_n,r,r)=0$.

Definition 1.5. [9]. In a G-metric space, a sequence $\{r_n\}$ is called a G-Cauchy if for any $\varepsilon>0$, there is natural number n_1 with

$$G(r_n,r_m,r_l)<\varepsilon \quad for\ all\ \ n,m,l \geq n_1.$$

Proposition 1.6. [9]. *A sequence* $\{r_n\}$ *in a* G-*metric space is* G-*Cauchy if and only if,* $G(r_n,r_m,r_m)<\varepsilon$ *where* $\varepsilon>0$ *and for all* $m,n \geq n_1$.

Definition 1.7. [9]. A G-metric space is said to be symmetric if

$$\mathrm{G}(r,s,s) = \mathrm{G}(r,r,s), \quad for\ all\ r,s \in X.$$

Proposition 1.8. [9]. *Every* G*-metric space* (X,G) *will define a metric space* (X,d_{G}) *by*

$$d_{\mathrm{G}}(r,s) = \mathrm{G}(r,s,s) + \mathrm{G}(s,r,r), \ \forall\, r,s \in X.$$

Note that if (X,G) *is a symmetric* G*-metric space, then*

$$d_{\mathrm{G}}(r,s) = 2\mathrm{G}(r,s,s), \quad \forall\, r,s \in X. \tag{1}$$

Definition 1.9. [9]. In a G-metric space, if every G-Cauchy sequence is G-convergent then it is called a complete metric space. Further a G-metric space is continuous in all its three variables.

Proposition 1.10. [9]. *Let* (X,G) *and* $(X^{'},\mathrm{G}^{'})$ *be* G*-metric spaces, then a function* $f : X \to X'$ *is* G*-continuous at a point* $r \in X$ *if and only if it is* G*-sequentially continuous at r; that is, whenever* $\{r_n\}$ *is* G*-convergent to r,* $\{f(r_n)\}$ *is* G*-convergent to f(r).*

Definition 1.11. [5] A pair (f,g) of self mappings of metric space (X,d) is said to be weakly compatible if the mappings commute at all of their coincidence points, that is, $fr = gr$ for some $r \in X$ implies $fgr = gfr$.

Definition 1.12. [1] A point $r \in X$ is called a coincidence point of two self mappings f and g if $w = fr = gr$ and the point w is called a point of coincidence of f and g.

Proposition 1.13. [1] *If two weakly compatible self mappings f and g have a unique point of coincidence, then that point is the unique common fixed point of f and g.*

2. Main Results

Theorem 2.1. *Let* (X,G) *be a complete* G*-metric space and* $f,g\colon X \to X$ *be mappings satisfying one of the following conditions:*

$$\mathrm{G}(fr,fs,ft) \le a_1\mathrm{G}(gr,gs,gt) + a_2\mathrm{G}(gr,fr,fr) + a_3\mathrm{G}(gs,fs,fs) + a_4\mathrm{G}(gt,ft,ft) \tag{2}$$

or

$$\mathrm{G}(fr,fs,ft) \le a_1\mathrm{G}(gr,gs,gt) + a_2\mathrm{G}(gr,gr,fr) + a_3\mathrm{G}(gs,gs,fs) + a_4\mathrm{G}(gt,gt,ft) \tag{3}$$

for all $r,s,t \in X$ where $0 \leq a_1+a_2+a_3+a_4 < 1$.

Further if range of X under f is contained in the range of X under g and g(X) is taken as complete subspace, then f and g have one and only one point of coincidence. Moreover if f and g satisfy weakly compatibility, then f and g have one and only one fixed point (say p) and f is G-*continuous at p.*

Proof. Suppose that f and g satisfy condition (2), then for all $r,s \in X$, we have

$$\mathrm{G}(fr,fs,fs) \leq a_1\mathrm{G}(gr,gs,gs)+a_2\mathrm{G}(gr,fr,fr)+(a_3+a_4)\mathrm{G}(gs,fs,fs),$$

$$\mathrm{G}(fs,fr,fr) \leq a_1\mathrm{G}(gs,gr,gr)+a_2\mathrm{G}(gs,fs,fs)+(a_3+a_4)\mathrm{G}(gr,fr,fr).$$

Suppose that (X,G) is symmetric, then by definition of metric (X,d_{G}) and (1), we get

$$d_{\mathrm{G}}(fr,fs) \leq a_1 d_{\mathrm{G}}(gr,gs)+\frac{a_3+a_4+a_2}{2}d_{\mathrm{G}}(gr,fr)+\frac{a_3+a_4+a_2}{2}d_{\mathrm{G}}(gs,fs),\ \forall\, r,s \in X.$$

In this line, since $0 < a_1+a_2+a_3+a_4 < 1$, then the existence and uniqueness of the fixed point follows from well-known theorem in metric space (X,d_{G}) (see [2]).

Suppose (X,G) is not symmetric. Let r_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, there is $r_1 \in X$ such that $gr_1 = fr_0$. Continuing the same process, we can construct a sequence $\{gr_n\}$ such that $gr_{n+1} = fr_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gr_n = gr_{n+1}$, then f and g have a point of coincidence. Let $gr_n \neq gr_{n+1}$ for all $n \in \mathbb{N}$. By (2), we have

$$\begin{aligned}\mathrm{G}(gr_n,gr_{n+1},gr_{n+1}) &= \mathrm{G}(fr_{n-1},fr_n,fr_n)\\ &\leq a_1\mathrm{G}(gr_{n-1},gr_n,gr_n)+a_2\mathrm{G}(gr_{n-1},gr_n,gr_n)\\ &+(a_3+a_4)\mathrm{G}(gr_n,gr_{n+1},gr_{n+1}),\end{aligned}$$

then

$$\mathrm{G}(gr_n,gr_{n+1},gr_{n+1}) \leq \frac{a_1+a_2}{1-(a_3+a_4)}\mathrm{G}(gr_{n-1},gr_n,gr_n).$$

Let $q = (a_1+a_2)/(1-(a_3+a_4))$, then $0 \leq q < 1$ since $0 \leq a_1+a_2+a_3+a_4 < 1$. So,

$$\mathrm{G}(gr_n,gr_{n+1},gr_{n+1}) \leq q\,\mathrm{G}(gr_{n-1},gr_n,gr_n).$$

Continuing in the same argument, we will get

$$G(gr_n, gr_{n+1}, gr_{n+1}) \leq q^n \, G(gr_0, gr_1, gr_1).$$

Also for all natural numbers n, m with $n < m$, the rectangle inequality gives

$$\begin{aligned} G(gr_n, gr_m, gr_m) &\leq G(gr_n, gr_{n+1}, gr_{n+1}) + G(gr_{n+1}, gr_{n+2}, gr_{n+2}) \\ &\quad + \ldots + G(gr_{m-1}, gr_m, gr_m) \\ &\leq (q^n + q^{n+1} + \ldots + q^{m-1]})G(gr_0, gr_1, gr_1) \\ &\leq \frac{q^n}{1-q} G(gr_0, gr_1, gr_1). \end{aligned}$$

Thus $\lim_{n,m\to\infty} G(gr_n, gr_m, gr_m) = 0$, proving $\{gr_n\}$ as G-Cauchy sequence. Since the space g(X) is complete, there exists $q \in g(X)$ and there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. Let $gp \neq fp$. By (2), we have

$$G(gr_n, fp, fp) \leq a_1 G(gr_{n-1}, gp, gp) + a_2 G(gr_{n-1}, gr_n, gr_n) + (a_3 + a_4) G(gp, fp, fp),$$

taking the limit as $n \to \infty$, and using the fact that the function G is continuous, then

$$G(gp, fp, fp) \leq (a_3 + a_4) G(gp, fp, fp).$$

This contradiction implies that $gp = fp$.

If $p^{'}$ is another coincidence point of f and g, then

$$\begin{aligned} G(gp, gp^{'}, gp^{'}) &\leq a_1 G(gp, gp^{'}, gp^{'}) + a_2 G(gp, fp, fp) + (a_3 + a_4) G(gp^{'}, fp^{'}, fp^{'}) \\ &= a_1 G(gp, gp^{'}, gp^{'}) \end{aligned}$$

which is a contradiction, so $gp = gp^{'}$. Thus f and g have one and only one coincident point. The uniqueness of fixed point follows by Proposition 1.13. Let $\{gs_n\}$ be a sequence in X with $\lim_{n\to\infty}(gs_n) = gp$. Now

$$\begin{aligned} G(gp, fs_n, fs_n) &\leq a_1 G(gp, gs_n, gs_n) + a_2 G(gp, fp, fp) + (a_3 + a_4) G(gs_n, fs_n, fs_n) \\ &= a_1 G(gp, gs_n, gs_n) + (a_3 + a_4) G(gs_n, fs_n, fs_n), \end{aligned}$$

and since

$$G(gs_n, fs_n, fs_n) \leq G(gs_n, gp, gp) + G(gp, fs_n, fs_n),$$

we have that

$$G(gp, fs_n, fs_n) \leq \frac{a_1}{1-(a_3+a_4)} G(gp, gs_n, gs_n) + \frac{a_3+a_4}{1-(a_3+a_4)} G(gs_n, gp, gp).$$

Taking the limit as $n \to \infty$, from which we see that $G(gp, fs_n, fs_n) \to 0$ and so, by Proposition 1.10, $f(s_n) \to gp = fp$, proving the continuity of f at p.

The similar arguments are applied if f and g fulfil (3). Now the only thing to be proved is $\{gr_n\}$ is a G-cauchy sequence. We have

$$G(gr_n, gr_n, gr_{n+1}) \leq a_1\, G(gr_{n-1}, gr_{n-1}, gr_n) + (a_2+a_3)\, G(gr_{n-1}, gr_{n-1}, gr_n)$$
$$+ a_4\, G(gr_n, gr_n, gr_{n+1}),$$

then

$$G(gr_n, gr_n, gr_{n+1}) \leq \frac{a_1+a_2+a_3}{1-a_4} G(gr_n, gr_n, gr_{n+1}).$$

Let $q = \frac{a_1+a_2+a_3}{1-a_4}$, then $0 \leq q < 1$ since $0 \leq a_1+a_2+a_3+a_4 < 1$.

Continuing in the same way, we find that

$$G(gr_n, gr_n, gr_{n+1}) \leq q^n\, G(gr_0, gr_0, gr_1).$$

Then for all $n, m \in \mathbb{N}$; $n < m$, we have by repeated use of the rectangle inequality

$$G(gr_n, gr_n, gr_m) \leq (q^n/1-q)\, G(gr_0, gr_0, gr_1).$$

□

Example 2.2. Let $X = [0,2]$, $G(r,s,t) = \max\{|r-s|, |s-t|, ||r-t|\}$.

Define $f, g \colon X \to X$ by

$$fr = 1 \quad and \quad gr = 2-r,$$

The functions f and g satisfy the condition (2) in Theorem 2.1. Indeed, we have

$$G(fr, fs, ft) = 0,$$

Further, $f(X) \subset g(X)$ and g(X) is a complete subspace of (X, G), f and g are weakly compatible. Thus all the axioms in Theorem 2.1 are fulfilled. This gives f and g have a unique common fixed point which is $r = 1$. The similar arguments if condition (3) is satisfied.

Corollary 2.3. [7, Theorem 2.1] *Let* (X, G) *be a complete* G*-metric space and* $f\colon X \to X$ *be a mapping satisfying one of the following conditions:*

$$\mathrm{G}(fr, fs, ft) \le a_1\mathrm{G}(r,s,t) + a_2\mathrm{G}(r, fr, fr) + a_3\mathrm{G}(s, fs, fs) + a_4\mathrm{G}(t, ft, ft)$$

or

$$\mathrm{G}(fr, fs, ft) \le a_1\mathrm{G}(r,s,t) + a_2\mathrm{G}(r, r, fr) + a_3\mathrm{G}(s, s, fs) + a_4\mathrm{G}(t, t, ft)$$

for all $r,s,t \in X$ *where* $0 \le a_1 + a_2 + a_3 + a_4 < 1$, *then f has a unique fixed point (say p, i.e., fp = p), and f is* G*-continuous at u.*

Proof. It follows by taking $g = I_X$ (Identity mapping) in Theorem 2.1. □

Corollary 2.4. [13, Corollary 3.4] *Let X be a complete* G*-metric space. Suppose there is* $k \in [0,1)$ *such that the map* $f\colon X \to X$ *satisfies*

$$\mathrm{G}(fr, fs, ft) \le k\,\mathrm{G}(r,s,t),$$

for all $r,s,t \in X$. *Then f has a unique fixed point (say p) and* f *is* G*-continuous at p.*

Proof. From Theorem 2.1, taking $g = I_X$ and $a_2 = a_3 = a_4 = 0$. □

Corollary 2.5. [10, Theorem 2.3] *Let* (X, G) *be a complete* G*-metric space and* $f\colon X \to X$ *be a mapping satisfying, for all* $r,s,t \in X$

$$\mathrm{G}(fr, fs, ft) \le a_1\mathrm{G}(r, fr, fr) + a_2\mathrm{G}(s, fs, fs) + a_3\mathrm{G}(t, ft, ft)$$

where $0 < a_1 + a_2 + a_3 < 1$, *then* f *has a fixed point, say p, and* f *is* G*-continuous at p.*

Proof. If we assume $g = I_X$ and $a_1 = 0$, in Theorem 2.1 we get the result. □

Corollary 2.6. *Let* (X, G) *be a complete* G*-metric space and let* $f, g\colon X \to X$ *be mappings satisfying one of the following conditions:*

$$\mathrm{G}(f^m(r), f^m(s), f^m(t)) \le a_1\mathrm{G}(gr, gs, gs) + a_2\mathrm{G}(gr, f^m(r), f^m(r)) + a_3\mathrm{G}(gs, f^m(s), f^m(s)) + a_4\mathrm{G}(gt, f^m(t), f^m(t)) \tag{4}$$

or

$$G(f^m(r), f^m(s), f^m(t)) \le a_1 G(gr, gs, gs) + a_2 G(gr, gr, f^m(r)) + a_3 G(gs, gs, f^m(s)) + a_4 G(gt, gt, f^m(t)) \tag{5}$$

for all $r, s, t \in X$*, where* $0 \le a_1 + a_2 + a_3 + a_4 < 1$*. Then f has a unique fixed point (say p), and* f^m *is* G*-continuous at p.*

Further if range of X under f is contained in the range of X under g and g(X) is taken as complete subspace, then f and g have one and only one point of coincidence. Moreover if f and g satisfy weakly compatibility, then f and g have one and only one fixed point (say p) and f is G*-continuous at p.*

Proof. From Theorem 2.1, we see that f^m has a unique fixed point (say p), that is, $f^m(p) = p$. But $f(p) = f(f^m(p)) = f^{m+1}(p) = f^m(f(p))$, so $f(p)$ is another fixed point for f^m and by uniqueness $fp = p$. □

Corollary 2.7. [7, Corollary 2.2] *Let* (X, G) *be a complete* G*-metric space and let* $f : X \to X$ *be a mapping satisfying one of the following conditions:*

$$G(f^m(r), f^m(s), f^m(t)) \le a_1 G(r, s, s) + a_2 G(r, f^m(r), f^m(r)) + a_3 G(s, f^m(s), f^m(s)) + a_4 G(t, f^m(t), f^m(t)) \tag{6}$$

or

$$G(f^m(r), f^m(s), f^m(t)) \le a_1 G(r, s, s) + a_2 G(r, r, f^m(r)) + a_3 G(s, s, f^m(s)) + a_4 G(t, t, f^m(t)) \tag{7}$$

for all $r, s, t \in X$*, where* $0 \le a_1 + a_2 + a_3 + a_4 < 1$*. Then f has a unique fixed point (say p), and* f^m *is* G*-continuous at p.*

Proof. It follows by taking $g = I_X$ in Corollary 2.6. □

Theorem 2.8. *Let* (X, G) *be a complete* G*-metric space and let* $f, g : X \to X$ *be mappings satisfying one of the following conditions:*

$$G(fr, fs, ft) \le k \max\{G(gr, fr, fr), G(gs, fs, fs), G(gt, ft, ft)\} \tag{8}$$

or

$$\text{(9)} \qquad \mathrm{G}(fr, fs, ft) \le k \max\{\mathrm{G}(gr, gr, fr), \mathrm{G}(gs, gs, fs), \mathrm{G}(gt, gt, ft)\}$$

for all $r, s, t \in X$, where $0 \le k < 1$. Then f has a unique fixed point (say p), and f is G*-continuous at p.*

Further if range of X under f is contained in the range of X under g and g(X) is taken as complete subspace, then f and g have one and only one point of coincidence. Moreover if f and g satisfy weakly compatibility, then f and g have one and only one fixed point (say p) and f is G*-continuous at p.*

Proof. Suppose that f and g satisfy condition (8), then for all $r, s \in X$,

$$\mathrm{G}(fr, fs, fs) \le k \max\{\mathrm{G}(gr, fr, fr), \mathrm{G}(gs, fs, fs)\}$$

$$\mathrm{G}(fs, fr, fr) \le k \max\{\mathrm{G}(gs, fs, fs), \mathrm{G}(gr, fr, fr)\}$$

Suppose that (X, G) is symmetric, then by definition of the metric (X, d_{G}) and (1) we get

$$d_{\mathrm{G}}(fr, fs) \le k \max\{d_{\mathrm{G}}(gr, fr), d_{\mathrm{G}}(gs, fs)\}, \quad \forall\, r, s \in X.$$

Since $k < 1$, then the existence and uniqueness of the fixed point follows from a theorem in metric space (X, d_{G}) (see [2]).

Suppose (X, G) is not symmetric.Let r_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, there is $r_1 \in X$ such that $gr_1 = fr_0$. Continuing the same process, we can construct a sequence $\{gr_n\}$ such that $gr_{n+1} = fr_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gr_n = gr_{n+1}$, then f and g have a point of coincidence. Let $gr_n \ne gr_{n+1}$ for all $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$. By (8), we have

$$\begin{aligned} \mathrm{G}(gr_n, gr_{n+1}, gr_{n+1}) &\le k \max\{\mathrm{G}(gr_{n-1}, gr_n, gr_n), \mathrm{G}(gr_n, gr_{n+1}, gr_{n+1})\} \\ &= k\, \mathrm{G}(gr_{n-1}, gr_n, gr_n) \quad \textit{since}(0 \le k < 1). \end{aligned}$$

Continuing in the same argument, we will find

$$\mathrm{G}(gr_n, gr_{n+1}, gr_{n+1}) \le k^n\, \mathrm{G}(gr_0, gr_1, gr_1).$$

For all $n, m \in \mathbb{N}$; $n < m$, we have by rectangle inequality that

$$\begin{aligned} G(gr_n, gr_m, gr_m) &\leq G(gr_n, gr_{n+1}, gr_{n+1}) + G(gr_{n+1}, gr_{n+2}, gr_{n+2}) \\ &\leq (k^n + k^{n+1} + ... + k^{m-1]})G(gr_0, gr_1, gr_1) \\ &\leq \frac{k^n}{1-k} G(gr_0, gr_1, gr_1). \end{aligned}$$

Then, $\lim G(gr_n, gr_m, gr_m) = 0$, as $n, m \to \infty$, and thus $\{gr_n\}$ is G-Cauchy sequence. Due to the completeness of (X, G), there exists $q \in g(X)$ and $p \in X$ such that $gp = q$. We will show that $gp = fp$. Let $gp \neq fp$. By (8), we have

$$G(gr_{n+1}, fp, fp) \leq k \max\{G(gr_{n+1}, gr_{n+2}, gr_{n+2}), G(gp, fp, fp)\}$$

and by taking the limit as $n \to \infty$, and using the fact that the function G is continuous, we get that

$$G(gp, fp, fp) \leq k\, G(gp, fp, fp).$$

This contradiction implies that $gp = f(p)$.

To prove uniqueness, suppose that $p \neq p^{'}$ such that $f(p^{'}) = gp^{'}$, then

$$G(gp, gp^{'}, gp^{'}) \leq k \max\{G(gp^{'}, gp^{'}, gp^{'}), G(gp, gp, gp)\} = 0$$

which implies that $gp = fp^{'}$.

To show that f is G-continuous at p, let $\{gs_n\} \subseteq X$ be a sequence such that $\lim_{n\to\infty}(gs_n) = gp$, then

$$\begin{aligned} G(gp, fs_n, fs_n) &\leq \max\{G(gp, fp, fp), G(gs_n, fs_n, fs_n)\} \\ &= k\, G(gs_n, fs_n, fs_n) \end{aligned}$$

But,

$$G(gs_n, fs_n, fs_n) \leq G(gs_n, gp, gp) + G(gp, fs_n, fs_n),$$

then

$$G(gp, fs_n, fs_n) \leq \frac{k}{1-k} G(gs_n, gp, gp).$$

Taking the limit as $n \to \infty$, from which wee see that $G(gp, fs_np, fs_n) \to 0$, and so by Proposition 1.10, $f(s_n) \to gp = fp$. So, f is G-continuous at p.

Corollary 2.9. [7, Theorem 2.3] *Let* (X,G) *be a complete* G*-metric space and let* $f\colon X\to X$ *be a mapping satisfying one of the following conditions:*

$$\mathrm{G}(fr,fs,ft)\le k\max\{\mathrm{G}(r,fr,fr),\mathrm{G}(s,fs,fs),\mathrm{G}(t,ft,ft)\}$$

or

$$\mathrm{G}(fr,fs,ft)\le k\max\{\mathrm{G}(r,r,fr),\mathrm{G}(s,s,fs),\mathrm{G}(t,t,ft)\}$$

for all $r,s,t\in X$*, where* $0\le k<1$*. Then* f *has a unique fixed point (say* p*), and* f *is* G*-continuous at* p*.*

Proof. It follows by taking $g=I_X$ in Theorem 2.8. □

Corollary 2.10. *Let* (X,G) *be a* G*-complete* G*-metric space and let* $f,g\colon X\to X$ *be mappings satisfying one of the following conditions, for all* $m\in\mathbb{N}$

$$\mathrm{G}(f^m(r),f^m(s),f^m(t)\le k\max\left\{\begin{array}{c}\mathrm{G}(gr,f^m(r),f^m(r)),\\ \mathrm{G}(gs,f^m(s),f^m(s)),\\ \mathrm{G}(gt,f^m(t),f^m(t))\end{array}\right\}\tag{10}$$

or

$$\mathrm{G}(f^m(r),f^m(s),f^m(t)\le k\max\left\{\begin{array}{c}\mathrm{G}(gr,gr,f^m(r)),\\ \mathrm{G}(gs,gs,f^m(s)),\\ \mathrm{G}(gt,gt,f^m(t))\end{array}\right\}\tag{11}$$

for all $r,s,t\in X$*. Then* f *has a unique fixed point (say* p*), and* f^m *is* G*-continuous at* p*.*

Further if range of X under f *is contained in the range of X under* g *and g(X) is taken as complete subspace, then f and g have one and only one point of coincidence. Moreover if f and g satisfy weakly compatibility, then* f *and* g *have one and only one fixed point (say p) and* f *is* G*-continuous at p.*

Proof. We use the same argument as in Corollary in 2.6. □

Corollary 2.11. [7, Corollary 2.4] *Let* (X,G) *be a* G*-complete* G*-metric space and let* $f\colon X\to X$ *be a mapping satisfying one of the following conditions, for all* $m\in\mathbb{N}$

$$\text{(12)} \qquad \mathrm{G}(f^m(r), f^m(s), f^m(t) \leq k \max \left\{ \begin{array}{c} \mathrm{G}(r, f^m(r), f^m(r)), \\ \mathrm{G}(s, f^m(s), f^m(s)), \\ \mathrm{G}(t, f^m(t), f^m(t)) \end{array} \right\}$$

or

$$\text{(13)} \qquad \mathrm{G}(f^m(r), f^m(s), f^m(t) \leq k \max \left\{ \begin{array}{c} \mathrm{G}(r, r, f^m(r)), \\ \mathrm{G}(s, s, f^m(s)), \\ \mathrm{G}(t, t, f^m(t)) \end{array} \right\}$$

for all $r,s,t \in X$. Then f has a unique fixed point (say p), and f^m is G*-continuous at p.*

Proof. It follows by taking $g = I_X$ in Corollary 2.10. □

Theorem 2.12. *Let (X, G) be a complete* G*-metric space, and $f, g \colon X \to X$ be mappings satisfying one of the following conditions:*

$$\text{(14)} \qquad \mathrm{G}(fr, fs, fs) \leq k \max\{\mathrm{G}(gr, fs, fs), \mathrm{G}(gs, fr, fr), \mathrm{G}(gs, fs, fs)\}$$

or

$$\text{(15)} \qquad \mathrm{G}(fr, fs, fs) \leq k \max\{\mathrm{G}(gr, gr, fs), \mathrm{G}(gs, gs, fr), \mathrm{G}(gs, gs, fs)\}$$

for all $r,s,t \in X$, where $k \in [0,1)$. Then f has a unique fixed point (say p), and f is G*-continuous at p.*

Further if range of X under f is contained in the range of X under g and g(X) is taken as complete subspace, then f and g have one and only one point of coincidence. Moreover if f and g satisfy weakly compatibility, then f and g have one and only one fixed point (say p) and f is G*-continuous at p.*

Proof. Suppose that f and g satisfy condition (14), then for all $r, s \in X$,

$$\mathrm{G}(fr, fs, fs) \leq k \max\{\mathrm{G}(gr, fs, fs), \mathrm{G}(gs, fr, fr), \mathrm{G}(gs, fs, fs)\},$$

$$\mathrm{G}(fs, fs, fr) \leq k \max\{\mathrm{G}(gr, fs, fs), \mathrm{G}(gs, fr, fr), \mathrm{G}(gr, fr, fr)\}.$$

Suppose that (X,G) is symmetric, then by definition of the metric (X,d_{G}) and (1), we have

$$d_{\mathrm{G}}(fr,fs) \leq \frac{k}{2}\max\left\{\begin{array}{c} d_{\mathrm{G}}(gr,fs), \\ d_{\mathrm{G}}(gs,fr), \\ d_{\mathrm{G}}(gs,fs) \end{array}\right\} + \frac{k}{2}\max\left\{\begin{array}{c} d_{\mathrm{G}}(gr,fs), \\ d_{\mathrm{G}}(gs,fr), \\ d_{\mathrm{G}}(gr,fr) \end{array}\right\}$$

$$\leq k\max\{d_{\mathrm{G}}(gr,fs),d_{\mathrm{G}}(gs,fr),d_{\mathrm{G}}(gs,fs)\}, \quad \forall\, r,s\in X.$$

Since $0\leq k<1$, then the existence and uniqueness of the fixed point follows from a theorem in metric space (X,d_{G}) (see [2]).

Suppose (X,G) is not symmetric. Let $r_0\in X$ be arbitrary point, and define the sequence $\{gr_n\}$ by $gr_n=f^n(r_0)$, then by (14) and using $k<1$, we deduce that

$$\begin{aligned}\mathrm{G}(gr_n,gr_{n+1},gr_{n+1}) &\leq k\ \max\{\mathrm{G}(gr_{n-1},gr_{n+1},gr_{n+1}),\mathrm{G}(gr_n,gr_{n+1},gr_{n+1})\}\\ &= k\,\mathrm{G}(gr_{n-1},gr_{n+1},gr_{n+1})\end{aligned}$$

So,

$$\mathrm{G}(gr_n,gr_{n+1},gr_{n+1}) \leq k\,\mathrm{G}(gr_{n-1},gr_{n+1},gr_{n+1}),$$

and using

$$\begin{aligned}\mathrm{G}(gr_{n-1},gr_{n+1},gr_{n+1}) \leq k\ \max\{&\mathrm{G}(gr_{n-2},gr_{n+1},gr_{n+1}),\mathrm{G}(gr_n,gr_{n-1},gr_{n-1}),\\ &\mathrm{G}(gr_n,gr_{n+1},gr_{n+1})\},\end{aligned}$$

then,

$$\mathrm{G}(gr_n,gr_{n+1},gr_{n+1}) \leq k^2\max\{\mathrm{G}(gr_{n-2},gr_{n+1},gr_{n+1}),\mathrm{G}(gr_n,gr_{n-1},gr_{n-1})\}.$$

Continuing in this procedure, we will have

$$\mathrm{G}(gr_n,gr_{n+1},gr_{n+1}) \leq k^n\,\Gamma_n,$$

where $\Gamma n=\max\{\mathrm{G}(gr_i,gr_j,gr_j)$; for all $i,j\in\{0,1,...n+1\}\}$.

For $n,m\in\mathbb{N};n<m$, let $\Gamma=\max\{\Gamma_i$; for all $i=n,...,m-1\}$.

Then, for all $m,m\in\mathbb{N}$; $n<m$, we have by rectangle inequality

$$\mathrm{G}(gr_n,gr_m,gr_m) \leq \mathrm{G}(gr_n,gr_{n+1},gr_{n+1})+\mathrm{G}(gr_{n+1},gr_{n+2},gr_{n+2})$$

$$+\ldots+\mathrm{G}(gr_{m-1},gr_m,gr_m)$$
$$\leq k^n\Gamma_n+k^{n+1}\Gamma_{n+1}+\ldots+k^{m-1}\Gamma_{m-1}$$
$$\leq (K^n+k^{n+1}+\ldots+k^{m-1})\Gamma$$
$$\leq \frac{k^n}{1-k}\Gamma.$$

This prove that lim $\mathrm{G}(gr_n,gr_m,gr_m)=0$, as $n,m\to\infty$, and thus $\{gr_n\}$ is G-Cauchy sequence. Since (X,G) is G-complete then there exists $p\in X$ such that $\{gr_n\}$ is G-converge to p.
Suppose that $f(p)\neq gp$, then

$$\mathrm{G}(gr_n,fp,fp)\leq k\max\{\mathrm{G}(gr_{n-1},fp,fp),\mathrm{G}(gp,gr_{n+1},r_{n+1}),\mathrm{G}(gp,fp,fp)\}.$$

Taking the limit as $n\to\infty$, and using the fact that the function f is G-continuous, we get $\mathrm{G}(gp,fp,fp)\leq k\,\mathrm{G}(gp,fp,fp)$, this contradiction implies that $gp=fp$.
To prove the uniqueness, suppose that $gp\neq gp^{'}$ such that $f(p^{'})=gp^{'}$ So, by (14), we have that

$$\mathrm{G}(gp,gp^{'},gp^{'})\leq k\ \max\{\mathrm{G}(gp,gp^{'},gp^{'}),\mathrm{G}(gp^{'},gp,gp)\}$$
$$=k\,\mathrm{G}(gp^{'},gp,gp).$$

Again we will find $\mathrm{G}(gp^{'},gp,gp)\leq k\,\mathrm{G}(gp,gp^{'},gp^{'})$, so

$$\mathrm{G}(gp,gp^{'},gp^{'})\leq k^2\,\mathrm{G}(gp,gp^{'},gp^{'});$$

since $k<1$, this implies that $gp=gp^{'}$. To show that f is G-continuous at p, let $\{gs_n\}\subseteq X$ be a sequence such that $\lim(gs_n)=gp\ as\ n\to\infty$, then

$$\mathrm{G}(gp,f(s_n),f(s_n))\leq k\max\{\mathrm{G}(gp,f(s_n),f(s_n)),\mathrm{G}(g(s_n),fp,fp),\mathrm{G}(g(s_n),f(s_n),f(s_n))\}.$$

But,

$$\mathrm{G}(g(s_n),f(s_n),f(s_n))\leq k\max\{\mathrm{G}(g(s_n),gp,gp),\mathrm{G}(gp,f(s_n),f(s_n)),\mathrm{G}(g(s_n),f(s_n),f(s_n))\},$$

so,

$$\mathrm{G}(gp,f(s_n),f(s_n))\leq \frac{k}{1-k}\mathrm{G}(g(s_n),gp,gp).$$

Taking the limit as $n \to \infty$, from which we see that $G(gp, f(s_n), f(s_n)) \to 0$ and so, by Proposition 1.10, we have $f(s_n) \to gp = fp$ which implies that f is G-continuous at p. □

Corollary 2.13. [7, Theorem 2.5] *Let* (X, G) *be a complete* G*-metric space, and* $f: X \to X$ *be a mapping satisfying one of the following conditions:*

$$G(fr, fs, fs) \leq k \max\{G(r, fs, fs), G(s, fr, fr), G(s, fs, fs)\} \tag{16}$$

or

$$G(fr, fs, fs) \leq k \max\{G(r, r, fs), G(s, s, fr), G(s, s, fs)\} \tag{17}$$

for all $r, s, t \in X$*, where* $k \in [0, 1)$*. Then f has a unique fixed point (say p), and f is* G*-continuous at p.*

Proof. It follows by taking $g = I_X$ in Theorem 2.12. □

Corollary 2.14. *Let* (X, G) *be a complete* G*-metric space, and let* $f, g: X \to X$ *be mappings satisfying one of the following conditions:*

$$G(fr, fs, ft) \leq k \max\left\{\begin{array}{c} G(gr, fs, fs), G(gr, ft, ft), \\ G(gs, fr, fr), G(gs, ft, ft), \\ G(gt, fr, fr), G(gt, fs, fs) \end{array}\right\} \tag{18}$$

or

$$G(fr, fs, ft) \leq k \max\left\{\begin{array}{c} G(gr, gr, fs), G(gr, gr, ft), \\ G(gs, gs, fr), G(gs, gs, ft), \\ G(gt, gt, fr), G(gt, gt, fs) \end{array}\right\} \tag{19}$$

for all $r, s, t \in X$*, where* $k \in [0, 1)$*. Then f has a unique fixed point (say p), and f is* G*-continuous at p.*

Further if range of X under f is contained in the range of X under g and g(X) is taken as complete subspace, then f and g have one and only one point of coincidence. Moreover if f and g satisfy weakly compatibility, then f and g have one and only one fixed point (say p) and f is G*-continuous at p.*

Proof. If we let $s = t$ in conditions (18) and (19), then they become conditions (14) and (15), respectively, in Theorem 2.12; so the proof follows from Theorem 2.12. □

Corollary 2.15. [7, Corollary 2.6] *Let* (X, G) *be a complete* G*-metric space, and let* $f\colon X \to X$ *be a mapping satisfying one of the following conditions:*

$$\mathrm{G}(fr, fs, ft) \le k \max \left\{ \begin{array}{c} \mathrm{G}(r, fs, fs), \mathrm{G}(r, ft, ft), \\ \mathrm{G}(s, fr, fr), \mathrm{G}(s, ft, ft), \\ \mathrm{G}(t, fr, fr), \mathrm{G}(t, fs, fs) \end{array} \right\} \tag{20}$$

or

$$\mathrm{G}(fr, fs, ft) \le k \max \left\{ \begin{array}{c} \mathrm{G}(r, r, fs), \mathrm{G}(r, r, ft), \\ \mathrm{G}(s, s, fr), \mathrm{G}(s, s, ft), \\ \mathrm{G}(t, t, fr), \mathrm{G}(t, t, fs) \end{array} \right\} \tag{21}$$

for all $r, s, t \in X$*, where* $k \in [0, 1)$*. Then* f *has a unique fixed point (say* p*), and* f *is* G*-continuous at* p*.*

Proof. It follows by taking $g = I_X$ in Corollary 2.14. □

Corollary 2.16. *Let* (X, G) *be a complete* G*-metric space, and let* $f, g\colon X \to X$ *be mappings satisfying one of the following conditions:*

$$\mathrm{G}(f^m(r), f^m(s), f^m(t)) \le k \max \left\{ \begin{array}{c} \mathrm{G}(gr, f^m(s), f^m(s)), \mathrm{G}(gr, f^m(t), f^m(t)), \\ \mathrm{G}(gs, f^m(r), f^m(r)), \mathrm{G}(gs, f^m(t), f^m(t)), \\ \mathrm{G}(gt, f^m(r), f^m(r)), \mathrm{G}(gt, f^m(s), f^m(s)) \end{array} \right\}$$

$$\mathrm{G}(f^m(r), f^m(s), f^m(t)) \le k \max \left\{ \begin{array}{c} \mathrm{G}(gr, gr, f^m(s)), \mathrm{G}(gr, gr, f^m(t)), \\ \mathrm{G}(gs, gs, f^m(r)), \mathrm{G}(gs, gs, f^m(t)), \\ \mathrm{G}(gt, gt, f^m(r)), \mathrm{G}(gt, gt, f^m(s)) \end{array} \right\}$$

$$\mathrm{G}(f^m(r), f^m(s), f^m(s)) \le k \max \left\{ \begin{array}{c} \mathrm{G}(gr, f^m(s), f^m(s)), \\ \mathrm{G}(gs, f^m(r), f^m(r)), \\ \mathrm{G}(gs, f^m(s), f^m(s)) \end{array} \right\} \tag{22}$$

or

$$\mathrm{G}(f^m(r), f^m(s), f^m(s)) \leq k \max \left\{ \begin{array}{c} \mathrm{G}(gr, gr, f^m(s)), \\ \mathrm{G}(gs, gs, f^m(r)), \\ \mathrm{G}(gs, gs, f^m(s)) \end{array} \right\} \tag{23}$$

for all $r, s, t \in X$, for some $n \in \mathbb{N}$, where $k \in [0,1)$, then f has a unique fixed point (say p), and f^m is G*-continuous at p.*

Further if range of X under f is contained in the range of X under g and g(X) is taken as complete subspace, then f and g have one and only one point of coincidence. Moreover if f and g satisfy weakly compatibility, then f and g have one and only one fixed point (say p) and f is G*-continuous at p.*

Proof. The proof follows from Theorem 2.12, Corollary 2.14, and from an argument similar to that used in Corollary 2.6. □

Corollary 2.17. [7, Corollary 2.7] *Let (X, G) be a complete* G*-metric space, and let $f\colon X \to X$ be a mapping satisfying one of the following conditions:*

$$\mathrm{G}(f^m(r), f^m(s), f^m(t)) \leq k \max \left\{ \begin{array}{c} \mathrm{G}(r, f^m(s), f^m(s)), \mathrm{G}(r, f^m(t), f^m(t)), \\ \mathrm{G}(s, f^m(r), f^m(r)), \mathrm{G}(s, f^m(t), f^m(t)), \\ \mathrm{G}(t, f^m(r), f^m(r)), \mathrm{G}(t, f^m(s), f^m(s)) \end{array} \right\}$$

$$\mathrm{G}(f^m(r), f^m(s), f^m(t)) \leq k \max \left\{ \begin{array}{c} \mathrm{G}(r, r, f^m(s)), \mathrm{G}(r, r, f^m(t)), \\ \mathrm{G}(s, s, f^m(r)), \mathrm{G}(s, s, f^m(t)), \\ \mathrm{G}(t, t, f^m(r)), \mathrm{G}(t, t, f^m(s)) \end{array} \right\}$$

$$\mathrm{G}(f^m(r), f^m(s), f^m(s) \leq k \max \left\{ \begin{array}{c} \mathrm{G}(r, f^m(s), f^m(s)), \\ \mathrm{G}(s, f^m(r), f^m(r)), \\ \mathrm{G}(s, f^m(s), f^m(s)) \end{array} \right\} \tag{24}$$

or

$$\mathrm{G}(f^m(r), f^m(s), f^m(s)) \leq k \max \left\{ \begin{array}{c} \mathrm{G}(r, r, f^m(s)), \\ \mathrm{G}(s, s, f^m(r)), \\ \mathrm{G}(s, s, f^m(s)) \end{array} \right\} \tag{25}$$

for all $r,s,t \in X$, for some $n \in \mathbb{N}$, where $k \in [0,1)$, then f has a unique fixed point (say p), and f^m is G*-continuous at p.*

Proof. It follows by taking $g = I_X$ in Corollary 2.16. □

Theorem 2.18. *Let (X,G) be a complete* G*-metric space, and let $f,g\colon X \to X$ be mappings satisfying one of the following conditions:*

$$\mathrm{G}(fr,fs,fs) \leq k\max\{\mathrm{G}(gr,fs,fs),\mathrm{G}(gs,fr,fr)\} \tag{26}$$

or

$$\mathrm{G}(fr,fs,fs) \leq k\max\{\mathrm{G}(gr,gr,fs),\mathrm{G}(gs,gs,fr)\} \tag{27}$$

for all $r,s,t \in X$, where $k \in [0,1)$, then f has a unique fixed point (say p), and f is G*-continuous at p.*

Further if range of X under f is contained in the range of X under g and g(X) is taken as complete subspace, then f and g have one and only one point of coincidence. Moreover if f and g satisfy weakly compatibility, then f and g have one and only one fixed point (say p) and f is G*-continuous at p.*

Proof. Since whenever the mapping satisfy condition (26), or (27), then it satisfy condition (18), or (19), respectively, in Theorem 2.12. Then the proof follows from Theorem 2.12. □

Corollary 2.19. [7, Theorem 2.8] *Let (X,G) be a complete* G*-metric space, and let $f\colon X \to X$ be a mapping satisfying one of the following conditions:*

$$\mathrm{G}(fr,fs,fs) \leq k\max\{\mathrm{G}(r,fs,fs),\mathrm{G}(s,fr,fr)\} \tag{28}$$

or

$$\mathrm{G}(fr,fs,fs) \leq k\max\{\mathrm{G}(r,r,fs),\mathrm{G}(s,s,fr)\} \tag{29}$$

for all $r,s,t \in X$, where $k \in [0,1)$, then f has a unique fixed point (say p), and f is G*-continuous at p.*

Proof. It follows by taking $g = I_X$ in Theorem 2.18. □

Theorem 2.20. *Let (X,G) be a complete G-metric space, and let $f,g\colon X\to X$ be mappings satisfying one of the following conditions:*

$$\mathrm{G}(fr,fs,fs)\le a_1\{\mathrm{G}(gr,fs,fs)+\mathrm{G}(gs,fr,fr)\} \tag{30}$$

or

$$\mathrm{G}(fr,fs,fs)\le a_1\{\mathrm{G}(gr,gr,fs)+\mathrm{G}(gs,gs,fr)\} \tag{31}$$

for all $r,s,t\in X$, where $a_1\in[0,\frac{1}{2})$. then f has a unique fixed point (say p), and f is G*-continuous at p.*

Further if range of X under f is contained in the range of X under g and $g(X)$ is taken as complete subspace, then f and g have one and only one point of coincidence. Moreover if f and g satisfy weakly compatibility, then f and g have one and only one fixed point (say p) and f is G*-continuous at p.*

Proof. Suppose that f and g satisfy condition (30), then we have

$$\mathrm{G}(fr,fs,fs)\le a_1\{\mathrm{G}(gs,fr,fr)+\mathrm{G}(gr,fs,fs)\}$$

$$\mathrm{G}(fs,fr,fr)\le a_1\{\mathrm{G}(gr,fs,fs)+G(gs,fr,fr)\}$$

for all $r,s\in X$.

Suppose that (X,G) is symmetric, then by definition of the metric (X,d_{G}) and (1), we get

$$d_{\mathrm{G}}(fr,fs)\le a_1\{d_{\mathrm{G}}(gr,fs)+d_{\mathrm{G}}(gs,fr)\}\quad\forall\, x,y\in X.$$

Since $0\le 2a_1<1$, then the existence and uniqueness of the fixed point follow from a theorem in metric space (X,d_{G}) (see [2]).

Suppose (X,G) is not symmetric. Let $r_0\in X$ be arbitrary point, and define the sequence $\{gr_n\}$ by $gr_n=f^n(x_0)$, then by (30), we have

$$\begin{aligned}\mathrm{G}(gr_n,gr_{n+1},gr_{n+1})&\le a_1\{\mathrm{G}(gr_{n-1},gr_{n+1},gr_{n+1})+\mathrm{G}(gr_n,gr_n,gr_n)\}\\&=a_1\mathrm{G}(gr_{n-1},gr_{n+1},gr_{n+1}).\end{aligned}$$

But

$$\mathrm{G}(gr_{n-1},gr_{n+1},gr_{n+1})\le a_1\mathrm{G}(gr_{n-1},gr_n,gr_n)+a\mathrm{G}(gr_n,gr_{n+1},gr_{n+1}),$$

thus we have

$$G(gr_n, gr_{n+1}, gr_{n+1}) \leq \frac{a_1}{1-a_1} G(gr_{n-1}, gr_n, gr_n).$$

Let $k = a_1/(1-a_1)$, hence $0 \leq k < 1$ then continue in this procedure, we will get that

$$G(gr_n, gr_{n+1}, gr_{n+1}) \leq k^n G(gr_0, gr_1, gr_1).$$

For all $n, m \in N; n < m$, we have by rectangle inequality

$$\begin{aligned} G(gr_n, gr_m, gr_m) &\leq G(gr_n, gr_{n+1}, gr_{n+1}) + G(gr_{n+1}, gr_{n+2}, gr_{n+2}) \\ &+ \ldots + G(gr_{m-1}, gr_m, gr_m) \\ &\leq (K^n + k^{n+1} + \ldots + k^{m-1}) G(gr_0, gr_1, gr_1) \\ &\leq \frac{k^n}{1-k} G(gr_0, gr_1, gr_1). \end{aligned}$$

Then, $\lim G(gr_n, gr_m, gr_m) = 0$, as $n, m \to \infty$, and so, $\{gr_n\}$ is G-Cauchy completeness of (X, G), there exists $p \in X$ such that $\{gr_n\}$ is G-converge to p.

Suppose that $f(p) \neq gp$, then

$$G(gr_n, fp, fp) \leq a_1 \{G(gr_{n-1}, fp, fp) + G(gp, gr_n, gr_n)\}.$$

Taking the limit as $n \to \infty$, and using the fact that the function is G-continuous, then

$$G(gp, fp, fp) \leq a_1 G(gp, fp, fp).$$

This contradiction implies that $gp = fp$.

To prove uniqueness, suppose that $gp \neq gp^{'}$ such that $f(p^{'}) = gp^{'}$, then

$$G(gp, gp^{'}, gp^{'}) \leq a\{G(gp, gp^{'}, gp^{'}) + G(gp^{'}, gp, gp),$$

so

$$G(gp, gp^{'}, gp^{'}) \leq \left(k = \frac{a_1}{1-a_1}\right) G(gp^{'}, gp, gp)$$

again by the same argument, we can verify that $G(gp, gp^{'}, gp^{'}) \leq k^2 G(gp, gp^{'}, gp^{'})$, which implies that $gp = gp^{'}$.

To show that f is G-continuous at u, let $\{gs_n\} \subseteq X$ be a sequence such that $\lim(gs_n) = gp$, then

$$\mathrm{G}(gp, f(s_n), f(s_n)) \leq a\{\mathrm{G}(gp, f(s_n), f(s_n)) + \mathrm{G}(g(s_n), fp, fp)\},$$

and so $\mathrm{G}(gp, f(s_n), f(s_n)) \leq (a_1/(1-a_1))\mathrm{G}(g(s_n), fp, fp)$.
Taking the limit as $n \to \infty$, from which we see that $\mathrm{G}(gp, f(s_n), f(s_n) \to 0$. By Proposition 1.10, we have $f(s_n) \to gp = fp$ which implies that f is G-continuous at p. □

Corollary 2.21. [7, Theorem 2.9] *Let (X, G) be a complete* G*-metric space, and let $f\colon X \to X$ be a mapping satisfying one of the following conditions:*

(32) $$\mathrm{G}(fr, fs, fs) \leq a_1\{\mathrm{G}(r, fs, fs) + \mathrm{G}(s, fr, fr)\}$$

or

(33) $$\mathrm{G}(fr, fs, fs) \leq a_1\{\mathrm{G}(r, r, fs) + \mathrm{G}(s, s, fr)\}$$

for all $r, s, t \in X$, where $a_1 \in [0, \frac{1}{2})$. then f has a unique fixed point (say p), and f is G*-continuous at p.*

Proof. It follows by taking $g = I_X$ in Theorem 2.20. □

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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