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AUXILIARY PRINCIPLE TECHNIQUE FOR GENERAL HARMONIC VARIATIONAL INEQUALITIES

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Abstract. In this Paper, we have introduced and analyzed certain new types of harmonic variational inequalities, such as Auxiliary general Harmonic variational inequalities. The optimality conditions of general harmonic convex functions are shown to be specified by auxiliary general harmonic variational inequalities. Using the harmonic variational inequality, it is demonstrated that the harmonic convex set's minimum of a differentiable harmonic convex function may be described. In addition to the auxiliary principle technique, other hybrid proximal point methods are proposed. The convergence of the proposed method is shown to depend only on the operator being pseudo-monotonic, which is a weaker criterion than monotonicity. The findings presented in this work may encourage more study in this area.

Keywords: harmonic general variational inequalities; harmonic convex functions; auxiliary principle technique.

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1. Introduction

A number of generalizations and extensions for classical convexity have been proposed recently. Harmonic functions are a notable extension of convex functions. Harmonic convex functions have been the subject of various investigations by Anderson et al. [9].

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Convexity theory has played a major role in the development of every branch of pure and applied research. Its many important uses in resolving complex and difficult issues have prompted its extension and generalization through the use of novel and imaginative ideas. For example, Stampacchia [7] showed that the optimality conditions of the differentiable convex functions on the convex set can be described by an inequality called the variational inequality. Variational inequalities can be viewed as novel applications of variational concepts, such as the Riesz-Frechet theorem and the Lax-Milgram lemma. For applications, formulation, numerical methods, and other aspects of variational inequalities, see [5, 3, 4, 39, 1, 13], [14, 17, 18, 19, 37, 27, 21, 22], [32, 36, 30, 31, 33, 34, 35], [7, 16, 28] and the references therein.

It's incredible how useful the harmonic means are in electrical systems. The study of Asian options with harmonic average by Al-Azemi et al. [6] can be seen as a new approach to the study of risk analysis and financial mathematics. Noor [38] proposed various iterative techniques for solving nonlinear equations using the harmonic mean. Harmonic convex functions have been the subject of multiple investigations by Anderson et al. [6, 9]. Numerous Hermite-Hadamard type integral inequalities for the harmonic convex functions and their variation forms have been derived by Iscan [12] and Noor et al. [27, 24], [20, 23, 25]. Noor et al. [27, 21], [23, 25, 26] looked at how the differentiable harmonic convex functions were characterized.

Novel generalizations and expansions of the conventional variational inequalities may be seen in harmonic and variational inequality. Studying these courses in a cohesive framework makes sense. Inspired and motivated by these initiatives, we present and examine the general variational inequalities which are harmonic (HGVI). The harmonic variational inequalities cannot be solved iteratively using projection or resolvent methods. In this research, we propose a hybrid proximal point method for solving the HGVI using the auxiliary principle technique [39, 13, 18, 16, 28]. Weaker condition of operator pseudomonotonicity is used to examine convergence of the offered approaches. There are also some specific examples of how the suggested techniques can be used.

Convex sets and the convex function have been found to be ineffective in solving certain issues because of nonlinear structure and other limitations. This shortcoming has been addressed by considering a number of new convex sets and convex functions with regard to arbitrary functions. Noor [36] presented and examined the general convex functions and general (g-convex) sets, also referred to as Noor-convexity. The m-convex sets and m-convex functions are special cases of general convex sets and convex functions, as defined by Toader [25]. Cristescu et al. [13] have investigated the various uses of Noor-convex sets in optimization issues, including data analysis, computer-aided design, ecology-economic efficiency, railway transport systems, picture processing, and machine learning.

In this paper, we propose and examine various new hybrid inertial iterative techniques for solving equilibrium general harmonic variational inequalities using the auxiliary principle technique with an arbitrary operator. Additionally, we demonstrate that pseudomonotonicity a weaker condition than monotonicity is necessary for the convergence of these novel approaches. For harmonic variational inequalities, variational inequalities and associated optimization issues, we have identified a number of known and novel examples.

2. Preliminaries

Let U_{hg} be a nonempty closed and harmonic convex set in the real Hilbert space H. The inner product and norm are denote by $\langle .,. \rangle$ and $\|.\|$, respectively.

For a given non-linear operator T, consider the problem of finding $u \in U_{hg}$, such that:

$$\left\langle \mathscr{T}u, \frac{g(u)g(v)}{g(u)-g(v)} \right\rangle + \langle w-u, v-w \rangle \geq 0 \qquad \forall g(v) \in U_{hg} \tag{1}$$

which is called auxiliary general harmonic Variational inequality.

We provide the necessary information for completeness and to help communicate our main points.

Definition 1. A set $U \subseteq H$ is said to be convex set, if

$$u + \mu(v-u) \in U, \qquad \forall u, v \in U.$$

Definition 2. A general convex set with respect to H is defined as set $U_g \subseteq H$. in relation to any function g, if

$$g(u) + \mu(g(v) - g(u)) \in U_g, \qquad \forall u, v \in U_g.$$

Definition 3. [9, 12]. A set U_h is said to be a harmonic convex set, if

$$\frac{uv}{v+\mu(u-v)} \in U_h, \qquad \forall u,v \in U_h, \quad \mu \in [0,1].$$

Definition 4. (cf. [9]). A set U_{hg} is said to be a harmonic general convex set, if

$$\frac{g(u)g(v)}{g(v)+\mu(g(u)-g(v))} \in U_{hg}, \qquad \forall u,v \in U_h, \quad \mu \in [0,1].$$

Definition 5. A set U_{hg} is said to be a harmonic general convex function, if

$$C\left(\frac{g(u)g(v)}{g(v)+\mu(g(u)-g(v))}\right) \leq (1-\mu)C(g(u))+\mu C(g(v)), \qquad \forall u,v \in U_h, \quad \mu \in [0,1].$$

The function C is said to be harmonic concave, if and only if $-C$ is harmonic convex.

Definition 6. The differentiable function C on U_{hg} is said to be an harmonic invex function, if

$$C(v)-C(u) \geq \left\langle C'(u), \frac{g(u)g(v)}{g(u)-g(v)} \right\rangle + \langle w-u, v-w \rangle, \qquad \forall u,v \in U_h, \quad \mu \in [0,1].$$

where C'(u) is the differential of C at u.

Definition 7. A function C is said to be a quasi-harmonic general function if

$$C\left(\frac{g(u)g(v)}{g(v)+\mu(g(u)-g(v))}\right) \leq max\{C(u),C(v)\} \qquad \forall u,v \in U_h, \quad \mu \in [0,1].$$

Here we have given some theorems related to differential harmonic general convex function.

Theorem 1. Let C be a differentiable harmonic general convex function in the harmonic general convex set U_{hg}. Then $u \in U_{hg}$ is a minimum of C, if and only if u $\in U_{hg}$ is the solution of the inequality.

$$\left\langle C'g(u), \frac{g(u)g(v)}{g(u)-g(v)} \right\rangle \geq 0 \qquad \forall g(v) \in U_{hg} \tag{2}$$

Which is called the new harmonic general variational inequality.

Proof. Let $g(v) \in U_{hg}$ be a minimum of a harmonic general convex function C. Then

$$C(g(u)) \leq C(g(v))\forall \qquad g(v) \in U_{hg} \tag{3}$$

Since U_{hg} is a harmonic general convex set, so $\forall\ u,v \in U_{hg}$ and $\mu \in [0,1]$.

$$V_\mu = \frac{g(u)g(v)}{g(v)+\mu(g(u)-g(v))} \in U_{hg}.$$

Replacing $g(v)$ by $g(v_\mu)$ in (3), we have

$$\frac{C(\frac{g(u)g(v)}{g(v)+\mu(g(u)-g(v))}) - C(g(u))}{\mu} \geq 0$$

Since C is a differentiable function, taking the limit in the above inequality, as $\mu \to 0$, we have

$$\left\langle C'g(u), \frac{g(u)g(v)}{g(u)-g(v)} \right\rangle \geq 0 \qquad \forall g(v) \in U_{hg} \tag{4}$$

Which is required result (1).

Conversely, let $u \in U_{hg}$ satisfy (3). Then, we have to show that $u \in U_{hg}$ is the minimum of the function C on the harmonic general convex function.

$$\begin{aligned} C(g(v)) + \mu(g(u)-g(v)) \leq& (1-\mu)C(g(u)) + \mu C(g(v)) \\ =& C(g(u)) + \mu(C(g(v)) - C(g(u))) \end{aligned}$$

which implies that

$$\begin{aligned} C(g(v)) - C(g(u)) \geq& \lim_{\mu\to\infty} \frac{C(\frac{g(u)g(v)}{g(v)+\mu(g(u)-g(v))}) - C(u)}{\mu} \\ =& \left\langle C(g(u))', \frac{g(u)g(v)}{g(u)-g(v)} \right\rangle \geq 0 \qquad (using (1)) \end{aligned} \tag{5}$$

Consequently, it follows that

$$C(g(u)) \leq C(g(v)) \qquad \forall g(v) \in U_{hg}$$

This shows that $u \in U_{hg}$ is the minimum of the differentiability of the general haemonic convex function. □

We now examine a few additional characteristics of the differentiable harmonic convex functions. We have the following in this regard:

Theorem 2. Let C be a differentiable harmonic general convex function in the harmonic general convex set U_{hg}. Then

$$\textbf{(a)} \quad C(g(v)) - C(g(u)) \geq \left\langle C'(g(u)), \frac{g(u)g(v)}{g(u)-g(v)} \right\rangle, \quad \forall u, v \in U_{hg}.$$

$$\textbf{(b)} \left\langle C'(g(u)) - C'(g(v)), \frac{g(u)g(v)}{g(u)-g(v)} \right\rangle \leq 0 \quad \forall u, v \in U_{hg}.$$

where C'(u) is the differential of C at u in the direction $\frac{g(u)g(v)}{g(u)-g(v)}$

Proof **(a)** Let C be a harmonic convex function. Then

$$C(\frac{g(u)g(v)}{g(v)+\psi(g(u)-g(v))}) \leq (1-\mu)C(g(u)) + \mu C(g(v)), \tag{6}$$

From which, we have

$$C(g(v)) - C(g(u)) \geq \frac{C(\frac{g(u)g(v)}{g(v)+\mu(g(u)-g(v))}) - C(g(u))}{\mu}$$

Since C is a differentiable function, so taking the limit in the above inequality as $\mu \to 0$, we have

$$C(g(v)) - C(g(u)) \geq \left\langle C'(g(u)), \frac{g(u)g(v)}{g(u)-g(v)} \right\rangle, \quad \forall u, v \in U_{hg}. \tag{7}$$

which is required result first.

(b) Interchanging v and u, in (7), we obtain

$$C(g(u)) - C(g(v)) \geq \left\langle C'(g(v)), \frac{g(v)g(u)}{g(v)-g(u)} \right\rangle \tag{8}$$

taking $u_{n+1} = u_n, v_{n+1} = v_n$ in (7) and (8) and Adding, we obtain

$$\left\langle C'(g(u)) - C'(g(v)), \frac{g(u)g(v)}{g(u)-g(v)} \right\rangle \leq 0.$$

which is required result second. □

3. Main Results

It is crucial to note that finding the space projection and resolvent into the harmonic convex set is quite challenging. In certain situations, it is not possible to discover approximate solutions of the harmonic variational inequalities using the projection and resolvent approaches.

Glowinski et al. [22] created the auxiliary principle technique to investigate the existence of a solution to mixed variational inequalities in order to get around these problems. Lions and Stampacchia [13] investigated the existence of a variational inequality solution using this method. This method takes into account the auxiliary variational inequality problem that is connected to the original problem that is supplied and demonstrates that the auxiliary problem's solution is the variational inequality problem's answer.

This method gives us a broad framework within which we can propose and evaluate some creative and original iterative algorithms for solving optimization, equilibrium, and variational inequalities. Using this strategy, Noor [18] and Noor et al. [22, 15, 16, 28] proposed a broad class of inertial implicit (explicit) methods for solving different classes of variational inequalities. Polyak [2] presented the inertial-type methods to investigate the convergence requirements of iterative methods.

In this section, we using the auxiliary principle technique and propose a hybrid implicit method for solving the AGHVI and the convergence analysis.

For a given $u \in U_{hg}$ satisfying (1), consider the problem of finding $k \in U_{hg}$ such that

$$\left\langle \rho \mathscr{T}(k+\phi(u-k)), \frac{g(v)g(k)}{g(k)-g(v)} \rangle + \langle k-u, v-k \right\rangle \geq 0 \qquad \forall g(v) \in U_{hg} \tag{9}$$

where $\phi \in [0,1]$ is a constant. The inequality (9) is called the auxiliary GHVI. Obviously, if k = u, then w is a solution of the GHVI (1). This observation is used to suggest and analyze an implicit method for solving GHVI (1).

Algorithm 1. For a given $u_0 \in U_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\left\langle \rho \mathscr{T}(u_{n+1}+\phi(u_n-u_n+1)), \frac{g(v)g(u_{n+1})}{g(u_{n+1})-g(v)} \right\rangle + \langle u_{n+1}-u_n, v-u_{n+1} \rangle \geq 0 \qquad \forall v \in U_{hg}. \tag{10}$$

which is called the hybrid proximal-point (implicit) method.

We now discuss some special cases of Algorithm 1.

(I). For $\phi = 1$, The result of Algorithm 1 breakdown in the algorithm:

Algorithm 2. For a given $u_0 \in U_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\left\langle \rho \mathscr{T} u_n, \frac{g(v)g(u_{n+1})}{g(u_{n+1}) - g(v)} \right\rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0 \qquad \forall v \in U_{hg}.$$

which is called the proximal-point explicit method.

(II). For $\phi = \frac{1}{2}$, The result of Algorithm 1 breakdown in the algorithm:

Algorithm 3. For a given $u_0 \in U_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\left\langle \rho \mathscr{T}(\frac{u_{n+1} + u_n}{2}), \frac{g(v)g(u_{n+1})}{g(u_{n+1}) - g(v)} \right\rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0 \qquad \forall v \in U_{hg}.$$

which is called the mid-point proximal method.

(III). For $\phi = \frac{1}{4}$, The result of Algorithm 1 breakdown in the algorithm:

Algorithm 4. For a given $u_0 \in U_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\left\langle \rho \mathscr{T}(\frac{3u_{n+1} + u_n}{4}), \frac{g(v)g(u_{n+1})}{g(u_{n+1}) - g(v)} \right\rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0 \qquad \forall v \in U_{hg}.$$

which is called the quartile-point (lower) proximal method.

(IV). For $\phi = \frac{3}{4}$, The result of Algorithm 1 breakdown in the algorithm:

Algorithm 5. For a given $u_0 \in U_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\left\langle \rho \mathscr{T}(\frac{u_{n+1} + u_n}{4}), \frac{g(v)g(u_{n+1})}{g(u_{n+1}) - g(v)} \right\rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0 \qquad \forall v \in U_{hg}.$$

which is called the quartile-point (upper) proximal method.

(V). If $\phi = 0$, then Algorithm 1 reduces to :

Algorithm 6. For a given $u_0 \in U_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\left\langle \rho \mathscr{T} u_{n+1}, \frac{g(v)g(u_{n+1})}{g(u_{n+1}) - g(v)} \right\rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0 \qquad \forall g(v) \in U_{hg}. \tag{11}$$

which is called the proximal-point (implicit) method.

To implement Algorithm 6, one usually uses the predictor-corrector technique. Consequently, Algorithm 6 can be rewritten in the following equivalent form.

Algorithm 7. For a given $u_0 \in U_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\left\langle \rho \mathscr{T} u_n, \frac{g(v)g(J_n)}{g(J_n)-g(v)} \right\rangle + \langle J_n - u_n, v - J_n \rangle \geq 0 \qquad \forall g(v) \in U_{hg}.$$

$$\left\langle \rho \mathscr{T} J_n, \frac{g(v)g(u_{n+1})}{g(u_{n+1})-g(v)} \right\rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0 \qquad \forall g(v) \in U_{hg}.$$

Algorithm 7 is a two-step method for solving the harmonic variational inequalities. We would also like to point out that Algorithm 6 and Algorithm 7 are equivalent. This alternative equivalent formulation is applied to study the convergence analysis of Algorithm 6. We now study the convergence of the proposed Algorithm 6.

Theorem 3. Let $u \in U_{hg}$ be a solution of (1) and let u_{n+1} be the approximated solution obtained from Algorithm 1, If the operator $\mathscr{T}$ is harmonic general pseudomonotone, then

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{12}$$

Proof. Let $u \in U_{hg}$ be a solution of (1). Then

$$\left\langle \mathscr{T} u, \frac{g(u)g(v)}{g(u)-g(v)} \right\rangle + \langle u_{n+1} - u_n, u - u_{n+1} \rangle \geq 0, \qquad \forall g(v) \in U_{hg},$$

which implies that

$$\left\langle \mathscr{T} v, \frac{g(u)g(v)}{g(v)-g(u)} \right\rangle + \langle u_{n+1} - u_n, u - u_{n+1} \rangle \geq 0, \qquad \forall g(v) \in U_{hg}, \tag{13}$$

Since $\mathscr{T}$ is harmonic general pseudomonotone.

Taking $v = u_{n+1}$ in (13) and taking $v = u$ in (11) we have

$$\left\langle \mathscr{T} u_{n+1}, \frac{g(u)g(u_{n+1})}{g(u_{n+1})-g(u)} \right\rangle + \langle u_{n+1} - u_n, u - u_{n+1} \rangle \geq 0, \tag{14}$$

and

$$\left\langle \rho \mathscr{T} u_{n+1}, \frac{g(u)g(u_{n+1})}{g(u_{n+1})-g(u)} \right\rangle + \langle u_{n+1} - u_n, u - u_{n+1} \rangle \geq 0, \tag{15}$$

From (14) and (15), we have

$$2\langle u_{n+1}-u_n, u-u_{n+1}\rangle \geq \left\langle \rho\mathscr{T}u_{n+1}, \frac{g(u)g(u_{n+1})}{g(u_{n+1})-g(u)}\right\rangle \geq 0. \tag{16}$$

From (16) and using the inequality

$$2\langle u,v\rangle = \|u+v\|^2-\|u\|^2-\|v\|^2, \qquad \forall u,v\in H,$$

we obtain

$$\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2-\|u_n-u_{n+1}\|^2.$$

the required result.

Theorem 4. Let H be a finite dimensional Hilbert space and let $\mathscr{T}$ be general harmonic pseudomonotone operator. If u_{n+1} is the approximate solution obtained from Algorithm 1 and $u\in U_{hg}$ is a solution of the problem (1), then

$$\lim_{n\to\infty} u_n = \hat{u}.$$

Proof. Let $u\in U_{hg}$ be a solution of (1). From (14), we see that the sequence $\{\|u-u_n\|\}$ is nondecreasing and consequently, the sequence $\{u_n\}$ is bounded. Also from (14), we obtain

$$\lim_{n\to\infty}\|u_n-u_{n+1}\|^2 \leq \|u-u_0\|^2$$

which implies that

$$\|u_n-u_{n+1}\| = 0 \tag{17}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequent $\{u_{n_j}\}$ of the sequence converges to $\hat{u}\in U_{hg}$. Replacing u_n by $\{u_{n_j}\}$ in (13), taking the limit as $n_j\to\infty$ and using (17), we obtain

$$\left\langle \mathscr{T}\hat{u}, \frac{g(v)g(\hat{u})}{g(\hat{u})-g(v)}\right\rangle + \langle u_{n+1}-u_n, u-u_{n+1}\rangle \geq 0 \qquad \forall g(v)\in U_{hg}$$

Which shows that $\hat{u}$ is a solution of HGVI (1) and consequently

$$\|\hat{u}-u_{n+1}\|^2 \leq \|\hat{u}-u_n\|^2.$$

Using the above inequality, one can easily show that the sequence $\{u_n\}$ has exactly one cluster point and $\lim_{n\to\infty}u_n=\hat{u}$, the required result. $\square$

4. Conclusion

In this paper, we have introduced and considered a new class of general trifunction equilibrium variational inequalities. It is shown that the optimum of the sum of differentiable and directional differentiable nonconvex functions can be characterized by means of this class. we have used the auxiliary principle technique for suggesting and analyzing some explicit and inertial proximal point algorithms for solving the trifunction equilibrium variational inequality problem. some special cases are also discussed.Our findings can be seen as an enhancement and improvement over the outcomes that were previously known. Please take note that the projection and resolvent techniques are not used in this technique.

Conflict of Interests

The authors declare that there is no conflict of interests.

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