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STRONG CONVERGENCE ALGORITHMS FOR SPLIT COMMON FIXED POINT PROBLEMS INVOLVING DEMICONTRACTIVE MAPPINGS

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Abstract. We propose two novel iterative algorithms for solving the split common fixed point problem (SCFPP) involving demicontractive mappings. These algorithms incorporate the inertial technique, which significantly enhances the convergence rate without requiring prior knowledge of operator norms. By eliminating the dependency on operator norms, our methods offer greater flexibility and computational efficiency, making them suitable for large-scale applications. We establish the strong convergence of the proposed algorithms under mild assumptions. Our work extends and generalizes existing results by considering a broader class of mappings and providing a unified framework for solving SCFPPs.

Keywords: split common fixed point problem; demicontractive mapping; Hilbert Space.

2020 AMS Subject Classification: 47H09, 47H10.

1. INTRODUCTION

The split feasibility problem (SFP) was first introduced by Censor and Elfving [6] in 1994. In the setting of a real Hilbert space $\mathscr{H}$, the SFP involves finding an element u in a closed convex subset $\mathscr{C}$ of $\mathscr{H}$ such that its image under a bounded linear operator $G : \mathscr{H} \to \mathscr{H}$ belongs to

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another closed convex subset $\mathscr{Q}$ of $\mathscr{H}$. Formally, the problem can be stated as:

$$\text{Find } \zeta^\dagger \in \mathscr{C} \text{ such that } G(\zeta^\dagger) \in \mathscr{Q}.$$

A natural generalization of SFP is the split common fixed point (SCFP) problem, which aims to find an element in a fixed point set whose image under a bounded linear operator belongs to another fixed point set. Specifically, let $\mathscr{E}_1, \mathscr{E}_2$ be two real Banach spaces, and let $\Phi : \mathscr{E}_1 \to \mathscr{E}_1$ and $\Psi : \mathscr{E}_2 \to \mathscr{E}_2$ be two nonlinear operators and $G : \mathscr{E}_1 \to \mathscr{E}_2$ is a bounded linear operator. The SCFP is formulated as:

$$\zeta^\dagger \in \mathscr{E}_1, \quad \text{such that} \quad \zeta^\dagger \in F(\Phi) \quad \text{and} \quad G(\zeta^\dagger) \in F(\Psi),$$

where $F(\Phi)$ and $F(\Psi)$ denote the fixed point sets of Φ and Ψ, respectively. The SCFP generalizes the SFP and has garnered significant attention due to its applications in inverse problems, signal processing, and constrained optimization [4, 19, 29, 12, 14]. Despite its theoretical significance, solving SCFP remains challenging due to the computational complexity of projections onto fixed point sets, particularly when they lack explicit characterizations.

To address SFP, Censor and Elfving [6] initially proposed an iterative scheme that relied on matrix inversion. However, this approach proved computationally expensive, particularly for large-scale problems. Byrne [4] introduced the CQ algorithm, which avoids direct matrix inversion by leveraging orthogonal projections. This algorithm iterates as follows:

$$\zeta_{n+1} = P_{\mathscr{C}} \left[I - \gamma G^* (I - P_{\mathscr{Q}}) G \right] \zeta_n, \quad n \in \mathbb{N},$$

where the step size parameter γ is chosen from the interval $(0, 2/\|G\|^2)$. However, this method still requires knowledge of $\|G\|$, which is often impractical to compute explicitly in high-dimensional settings. Subsequently, researchers such as Moudafi [19] and Wang and Xu [29] extended the method to accommodate quasi-nonexpansive and directed operators, improving convergence properties under various conditions. Nonetheless, these extensions often relied on prior computation of operator norms, which remains a limiting factor.

Recognizing this limitation, Cui and Wang [12] introduced an algorithm that eliminates the dependency on the operator norm, ensuring weak convergence under appropriate conditions. In a similar vein, López et al. [14] proposed a self-adaptive step size:

$$\mu_n = \frac{\rho_n g(\zeta_n)}{\|\nabla g(\zeta_n)\|^2}, \quad n \geq 1,$$

where $g(\zeta_n) = \frac{1}{2}\|(I - P_{\mathscr{Q}})G(\zeta_n)\|^2$ and ρ_n satisfies suitable conditions. The split feasibility problem (SFP) and the split common fixed point problem (SCFPP) have been extensively studied in the context of various mathematical problems, including variational inequality problems, equilibrium problems, and monotone inclusion problems (see, e.g., [7, 11, 10, 26, 13, 28]). Beyond their theoretical significance, these problems have found practical applications in diverse fields such as medical imaging, astronomy, compressed sensing, radiation therapy treatment planning, and remote sensing (see, e.g., [9, 20, 25, 2]). Moreover, the algorithmic framework introduced by [8] has been widely generalized to accommodate broader classes of operators, including quasi-nonexpansive and demicontractive operators. These extensions have significantly enriched the applicability and versatility of the original methods. While the studies mentioned above guaranteed weak convergence results, achieving strong convergence required the development of more advanced algorithms. To this end, researchers combined Halpern algorithms with viscosity algorithms under mild conditions, as demonstrated in works such as [3], [16]. Building on these advancements, [28] and [31] proposed a novel iterative algorithm of the form:

$$\zeta_{n+1} = \zeta_n - \beta_n \left[(I - \Phi) + G^*(I - \Psi)G\right]\zeta_n, \quad \forall n \geq 1,$$

where $\{\beta_n\}$ is a self-adaptive stepsize sequence. This approach further enhanced the convergence properties of iterative methods, see also [23, 22].

Additionally, to improve the convergence rate of iterative algorithms, the concept of inertial effects has been explored in recent studies, such as those by [1] and [17], among others. These works laid the foundation for further developments in the field. The contributions of [1], [28], and [31] have been instrumental in advancing the understanding and application of iterative algorithms with strong convergence properties. While the studies mentioned above guaranteed weak convergence results, achieving strong convergence required the development of more advanced algorithms. To this end, researchers combined Halpern algorithms with viscosity algorithms under mild conditions, as demonstrated in works such as [3], [16], and [15]. The viscosity algorithm, originally introduced by [18], utilized contraction mappings to establish strong convergence.

On the other hand. Consider a finite family of nonexpansive mappings $\{\Phi_i\}_{i=1}^N$ with a nonempty common fixed-point set $F := \bigcap_{i=1}^N F(\Phi_i)$. Numerous authors (see [21, 24] and references therein) have proposed iterative methods to find an element of F that serves as an optimal solution to a specific minimization problem. For $n > N$, the mapping Φ_n is defined cyclically as $\Phi_{n \bmod N}$, where the modulo function takes values in $\{1,2,\dots,N\}$. Let $\zeta^\dagger$ be a fixed element in the Hilbert space $\mathscr{H}$. In 2003, Xu [30] established that the sequence $\{\zeta_n\}$, generated by the iteration

$$\zeta_{n+1} = (1-\varepsilon_n G)\Phi_{n+1}(\zeta_n) + \varepsilon_n \zeta^\dagger,$$

converges strongly to the solution of the quadratic minimization problem

$$\min_{\zeta \in F} \frac{1}{2}\langle G(\zeta), \zeta\rangle - \langle \zeta, \zeta^\dagger\rangle,$$

under appropriate conditions on the sequence $\{\varepsilon_n\}$ and the additional assumption that

$$F = F(\Phi_1\Phi_2\dots\Phi_N) = F(\Phi_N\Phi_1\dots\Phi_{N-1}) = \cdots = F(\Phi_2\Phi_3\dots\Phi_N\Phi_1).$$

This result highlights the convergence of the iterative scheme to the optimal solution of the minimization problem over the common fixed-point set F.

In [5] Cegielski considered the following problem: Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be real Hilbert spaces, and let $G : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator with $\|G\| > 0$. Suppose $\Phi_i : \mathscr{H}_1 \to \mathscr{H}_1$ for $i \in I := \{1,2,\dots,p\}$ and $\Psi_j : \mathscr{H}_2 \to \mathscr{H}_2$ for $j \in J := \{1,2,\dots,r\}$ are quasi-nonexpansive operators. Let $\mathscr{C} := \bigcap_{i\in I} \mathrm{Fix}(\Phi_i)$ and $\mathscr{Q} := \bigcap_{j\in J} \mathrm{Fix}(\Psi_j)$ denote the sets of fixed points of Φ_i and Ψ_j, respectively. The split common fixed point problem (SCFPP) is to:

$$\text{find } \zeta^* \in \bigcap_{i\in I} \mathrm{Fix}\,\Phi_i \text{ with } G(\zeta^*) \in \bigcap_{j\in J} \mathrm{Fix}\,\Psi_j. \tag{1.1}$$

Motivated by the aforementioned results, we consider a more general class of mappings-demicontractive mappings-in the context of problem (1.1). To address this, we first propose two novel iterative algorithms that incorporate the inertial technique. Notably, these algorithms do not rely on prior knowledge of the operator norms, making them more flexible and applicable in practical scenarios. The integration of the inertial technique is aimed at accelerating the convergence rate of the algorithms, addressing a common limitation in traditional iterative methods. We rigorously establish the strong convergence of the proposed algorithms to the solutions of the problem. Our theoretical analysis demonstrates that the algorithms are not only computationally efficient but also robust under mild assumptions. This work extends and generalizes existing results, providing a unified framework for solving split common fixed point problems involving demicontractive mappings.

2. Preliminaries

Definition 2.1. A mapping $\Phi : \mathscr{H} \to \mathscr{H}$ is said to be

(a) monotone, if

$$\langle \Phi(\zeta) - \Phi(\eta), \zeta - \eta \rangle \geq 0, \forall \zeta, \eta \in \mathscr{H};$$

(b) pseudomonotone, if

$$\langle \Phi(\zeta), \eta - \zeta \rangle \geq 0 \implies \langle \Phi(\eta), \eta - \zeta \rangle \geq 0, \forall \zeta, \eta \in \mathscr{H};$$

(c) contraction, if $\exists$ a constant $k \in (0,1)$ such that

$$\|\Phi(\zeta) - \Phi(\eta)\| \leq k\|\zeta - \eta\|, \forall \zeta, \eta \in \mathscr{H};$$

(d) L-Lipschitz continuous, if

$$\|\Phi(\zeta) - \Phi(\eta)\| \leq L\|\zeta - \eta\|, \forall \zeta, \eta \in \mathscr{H};$$

(e) μ-demicontractive mapping if $F(\Phi) \neq \emptyset$ and $\exists$ a constant $\mu \in (0,1)$, such that

$$\|\Phi(\zeta) - \zeta^{\dagger}\|^2 \leq \|\zeta - \zeta^{\dagger}\|^2 + \mu\|\zeta - \Phi(\zeta)\|^2, \forall \zeta \in \mathscr{H}, \zeta^{\dagger} \in F(\Phi).$$

Lemma 2.2. [27] *Suppose $\{\tau_n\}$ be a suquence of positive real numbers, $\{\rho_n\}$ be a sequence of real number in $(0,1)$ with $\sum_{n=1}^{\infty} = \infty$ and $\{\zeta_n\}$ be a sequence of real numbers. Suppose that*

$$\tau_{n+1} \leq (1 - \rho_n)\tau_n + \rho_n\zeta_n, \ \forall n \geq 1.$$

If $\limsup_{k\to\infty} \zeta_{n_k} \leq 0$ for each subsequence $\{\tau_{n_k}\}$ of $\{\tau_n\}$ satisfying $\liminf_{k\to\infty}(\tau_{n_k+1} - \tau_{n_k}) \geq 0$ then $\lim_{n\to\infty} \tau_n = 0$.

3. Main Results

Condition 3.1. *Suppose $\mathscr{E}_1 \neq \emptyset$ and $\mathscr{E}_2 \neq \emptyset$ are closed and convex subsets of real Hilbert spaces Σ_1 and Σ_2, respectively.*

(1) *The solution set $\Xi \neq \emptyset$.*

(2) *$\Phi_i : \Sigma_1 \to \Sigma_1$ and $\Psi_i : \Sigma_2 \to \Sigma_2$ are two finite families of demicontractive mappings with $k_1 \in [0,1)$ and $k_2 \in [0,1)$, respectively such that $I - \Phi_i$ and $I - \Psi_i$ are demiclosed at zero.*

(3) *The mapping $G : \Sigma_1 \to \Sigma_2$ is bounded and linear with its adjoint operator G^*.*

Condition 3.2. *Suppose $\rho > 0$, $0 < \gamma < \min\{1-k_1, 1-k_2\}$ and $\zeta_0, \zeta_1 \in \Sigma$. Suppose that $\{\beta_n\}$, $\{\theta_n\}$, $\{\rho_n\}$ are the sequence of positive real numbers satisfying the following:*

(1) *for some $b > 0$, $\{\theta_n\} \subset \left[b, \frac{1}{2}\right]$,*

(2) *$\{\beta_n\} \subset (0,1)$ and $\lim\limits_{n\to\infty}(1-\beta_n) = 0$, $\sum\limits_{n=1}^{\infty}(1-\beta_n) = \infty$,*

(3) *$\lim\limits_{n\to\infty} \frac{\rho_n}{1-\beta_n} = 0$.*

Algorithm 3.3.

$$\begin{aligned}
\vartheta_n &= \zeta_n + \rho_n(\zeta_n - \zeta_{n-1}),\\
\chi_n &= \vartheta_n - \Phi_i(\vartheta_n) + G^*(I-\Psi_i)G(\vartheta_n),\\
\zeta_{n+1} &= (1-\theta_n)(\beta_n\zeta_n) + \theta_n(\vartheta_n - \gamma\tau_n\chi_n),
\end{aligned}$$

here

$$\rho_n = \begin{cases} \min\left\{\rho, \frac{\rho_n}{\|\zeta_n - \zeta_{n-1}\|}\right\} \text{ if } \zeta_n \neq \zeta_{n-1},\\ \rho, \text{ otherwise,} \end{cases}$$

and

$$\tau_n = \frac{\|\vartheta_n - \Phi_i(\vartheta_n)\|^2 + \|(I-\Psi_i)G(\vartheta_n)\|^2}{\|\vartheta_n - \Phi_i(\vartheta_n) + G^*(I-\Psi_i)G(\vartheta_n)\|^2}.$$

Theorem 3.4. *Suppose the conditions 3.1 and 3.2 are satisfied. Then the sequence $\{\zeta_n\}$ generated by Algorithm 3.3 converges strongly to an element $\zeta^\dagger \in \Xi$, where $\|\zeta^\dagger\| = \min\{\|\zeta^\dagger\| : \zeta^\dagger \in \Xi\}$.*

Proof. Suppose $\zeta^\dagger \in \Xi$, first we prove that the sequence $\{\zeta_n\}$ is bounded, we get

$$\begin{aligned}
\langle \chi_n, \vartheta_n - \zeta^\dagger \rangle &= \langle \vartheta_n - \Phi_i(\vartheta_n) + G^*(I-\Psi_i)G(\vartheta_n), \vartheta_n - \zeta^\dagger \rangle \\
&= \langle \vartheta_n - \Phi_i(\vartheta_n), \vartheta_n - \zeta^\dagger \rangle + \langle G^*(I-\Psi_i)G(\vartheta_n), \vartheta_n - \zeta^\dagger \rangle \\
&= \langle \vartheta_n - \Phi_i(\vartheta_n), \vartheta_n - \zeta^\dagger \rangle + \langle (I-\Psi_i)G(\vartheta_n), G(\vartheta_n) - G(\zeta^\dagger) \rangle \\
&\geq \frac{1-k_1}{2}\|\vartheta_n - \Phi_i(\vartheta_n)\|^2 + \frac{1-k_2}{2}\|(I-\Psi_i)G(\vartheta_n)\|^2 \\
&\geq \frac{1}{2}\min\{1-k_1, 1-k_2\}(\|\vartheta_n - \Phi_i(\vartheta_n)\|^2 + \|(I-\Psi_i)G(\vartheta_n)\|^2).
\end{aligned}$$

Assume $\eta_n = \vartheta_n - \gamma\tau_n\chi_n$, then

$$\begin{aligned}
\|\eta_n - \zeta^\dagger\|^2 &= \|\vartheta_n - \gamma\tau_n\chi_n - \zeta^\dagger\|^2 \\
&= \|\vartheta_n - \zeta^\dagger\|^2 - 2\gamma\tau_n\langle \chi_n, \vartheta_n - \zeta^\dagger \rangle + \gamma^2\tau_n^2\|\chi_n\|^2 \\
&\leq \|\vartheta_n - \zeta^\dagger\|^2 - \gamma\min\{1-k_1, 1-k_2\}\frac{\left(\|\vartheta_n - \Phi_i(\vartheta_n)\|^2 + \|(I-\Psi_i)G(\vartheta_n)\|^2\right)^2}{\|\vartheta_n - \Phi_i(\vartheta_n) + G^*(I-\Psi_i)G(\vartheta_n)\|^2} \\
&+ \gamma^2\frac{\left(\|\vartheta_n - \Phi_i(\vartheta_n)\|^2 + \|(I-\Psi_i)G(\vartheta_n)\|^2\right)^2}{\|\vartheta_n - \Phi_i(\vartheta_n) + G^*(I-\Psi_i)G(\vartheta_n)\|^2} \\
&= \|\vartheta_n - \zeta^\dagger\|^2 - \gamma(\min\{1-k_1, 1-k_2\} - \gamma)\frac{\left(\|\vartheta_n - \Phi_i(\vartheta_n)\|^2 + \|(I-\Psi_i)G(\vartheta_n)\|^2\right)^2}{\|\vartheta_n - \Phi_i(\vartheta_n) + G^*(I-\Psi_i)G(\vartheta_n)\|^2}.
\end{aligned}$$

It gives that

$$\|\eta_n - \zeta^\dagger\| \leq \|\vartheta_n - \zeta^\dagger\|. \tag{3.1}$$

We also have

$$\begin{aligned}
\|\vartheta_n - \zeta^\dagger\| &= \|\zeta_n + \rho_n(\zeta_n - \zeta_{n-1}) - \zeta^\dagger\| \\
&\leq \|\zeta_n - \zeta^\dagger\| + \rho_n\|\zeta_n - \zeta_{n-1}\| \\
&= \|\zeta_n - \zeta^\dagger\| + (1-\beta_n)\frac{\rho_n}{(1-\beta_n)}\|\zeta_n - \zeta_{n-1}\|.
\end{aligned}$$

Using condition 3.2, we can get $\lim_{n\to\infty}\frac{\rho_n}{(1-\beta_n)}\|\zeta_n - \zeta_{n-1}\| = 0$ thus $\exists$ a constant $M_1 > 0$, such that

$$\frac{\rho_n}{(1-\beta_n)}\|\zeta_n - \zeta_{n-1}\| \leq M_1, \forall n \geq 1.$$

And we get

$$\|\eta_n - \zeta^\dagger\| \leq \|\vartheta_n - \zeta^\dagger\| \leq \|\zeta_n - \zeta^\dagger\| + (1-\beta_n)M_1. \tag{3.2}$$

We can also have

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\| &= \|(1-\theta_n)(\beta_n\zeta_n) + \theta_n\eta_n - \zeta^\dagger\| \\
&= \|(1-\theta_n)\beta_n(\zeta_n - \zeta^\dagger) + \theta_n(\eta_n - \zeta^\dagger) - (1-\beta_n)(1-\theta_n)\zeta^\dagger\| \\
&\leq \|(1-\theta_n)\beta_n(\zeta_n - \zeta^\dagger) + \theta_n(\eta_n - \zeta^\dagger)\| + (1-\beta_n)(1-\theta_n)\|\zeta^\dagger\|. \qquad (3.3)
\end{aligned}$$

Now

$$\begin{aligned}
&\|(1-\theta_n)\beta_n(\zeta_n - \zeta^\dagger) + \theta_n(\eta_n - \zeta^\dagger)\|^2 \\
&= (1-\theta_n)^2\beta_n^2\|\zeta_n - \zeta^\dagger\|^2 + 2(1-\theta_n)\beta_n\theta_n\langle \zeta_n - \zeta^\dagger, \eta_n - \zeta^\dagger\rangle + \theta_n^2\|\eta_n - \zeta^\dagger\|^2 \\
&\leq (1-\theta_n)^2\beta_n^2\|\zeta_n - \zeta^\dagger\|^2 + 2(1-\theta_n)\beta_n\theta_n\|\zeta_n - \zeta^\dagger\|\|\eta_n - \zeta^\dagger\| + \theta_n^2\|\eta_n - \zeta^\dagger\|^2 \\
&= [(1-\theta_n)\beta_n\|\zeta_n - \zeta^\dagger\| + \theta_n\|\eta_n - \zeta^\dagger\|]^2.
\end{aligned}$$

Thus, we have

$$\|(1-\theta_n)\beta_n(\zeta_n - \zeta^\dagger) + \theta_n(\eta_n - \zeta^\dagger)\| \leq (1-\theta_n)\beta_n\|\zeta_n - \zeta^\dagger\| + \theta_n\|\eta_n - \zeta^\dagger\| \tag{3.4}$$

Using (3.2) in (3.4), we get

$$\begin{aligned}
&\|(1-\theta_n)\beta_n(\zeta_n - \zeta^\dagger) + \theta_n(\eta_n - \zeta^\dagger)\| \\
&\leq (1-\theta_n)\beta_n\|\zeta_n - \zeta^\dagger\| + \theta_n\|\zeta_n - \zeta^\dagger\| + (1-\beta_n)M_1 \\
&= (1-(1-\beta_n)(1-\theta_n))\|\zeta_n - \zeta^\dagger\| + (1-\beta_n)\theta_n M_1 \\
&\leq (1-(1-\beta_n)(1-\theta_n))\|\zeta_n - \zeta^\dagger\| + (1-\beta_n)(1-\theta_n)M_1. \qquad (3.5)
\end{aligned}$$

Now from (3.3), we get

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\| &\leq (1-(1-\beta_n)(1-\theta_n))\|\zeta_n - \zeta^\dagger\| + (1-\beta_n)(1-\theta_n)(\|\zeta^\dagger\| + M_1) \\
&\leq \max\{\|\zeta_n - \zeta^\dagger\|, \|\zeta^\dagger\| + M_1\} \\
&\leq \cdots \\
&\leq \max\{\|\zeta_0 - \zeta^\dagger\|, \|\zeta^\dagger\| + M_1\}.
\end{aligned}$$

Thus the sequence $\{\zeta_n\}$ is bounded. Next we also have

$$\begin{aligned}
\|\zeta_{n+1}-\zeta^\dagger\|^2 &= \|(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger)-(1-\beta_n)(1-\theta_n)\zeta^\dagger\|^2\\
&= \|(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger)\|^2+(1-\beta_n)^2(1-\theta_n)^2\|\zeta^\dagger\|^2\\
&-2(1-\beta_n)(1-\theta_n)\langle(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger),\zeta^\dagger\rangle\\
&\leq \|(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger)\|^2+(1-\beta_n)(1-\theta_n)\Big[(1-\beta_n)(1-\theta_n)\|\zeta^\dagger\|\\
&+2\|(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger)\|\|\zeta^\dagger\|\Big]\\
&\leq \|(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger)\|^2+(1-\beta_n)(1-\theta_n)M_2. \qquad (3.6)
\end{aligned}$$

Since the sequences $\{\zeta_n\}$, $\{\eta_n\}$, $\{\theta_n\}$, and $\{\beta_n\}$ are bounded $\exists\, M_2>0$, such that

$$(1-\beta_n)(1-\theta_n)\|\zeta^\dagger\|+2\|(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger)\|\|\zeta^\dagger\|\leq M_2.$$

Now,

$$\begin{aligned}
\|(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger)\|^2 &= (1-\theta_n)^2\beta_n^2\|\zeta_n-\zeta^\dagger\|^2\\
&+2(1-\theta_n)\beta_n\theta_n\langle\zeta_n-\zeta^\dagger,\eta_n-\zeta^\dagger\rangle+\theta_n^2\|\eta_n-\zeta^\dagger\|^2\\
&\leq (1-\theta_n)^2\beta_n^2\|\zeta_n-\zeta^\dagger\|^2\\
&+2(1-\theta_n)\beta_n\theta_n\|\zeta_n-\zeta^\dagger\|\|\eta_n-\zeta^\dagger\|+\theta_n^2\|\eta_n-\zeta^\dagger\|^2\\
&\leq (1-\theta_n)^2\beta_n^2\|\zeta_n-\zeta^\dagger\|^2+(1-\theta_n)\beta_n\theta_n\|\zeta_n-\zeta^\dagger\|^2\\
&+(1-\theta_n)\beta_n\theta_n\|\eta_n-\zeta^\dagger\|^2+\theta_n^2\|\eta_n-\zeta^\dagger\|^2\\
&= (1-\theta_n)\beta_n(1-(1-\theta_n)(11-\beta_n))\|\zeta_n-\zeta^\dagger\|^2\\
&+\theta_n(1-(1-\theta_n)(1-\beta_n))\|\eta_n-\zeta^\dagger\|^2.
\end{aligned}$$

And we get

$$\begin{aligned}
&\|(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger)\|^2\\
&\leq (1-\theta_n)\beta_n(1-(1-\theta_n)(1-\beta_n))\|\zeta_n-\zeta^\dagger\|^2+\theta_n(1-(1-\theta_n)(1-\beta_n))\|\eta_n-\zeta^\dagger\|^2\\
&\leq (1-\theta_n)\beta_n(1-(1-\theta_n)(1-\beta_n))\|\zeta_n-\zeta^\dagger\|^2+\theta_n(1-(1-\theta_n)(1-\beta_n))\|\vartheta_n-\zeta^\dagger\|^2
\end{aligned}$$

$$- \theta_n(1-(1-\theta_n)(1-\beta_n))\gamma(\min\{1-k_1,1-k_2\}-\gamma)\frac{\left(\|\vartheta_n-\Phi_i(\vartheta_n)\|^2+\|(I-\Psi_i)G(\vartheta_n)\|^2\right)^2}{\|\vartheta_n-\Phi_i(\vartheta_n)+G^*(I-\Psi_i)G(\vartheta_n)\|^2}$$

$$\leq (1-\theta_n)\beta_n(1-(1-\theta_n)(1-\beta_n))\|\zeta_n-\zeta^\dagger\|^2+\theta_n(1-(1-\theta_n)(1-\beta_n))\|\zeta_n-\zeta^\dagger\|^2$$

$$+\theta_n(1-(1-\theta_n)(1-\beta_n))M_1$$

$$- \theta_n(1-(1-\theta_n)(1-\beta_n))\gamma(\min\{1-k_1,1-k_2\}-\gamma)\frac{\left(\|\vartheta_n-\Phi_i(\vartheta_n)\|^2+\|(I-\Psi_i)G(\vartheta_n)\|^2\right)^2}{\|\vartheta_n-\Phi_i(\vartheta_n)+G^*(I-\Psi_i)G(\vartheta_n)\|^2}$$

$$\leq (1-(1-\theta_n)(1-\beta_n))^2\|\zeta_n-\zeta^\dagger\|^2+\theta_n(1-(1-\theta_n)(1-\beta_n))M_1$$

(3.7)

$$- \theta_n(1-(1-\theta_n)(1-\beta_n))\gamma(\min\{1-k_1,1-k_2\}-\gamma)\frac{\left(\|\vartheta_n-\Phi_i(\vartheta_n)\|^2+\|(I-\Psi_i)G(\vartheta_n)\|^2\right)^2}{\|\vartheta_n-\Phi_i(\vartheta_n)+G^*(I-\Psi_i)G(\vartheta_n)\|^2}.$$

Now substituting (3.7) to (3.6), we get

$$\|\zeta_{n+1}-\zeta^\dagger\|^2$$

$$\leq (1-(1-\theta_n)(1-\beta_n))^2\|\zeta_n-\zeta^\dagger\|^2+(1-\beta_n)(\theta_n(1-(1-\theta_n)(1-\beta_n))M_1+(1-\theta_n)M_2)$$

$$- \theta_n(1-(1-\theta_n)(1-\beta_n))\gamma(\min\{1-k_1,1-k_2\}-\gamma)\frac{\left(\|\vartheta_n-\Phi_i(\vartheta_n)\|^2+\|(I-\Psi_i)G(\vartheta_n)\|^2\right)^2}{\|\vartheta_n-\Phi_i(\vartheta_n)+G^*(I-\Psi_i)G(\vartheta_n)\|^2}.$$

Since $(1-(1-\theta_n)(1-\beta_n))^2 \leq 1$ and $\theta_n \geq b$, we get

$$\|\zeta_{n+1}-\zeta^\dagger\|^2$$

$$\leq \|\zeta_n-\zeta^\dagger\|^2+(1-\beta_n)(\theta_n(1-(1-\theta_n)(1-\beta_n))M_1+(1-\theta_n)M_2)$$

$$- b(1-(1-\theta_n)(1-\beta_n))\gamma(\min\{1-k_1,1-k_2\}-\gamma)\frac{\left(\|\vartheta_n-\Phi_i(\vartheta_n)\|^2+\|(I-\Psi_i)G(\vartheta_n)\|^2\right)^2}{\|\vartheta_n-\Phi_i(\vartheta_n)+G^*(I-\Psi_i)G(\vartheta_n)\|^2}.$$

This gives us

$$b(1-(1-\theta_n)(1-\beta_n))\gamma(\min\{1-k_1,1-k_2\}-\gamma)\frac{\left(\|\vartheta_n-\Phi_i(\vartheta_n)\|^2+\|(I-\Psi_i)G(\vartheta_n)\|^2\right)^2}{\|\vartheta_n-\Phi_i(\vartheta_n)+G^*(I-\Psi_i)G(\vartheta_n)\|^2}$$

(3.8)

$$\leq \|\zeta_n-\zeta^\dagger\|^2-\|\zeta_{n+1}-\zeta^\dagger\|^2+(1-\beta_n)[\theta_n(1-(1-\theta_n)(1-\beta_n))M_1+(1-\theta_n)M_2].$$

Now we claim,

$$\|\zeta_{n+1}-\zeta^\dagger\|^2 \leq (1-(1-\theta_n)(1-\beta_n))\|\zeta_n-\zeta^\dagger\|^2$$

$$+(1-\theta_n)(1-\beta_n)\left[\frac{\rho_n}{1-\beta_n}\|\zeta_n-\zeta_{n-1}\|(1-(1-\theta_n)(1-\beta_n))\frac{\theta_n}{(1-\theta_n)}M_3-2\langle\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\right],$$

for any positive M_3. We have

$$\begin{aligned}\|\zeta_{n+1}-\zeta^\dagger\|^2 &= \|(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger)-(1-\beta_n)(1-\theta_n)\zeta^\dagger\|^2\\ &\leq \|(1-\theta_n)\beta_n(\zeta_n-\zeta^\dagger)+\theta_n(\eta_n-\zeta^\dagger)\|^2-2(1-\beta_n)(1-\theta_n)\langle\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &\leq (1-\theta_n)\beta_n(1-(1-\theta_n)(1-\beta_n))\|\zeta_n-\zeta^\dagger\|^2\\ &+\theta_n(1-(1-\theta_n)(1-\beta_n))\|\vartheta_n-\zeta^\dagger\|^2-2(1-\beta_n)(1-\theta_n)\langle\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle. \end{aligned}\tag{3.9}$$

We also have

$$\begin{aligned}\|\vartheta_n-\zeta^\dagger\|^2 &= \|\zeta_n+\rho_n(\zeta_n-\zeta_{n-1})-\zeta^\dagger\|^2\\ &= \|\zeta_n-\zeta^\dagger\|^2+\rho_n^2\|\zeta_n-\zeta_{n-1}\|^2+2\rho_n\langle\zeta_n-\zeta^\dagger,\zeta_n-\zeta_{n-1}\rangle\\ &\leq \|\zeta_n-\zeta^\dagger\|^2+\rho_n^2\|\zeta_n-\zeta_{n-1}\|^2+2\rho_n\|\zeta_n-\zeta^\dagger\|\|\zeta_n-\zeta_{n-1}\|\\ &\leq \|\zeta_n-\zeta^\dagger\|^2+\rho_n\|\zeta_n-\zeta_{n-1}\|(\rho_n\|\zeta_n-\zeta_{n-1}\|+2\|\zeta_n-\zeta^\dagger\|)\\ &\leq \|\zeta_n-\zeta^\dagger\|^2+\rho_n\|\zeta_n-\zeta_{n-1}\|M_4. \end{aligned}\tag{3.10}$$

For some $M_4>0$. Now substituting (3.10) in (3.9), we get

$$\begin{aligned}&\|\zeta_{n+1}-\zeta^\dagger\|^2\\ &\leq (1-\theta_n)\beta_n(1-(1-\theta_n)(1-\beta_n))\|\zeta_n-\zeta^\dagger\|^2+\theta_n(1-(1-\theta_n)(1-\beta_n))\{\|\zeta_n-\zeta^\dagger\|^2\\ &+\rho_n\|\zeta_n-\zeta_{n-1}\|M_4\}-2(1-\beta_n)(1-\theta_n)\langle\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &=(1-(1-\theta_n)(1-\beta_n))^2\|\zeta_n-\zeta^\dagger\|^2\\ &+(1-\theta_n)(1-\beta_n)\left(\frac{\alpha_n}{1-\beta_n}\|\zeta_n-\zeta_{n-1}\|(1-(1-\theta_n)(1-\beta_n))\frac{\theta_n}{1-\theta_n}M_4-2\langle\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\right)\\ &\leq (1-(1-\theta_n)(1-\beta_n))\|\zeta_n-\zeta^\dagger\|^2\\ &+(1-\theta_n)(1-\beta_n)\left(\frac{\alpha_n}{1-\beta_n}\|\zeta_n-\zeta_{n-1}\|(1-(1-\theta_n)(1-\beta_n))\frac{\theta_n}{1-\theta_n}M_4-2\langle\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\right). \end{aligned}$$

Now we prove that the sequence $\{\|\zeta_n-\zeta^\dagger\|^2\}$ converges to zero. To prove this using Lemma 2.2 it is sufficient to show that $\limsup_{k\to\infty}\langle\zeta^\dagger,\zeta_{n_k+1}-\zeta^\dagger\rangle\leq 0$ for each subsequence $\{\|\zeta_{n_k}-\zeta^\dagger\|\}$

of $\{\|\zeta_n - \zeta^\dagger\|\}$ satisfying

$$\liminf_{k\to\infty}(\|\zeta_{n_k+1} - \zeta^\dagger\| - \|\zeta_{n_k} - \zeta^\dagger\|) \geq 0.$$

For this let $\{\|\zeta_{n_k} - \zeta^\dagger\|\}$ is a subsequence $\{\|\zeta_n - \zeta^\dagger\|\}$ such that $\liminf_{k\to\infty}(\|\zeta_{n_k+1} - \zeta^\dagger\| - \|\zeta_{n_k} - \zeta^\dagger\|) \geq 0$. Then

$$\begin{aligned}
&\liminf_{k\to\infty}(\|\zeta_{n_k+1} - \zeta^\dagger\|^2 - \|\zeta_{n_k} - \zeta^\dagger\|^2)\\
&= \liminf_{k\to\infty}\Big[(\|\zeta_{n_k+1} - \zeta^\dagger\| - \|\zeta_{n_k} - \zeta^\dagger\|)(\|\zeta_{n_k+1} - \zeta^\dagger\| + \|\zeta_{n_k} - \zeta^\dagger\|)\Big] \geq 0.
\end{aligned}$$

From (3.8), we can have

$$\begin{aligned}
&\limsup_{k\to\infty} b(1-(1-\theta_{n_k})(1-\beta_{n_k}))\gamma(\min\{1-k_1, 1-k_2\}-\gamma)\times\\
&\times \frac{\left(\|\vartheta_{n_k} - \Phi_i(\vartheta_{n_k})\|^2 + \|(I-\Psi_i)G(\vartheta_{n_k})\|^2\right)^2}{\|\vartheta_{n_k} - \Phi_i(\vartheta_{n_k}) + G^*(I-\Psi_i)G(\vartheta_{n_k})\|^2}\\
&\leq \limsup_{k\to\infty}\Big\{\|\zeta_{n_k} - \zeta^\dagger\|^2 - \|\zeta_{n_k+1} - \zeta^\dagger\|^2\\
&+(1-\beta_{n_k})[\theta_{n_k}(1-(1-\theta_{n_k})(1-\beta_{n_k}))M_1 + (1-\theta_{n_k})M_2]\}\\
&\leq \limsup_{k\to\infty}\Big\{\|\zeta_{n_k} - \zeta^\dagger\|^2 - \|\zeta_{n_k+1} - \zeta^\dagger\|^2\Big\}\\
&+\limsup_{k\to\infty}\{(1-\beta_{n_k})[\theta_{n_k}(1-(1-\theta_{n_k})(1-\beta_{n_k}))M_1 + (1-\theta_{n_k})M_2]\}\\
&= \limsup_{k\to\infty}\Big\{\|\zeta_{n_k} - \zeta^\dagger\|^2 - \|\zeta_{n_k+1} - \zeta^\dagger\|^2\Big\}\\
&= -\liminf_{k\to\infty}\Big\{\|\zeta_{n_k+1} - \zeta^\dagger\|^2 - \|\zeta_{n_k} - \zeta^\dagger\|^2\Big\}\\
&\leq 0.
\end{aligned}$$

It gives us

$$\lim_{k\to\infty}\frac{\left(\|\vartheta_{n_k} - \Phi_i(\vartheta_{n_k})\|^2 + \|(I-\Psi_i)G(\vartheta_{n_k})\|^2\right)^2}{\|\vartheta_{n_k} - \Phi_i(\vartheta_{n_k}) + G^*(I-\Psi_i)G(\vartheta_{n_k})\|^2}. \tag{3.11}$$

Moreover

$$\frac{\left(\|\vartheta_{n_k} - \Phi_i(\vartheta_{n_k})\|^2 + \|(I-\Psi_i)G(\vartheta_{n_k})\|^2\right)^2}{\|\vartheta_{n_k} - \Phi_i(\vartheta_{n_k}) + G^*(I-\Psi_i)G(\vartheta_{n_k})\|^2} \geq \frac{\left(\|\vartheta_{n_k} - \Phi_i(\vartheta_{n_k})\|^2 + \|(I-\Psi_i)G(\vartheta_{n_k})\|^2\right)^2}{2\|\vartheta_{n_k} - \Phi_i(\vartheta_{n_k})\| + 2\|G\|^2\|(I-\Psi_i)G(\vartheta_{n_k})\|^2}$$

(3.12)

$$\geq \frac{\left(\|\vartheta_{n_k}-\Phi_i(\vartheta_{n_k})\|^2+\|(I-\Psi_i)G(\vartheta_{n_k})\|^2\right)^2}{2\max\{1,\|G\|^2\}}.$$

From (3.11) and (3.12), we have

$$\lim_{k\to\infty}\|\vartheta_{n_k}-\Phi_i(\vartheta_{n_k})\|=0, \text{ and } \lim_{k\to\infty}\|(I-\Psi_i)G(\vartheta_{n_k})\|=0. \tag{3.13}$$

Next, we prove that

$$\lim_{k\to\infty}\|\zeta_{n_k+1}-\zeta_{n_k}\|=0. \tag{3.14}$$

We have

$$\begin{aligned}\|\eta_{n_k}-\vartheta_{n_k}\| &= \gamma\tau_{n_k}\|\chi_{n_k}\|\\ &\leq \gamma(\|\vartheta_{n_k}-\Phi_i(\vartheta_{n_k})\|^2+\|(I-\Psi_i)G(\vartheta_{n_k})\|^2)\to 0,\end{aligned} \tag{3.15}$$

and

$$\|\zeta_{n_k}-\vartheta_{n_k}\|=\rho_{n_k}\|\zeta_{n_k}-\zeta_{n_k-1}\|=(1-\beta_{n_k})\frac{\rho_{n_k}}{1-\beta_{n_k}}\|\zeta_{n_k}-\zeta_{n_k-1}\|\to 0. \tag{3.16}$$

$$\lim_{k\to\infty}\|\eta_{n_k}-\zeta_{n_k}\|\leq \lim_{k\to\infty}\|\eta_{n_k}-\vartheta_{n_k}\|+\lim_{k\to\infty}\|\vartheta_{n_k}-\zeta_{n_k}\|=0.$$

Therefore, we get

$$\begin{aligned}\|\zeta_{n_k+1}-\zeta_{n_k}\| &= \|(1-\theta_{n_k})\beta_{n_k}\zeta_{n_k}+\theta_{n_k}\eta_{n_k}-\zeta_{n_k}\|\\ &= \|\theta_{n_k}(\eta_{n_k}-\zeta_{n_k})-(1-\beta_{n_k})(1-\theta_{n_k})\zeta_{n_k}\|\\ &\leq \theta_{n_k}\|\eta_{n_k}-\zeta_{n_k}\|+(1-\beta_{n_k})(1-\theta_{n_k})\|\zeta_{n_k}\|\to 0.\end{aligned}$$

Since the sequence $\{\zeta_{n_k}\}$ is bounded, it follows that $\exists$ a subsequence $\left\{\zeta_{n_{k_j}}\right\}$ of $\{\zeta_{n_k}\}$, which converges weakly to some $\zeta^*\in\Sigma$ such that

$$\limsup_{k\to\infty}\langle\zeta^\dagger,\zeta_{n_k}-\zeta^\dagger\rangle=\lim_{j\to\infty}\langle\zeta^\dagger,\zeta_{n_{k_j}}-\zeta^\dagger\rangle=\langle\zeta^\dagger,\zeta^*-\zeta^\dagger\rangle. \tag{3.17}$$

From (3.16), we get

$$\vartheta_{n_k}\rightharpoonup\zeta^*.$$

This together with (3.13), we get $\zeta^* \in \Xi$. Using (3.17) and the definition of $\zeta^\dagger = P_\Xi(0)$, we get

$$\limsup_{k\to\infty}\langle \zeta^\dagger, \zeta_{n_k} - \zeta^\dagger\rangle = \langle \zeta^\dagger, \zeta^* - \zeta^\dagger\rangle \le 0. \tag{3.18}$$

Combining (3.14) and (3.18), we get

$$\limsup_{k\to\infty}\langle \zeta^\dagger, \zeta_{n_k+1} - \zeta^\dagger\rangle \le \limsup_{k\to\infty}\langle \zeta^\dagger, \zeta_{n_k} - \zeta^\dagger\rangle = \langle \zeta^\dagger, \zeta^* - \zeta^\dagger\rangle \le 0. \tag{3.19}$$

Hence $\lim_{n\to\infty} \frac{\rho_n}{(1-\beta_n)}\|\zeta_n - \zeta_{n-1}\| = 0$. Applying Lemma 2.2, we get $\lim_{n\to\infty}\|\zeta_n - \zeta^\dagger\| = 0$. □

Condition 3.5. *Suppose $\mathscr{E}_1 \neq \emptyset$ and $\mathscr{E}_2 \neq \emptyset$ are closed and convex subsets of real Hilbert spaces Σ_1 and Σ_2, respectively.*

(1) *The solution set $\Xi \neq \emptyset$.*
(2) *$\Phi_i : \Sigma_1 \to \Sigma_1$ and $\Psi_i : \Sigma_2 \to \Sigma_2$ are two finite families of demicontractive mappings with $k_1 \in [0,1)$ and $k_2 \in [0,1)$, respectively such that $I - \Phi_i$ and $I - \Psi_i$ are demiclosed at zero.*
(3) *The mapping $G : \Sigma_1 \to \Sigma_2$ is bounded and linear with its adjoint operator G^*.*

Condition 3.6. *Suppose $\rho > 0$, $0 < \gamma < \min\{1-k_1, 1-k_2\}$ and $\zeta_0, \zeta_1 \in \Sigma$. Suppose that $\{\beta_n\}$, $\{\theta_n\}$, $\{\rho_n\}$ are the sequence of positive real numbers satisfying the following:*

(1) $\{\beta_n\} \subset (0,1)$ *and* $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$,
(2) $\{\theta_n\} \subset (a,b) \subset (0, 1-\beta_n)$,
(3) $\lim_{n\to\infty}\frac{\rho_n}{\beta_n} = 0$.

Algorithm 3.7.

$$\begin{aligned}
\vartheta_n &= \zeta_n + \rho_n(\zeta_n - \zeta_{n-1}),\\
\chi_n &= \vartheta_n - \Phi_i(\vartheta_n) + G^*(I - \Psi_i)G(\vartheta_n),\\
\zeta_{n+1} &= (1 - \theta_n - \beta_n)\zeta_n + \theta_n(\vartheta_n - \gamma\tau_n\chi_n),
\end{aligned}$$

here

$$\rho_n = \begin{cases} \min\left\{\rho, \frac{\rho_n}{\|\zeta_n - \zeta_{n-1}\|}\right\} \text{ if } \zeta_n \neq \zeta_{n-1},\\ \rho, \text{ otherwise,}\end{cases}$$

and

$$\tau_n = \frac{\|\vartheta_n - \Phi_i(\vartheta_n)\|^2 + \|(I - \Psi_i)G(\vartheta_n)\|^2}{\|\vartheta_n - \Phi_i(\vartheta_n) + G^*(I - \Psi_i)G(\vartheta_n)\|^2}.$$

Theorem 3.8. *Suppose the conditions 3.5 and 3.6 are satisfied. Then the sequence $\{\zeta_n\}$ generated by Algorithm 3.7 converges strongly to an element $\zeta^\dagger \in \Xi$, where $\|\zeta^\dagger\| = \min\{\|\zeta^\dagger\| : \zeta^\dagger \in \Xi\}$.*

Proof. If we follow the same proof as of Theorem 3.4, we can easily reach to the conclusion. □

4. Examples

Example 4.1. Let $\Sigma_1 = \Sigma_2 = \mathbb{R}$ and $\mathscr{E} = [0,1]$, define $G(\zeta) = \zeta$. Define mappings $\Phi : \mathscr{E} \to \mathscr{E}$ by

$$\Phi(\zeta) = \begin{cases} \frac{3}{4}, \text{ if } 0 \le \zeta < 1, \\ \frac{1}{2}, \text{ if } \zeta = 1, \end{cases}$$

and $\Psi : \mathscr{E} \to \mathscr{E}$ by

$$\Psi(\zeta) = \begin{cases} \frac{3}{4}, \text{ if } 0 \le \zeta \le \frac{3}{4}, \\ \frac{1}{9}, \text{ if } \frac{3}{4} < \zeta \le 1. \end{cases}$$

Here Φ and Ψ are strict demicontractive mappings and set of fixed point of the mapping Φ is $F(\Phi) = \{\frac{3}{4}\}$ and $G(\frac{3}{4}) = \frac{3}{4}$, which is a fixed point of the mapping Ψ.

Example 4.2. Suppose $\Sigma_1 = \Sigma_2 = \mathbb{R}$ be equipped with the usual inner product and norm. Define the mappings $\Phi_i : \Sigma_1 \to \Sigma_1$ as

$$\Phi_i(\zeta) = \begin{cases} \frac{\zeta}{2i} \sin \frac{1}{\zeta}, \text{ if } \zeta \ne 0, \\ 0, \text{ if } \zeta = 0, \end{cases}$$

and $\Psi_i : \Sigma_2 \to \Sigma_2$ as

$$\Psi_i(\zeta) = \frac{1}{1+i}\zeta, \ \ \forall \zeta \in \Sigma_2.$$

Define the mapping $G : \Sigma_1 \to \Sigma_2$ as $G(\zeta) = 2\zeta$. Here we have $\bigcap_{i=1}^{n} F(\Phi_i) = \{0\}$, $G(0) = 0$ which is the fixed point of the family of mappings Ψ_i because $\bigcap_{i=1}^{n} F(\Psi_i) = \{0\}$. Here all the conditions of the Theorem 3.4 are satisfied so the sequence generated by Algorithm 3.3 converges strongly to 0.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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