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THE EXISTENCE ANALYSIS OF NEW CLASS OF FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. In this article, we present a new class of nonlinear differential equations and prove the existence and uniqueness of their solutions. The proof is based on applying Krasnoselskii's fixed-point theorem and Banach's Contraction Mapping Theorem. We establish sufficient conditions for the existence of solutions within an appropriate Banach space and show that these solutions are unique. The combination of these two powerful theorems allows us to address the challenges posed by the nonlinear nature of the equations. Additionally, we provide an illustrative example to demonstrate the practical application of our results. The approach presented not only advances the theoretical understanding of nonlinear differential equations but also offers a robust framework for solving such equations in various applied settings.

Keywords: Banach space; Riemann-Liouville fractional derivative; boundary value problem; Krasnoselskii's fixed point theorem.

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1. INTRODUCTION

Fractional calculus has emerged as a powerful tool for modeling various processes in fields such as engineering, physics, and economics. In fact, fractional-order models often provide a

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more accurate representation of real-world situations compared to integer-order models. This is because fractional derivatives effectively capture the memory and hereditary properties of many materials and processes. Applications of fractional calculus span across diverse areas, including material science, transport phenomena, earthquakes, electrochemical reactions, wave propagation, signal processing, biology, electromagnetic theory, fluid dynamics, thermodynamics, mechanics, geology, astrophysics, economics, and control theory (see [8],[11], [20], [24]). These fields highlight the advantage of fractional differential equations over traditional integer-order models. Key challenges in this area involve understanding different types of fractional derivatives, such as Riemann-Liouville [22], Caputo [13], Hilfer [25], Erdelyi-Kober [15], and Hadamard derivatives [17]. Recent interest has grown in solving boundary value problems for nonlinear fractional differential equations, which are instrumental in modeling and analyzing non-homogeneous physical phenomena. The Caputo derivative was introduced by Michele Caputo in 1967 as a modification of the Riemann–Liouville fractional derivative. It has become widely used in fractional calculus, particularly for modeling physical systems with memory and hereditary properties. For more details, we refer to [23].

Ahmad et al [3] are establish the Existence and Uniqueness of solution for fractional LE in terms of generalized Liouville-Caputo derivatives with non-local boundary conditions involving generalized operators. Recently, a few solutions for the FDEs involving distinct boundary conditions were discussed in [5]-[14]. In [1] the authors discussed the existence and uniqueness of the problem

$$\begin{cases} {}^{C}D^{\nu}\vartheta(\tau) = \mathbf{f}(\tau,\vartheta(\tau)), \tau \in [a,b], \\ \vartheta(a) = \vartheta_0, \end{cases} \tag{1}$$

Hence, researchers are focused on researching many aspects including the existence theory and computational techniques of the stated class of DEs in terms of applications of [4]

$$\begin{cases} {}^{C}D^{\nu}\left(\vartheta(\tau) - \mathbf{g}(\tau,\vartheta(\tau))\right) = \mathbf{f}(\tau,\vartheta(\tau)), \tau \in J = [0,T], \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{2}$$

To the best of our knowledge, we are the first to consider the new class of fractional differential equations presented in this paper. Thus, inspired by the works mentioned above, in this manuscript, we focus on discussing the existence and uniqueness of solutions for the problem

$$\begin{cases} {}^{C}D^{\nu}\left(\frac{\vartheta(\tau)}{1+\lambda(\tau)|\vartheta(\tau)|}-\mathbf{g}(\tau,\vartheta(\tau))\right)=\mathbf{f}(\tau,\vartheta(\tau)), \tau\in J=[0,T], \\ \vartheta(0)=\vartheta_0, \end{cases} \tag{3}$$

The notation ${}^{C}D^{\nu}$ indicates a Caputo fractional derivative of order $0<\nu<1$, $\mathbf{g}: J\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given function, $\mathbf{f}: J\times\mathbb{R}\longrightarrow\mathbb{R}$ is a forcing term or source function, $\lambda: J\longrightarrow\mathbb{R}^{+}$ is continuous function, and ϑ_0 is a known constant.

Additionally, (3) covers various problems and encompasses several research studies in the Caputo operator setting that are present in the literature, such as:

(1) If $\lambda=0$ and $\mathbf{g}=0$ the problem (3) return to the equation (1)
(2) The problem (3) returns to the equation (2) if $\lambda=0$

The paper's rest is arranged as follows: In Section 2, we introduce fundamental definitions and properties of the fractional integral and Caputo fractional derivative, which will be referenced throughout the remainder of the paper. In Section 3, we demonstrate the existence of solutions to the Caputo type fractional problem (3) by applying Banach fixed point theorem and Krasnoselskii's fixed point theorem. Several illustrative examples are presented in Section 4, as an application.

2. Preliminary

In this section, we recall some basic definitions and results that will be used in this paper. Denote by $L^1(J,\mathbb{R})$ the space of Lebesgue integral real-valued functions on J and by $\mathscr{C}=C(J,\mathbb{R})$ the Banach space of continuous functions $\vartheta: J\longrightarrow\mathbb{R}$, with norm $\|\vartheta\|=\sup_{\tau\in J}|\vartheta(\tau)|$, for justifying this, we refer the readers to this work [22]

Definition 2.1. The Caputo fractional derivative with order ν, for a suitable function ϑ is defined as

$${}^{C}D^{\nu}\vartheta(\tau)=\frac{1}{\Gamma(n-\nu)}\int_0^{\tau}\vartheta^{(n)}(\tau)(\tau-z)^{n-\nu-1}dz$$

where $n=[\nu]+1$, with $[\nu]$ denotes integer part of ν. The associated fractional integral is defined by

$$I^{\nu}\vartheta(\tau)=\frac{1}{\Gamma(\nu)}\int_0^{\tau}\vartheta(s)(\tau-z)^{\nu-1}dz$$

Lemma 2.2. *([22]) Let $\nu > 0$ and $n = [\nu] + 1$, The following holds:*

(1)

$$I^{\nu}\left({}^{C}D^{\nu}\vartheta(\tau)\right) = \vartheta(\tau) - \sum_{p=0}^{n-1}\frac{\vartheta^{(p)}(0)}{p!}\tau^{p}$$

(2)

$${}^{C}D^{\nu}(I^{\nu}\vartheta(\tau)) = \vartheta(\tau)$$

Theorem 2.3. *(Krasnoselskii [2]). Let $\mathscr{M}$ be a closed convex non-empty subset of a Banach space $\mathscr{V}$. Suppose that $\mathscr{A}$ and $\mathscr{B}$ map $\mathscr{M}$ into $\mathscr{V}$ and that*

(1) $\mathscr{B}\vartheta_1 + \mathscr{A}\vartheta_2 \in \mathscr{M}$ $(\forall \vartheta_1, \vartheta_2 \in \mathscr{M})$,
(2) $\mathscr{B}$ is compact and continuous,
(3) $\mathscr{A}$ is a contraction mapping.

Then there exists ϑ in $\mathscr{M}$ such that $\mathscr{B}\vartheta + \mathscr{A}\vartheta = \vartheta$.

3. Main Results

In this section, before giving the main results, we assume the following assumptions throughout the rest of our paper.

$\mathscr{H}_1$) There exists a constant $\eta_{\mathbf{g}} > 0$ such that for all $\tau \in [0, T]$ and $\vartheta_1, \vartheta_2 \in \mathscr{C}$

$$|\mathbf{g}(\tau, \vartheta_1) - \mathbf{g}(\tau, \vartheta_2)| \leq \eta_{\mathbf{g}}|\vartheta_1 - \vartheta_2|$$

$\mathscr{H}_2$) There exists a constant $\eta_{\mathbf{f}} > 0$ such that

$$|\mathbf{f}(\tau, \vartheta_1) - \mathbf{f}(\tau, \vartheta_2)| \leq \eta_{\mathbf{f}}|\vartheta_1 - \vartheta_2|$$

$\mathscr{H}_3$) $\mathbf{f} : J \times \mathbb{R} \to \mathbb{R}$ continuous function, and there exists a continuous functions $\mu_1, \mu_2 : J \to \mathbb{R}^+$ such that

$$|\mathbf{f}(\tau, \vartheta)| \leq \mu_1(\tau)|\vartheta(\tau)| + \mu_2(\tau)$$

for all $\vartheta \in X$, and a.e. $\tau \in J$.

In the end, we set $\kappa_{\mathbf{f}} = \sup_{\tau\in J}|\mathbf{f}(\tau, 0)|$, $\kappa_{\mathbf{g}} = \sup_{\tau\in J}|\mathbf{g}(\tau, 0)|$, $\lambda^* = \sup_{\tau\in J}|\lambda(\tau)|$, $\gamma^* = \mathbf{g}(0, \vartheta_0)$, and $\delta^* = \frac{\vartheta_0}{1+\lambda(0)|\vartheta_0|}$

Lemma 3.1. *A function $\vartheta \in \mathscr{C}$ satisfies* (3) *if and only if ϑ satisfies the following fractional integral equation*

$$\vartheta(\tau) = (1+\lambda(\tau)|\vartheta(\tau)|)\left[\delta^* - \gamma^* + \mathbf{g}(\tau,\vartheta(\tau)) + \frac{1}{\Gamma(\nu)}\int_0^\tau (\tau - z)^{\nu-1}(\mathbf{f}(z,\vartheta(z)))dz\right] \tag{4}$$

Proof. First, we assume that $\vartheta \in \mathscr{C}$ satisfies (3). Applying the Riemann–Liouville fractional integer to both sides of (3) and making use of Lemma 2.2, we get

$$\frac{\vartheta(\tau)}{1+\lambda(\tau)|\vartheta(\tau)|} - \mathbf{g}(\tau,\vartheta(\tau)) = \frac{\vartheta_0}{1+\lambda(0)|\vartheta_0|} - \mathbf{g}(0,\vartheta_0) + \frac{1}{\Gamma(\nu)}\int_0^\tau (\tau - z)^{\nu-1}\mathbf{f}(s,\vartheta(s))dz$$

Therefore, by putting the values of γ^*, δ^* defined previously we get (4). Now, if ϑ satisfies (4), By dividing (4) by $1+\lambda(\tau)|\vartheta(\tau)|$, we get

$$\frac{\vartheta(\tau)}{1+\lambda(\tau)|\vartheta(\tau)|} = \frac{\vartheta_0}{1+\lambda(0)|\vartheta_0|} + \mathbf{g}(\tau,\vartheta(\tau)) - \mathbf{g}(0,\vartheta_0) + \frac{1}{\Gamma(\nu)}\int_0^\tau (\tau - z)^{\nu-1}\mathbf{f}(s,\vartheta(s))dz$$

By taking ${}^{C}D^{\nu}$ on both sides and by benefiting from Lemma 2.2, the required result is obtained. □

Next, we investigate the uniqueness result via Banach's fixed point theorem. So, we present the following result:

Lemma 3.2. *Let us consider the polynomial $\Pi(\mathfrak{R}) = \mathfrak{R}^2\lambda K_1 + \mathfrak{R}(\lambda L_1 + K_1 - 1) + L_1$. If only one of the following inequalities holds: $\sqrt{\lambda L_1} + \sqrt{K_1} \leq 1$ or $|\sqrt{\lambda L_1} - \sqrt{K_1}| \geq 1$, then Π has two roots $\mathfrak{R}_1 < \mathfrak{R}_2$. Moreover, if $\lambda^* L_1 + K_1 - 1 \leq \sqrt{\Delta_1}$, the set $[\mathfrak{R}_1, \mathfrak{R}_2] \cap [0, +\infty)$ is nonempty, where the constants K_1, L_1, and Δ_1 are defined below.*

Proof. Let us take the constants

$$K_1 = \eta_{\mathbf{g}} + \frac{1}{\Gamma(\nu+1)}T^{\nu}\eta_{\mathbf{f}}, \quad L_1 = |\delta^* - \gamma^*| + \kappa_{\mathbf{g}} + \frac{1}{\Gamma(\nu+1)}T^{\nu}\kappa_{\mathbf{f}}$$

and the discriminant of the equation $\Pi(\mathfrak{R}) = 0$

$$\begin{aligned}
\Delta_1 &= (\lambda^* L_1 + K_1 - 1)^2 - 4\lambda^* K_1 L_1 \\
&= \left(\lambda^* L_1 + K_1 - 1 + 2\sqrt{\lambda^* K_1 L_1}\right)\left(\lambda^* L_1 + K_1 - 1 - 2\sqrt{\lambda^* K_1 L_1}\right) \\
&= \left(\left(\sqrt{\lambda^* L_1} + \sqrt{K_1}\right)^2 - 1\right)\left(\left(\sqrt{\lambda^* L_1} - \sqrt{K_1}\right)^2 - 1\right)
\end{aligned}$$

By using the first condition of lemma, we get

$$\mathfrak{R}_1 = \frac{1-\lambda^* L_1 - K_1 - \sqrt{\Delta}}{2\lambda^* K_1}, \ \mathfrak{R}_2 = \frac{1-\lambda^* L_1 - K_1 + \sqrt{\Delta}}{2\lambda^* K_1}$$

and by the last condition we get $\mathfrak{R}_2$ is a positive number, which completes the proof. □

Theorem 3.3. *Assume that the hypothesis $\mathscr{H}_1) - \mathscr{H}_2)$ are satisfied and moreover $\frac{1}{2K_1}\left(\frac{1-K_1}{\lambda^*} - L_1\right) \in]\max(0,\mathfrak{R}_1), \mathfrak{R}_2]$. Then, the problem (3) has unique solution.*

Proof. We start our proof by defining the mapping $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ as below:

$$\mathscr{T}\vartheta(\tau) = (1+\lambda(\tau)|\vartheta(\tau)|)\left[\delta^* - \gamma^* + \mathbf{g}(\tau,\vartheta(\tau)) + \frac{1}{\Gamma(\nu)}\int_0^\tau (\tau-z)^{\nu-1}(\mathbf{f}(z,\vartheta(z)))dz\right].$$

Also the ball $\mathscr{F} = \{\vartheta \in \mathscr{C} / \|\vartheta\| \le \mathfrak{R}\}$ with $\mathfrak{R} \in]\max(0,\mathfrak{R}_1), \frac{1}{2K_1}\left(\frac{1-K_1}{\lambda^*} - L_1\right)[$. Let's prove that $\mathscr{T}(\mathscr{F}) \subset \mathscr{F}$. Let $\vartheta \in \mathscr{F}$; then

$$\begin{aligned}
|\mathscr{T}\vartheta(\tau)| &\le (1+\lambda(\tau)|\vartheta(\tau)|)\left[|\delta^* - \gamma^*| + |\mathbf{g}(\tau,\vartheta(\tau))| + \frac{1}{\Gamma(\nu)}\int_0^t (\tau-z)^{\nu-1}|\mathbf{f}(z,\vartheta(z))|dz\right] \\
&\le (1+\lambda^*\mathfrak{R})\,[|\delta^* - \gamma^*| + |\mathbf{g}(\tau,\vartheta(\tau)) - \mathbf{g}(\tau,0)| + |\mathbf{g}(\tau,0)| \\
&\quad + \frac{1}{\Gamma(\nu)}\int_0^\tau (\tau-z)^{\nu-1}(|\mathbf{f}(z,\vartheta(z)) - \mathbf{f}(z,0)| + |\mathbf{f}(z,0)|)dz\Big] \\
&\le (1+\lambda^*\mathfrak{R})\left[|\delta^* - \gamma^*| + \eta_{\mathbf{g}}|\vartheta(\tau)| + \kappa_{\mathbf{g}} + \frac{1}{\Gamma(\nu+1)}T^\nu(\eta_{\mathbf{f}}\mathfrak{R} + \kappa_{\mathbf{f}})\right] \\
&\le (1+\lambda^*\mathfrak{R})\left[\mathfrak{R}\left(\eta_{\mathbf{g}} + \frac{1}{\Gamma(\nu+1)}T^\nu\eta_{\mathbf{f}}\right) + |\delta^* - \gamma^*| + \kappa_{\mathbf{g}} + \frac{1}{\Gamma(\nu+1)}T^\nu\kappa_{\mathbf{f}}\right] \\
&\le \Pi(\mathfrak{R}) + \mathfrak{R}
\end{aligned}$$

By using the previous Lemma 3.2, we have $\Pi(\mathfrak{R}) \le \mathfrak{R}$. Therefore, $|\mathscr{T}\vartheta| \le \mathfrak{R}$. This means that $\mathscr{T}(\mathscr{F}) \subset \mathscr{F}$.

In the following, we show that the operator $\mathscr{T}$ is a contraction mapping on $\mathscr{F}$.

For any ϑ_1, ϑ_2 in $\mathscr{F}$, by $\mathscr{H}1)$ and $\mathscr{H}2)$, one obtains

$$\begin{aligned}
|\,\mathscr{T}\vartheta_1(\tau) - \mathscr{T}\vartheta_2(\tau)| &\le \lambda(\tau)\,|\delta^* - \gamma^*|\,||\vartheta_1(\tau)| - |\vartheta_2(\tau)|| + |\mathbf{g}(\tau,\vartheta_1(\tau)) - \mathbf{g}(\tau,\vartheta_2(\tau))| \\
&\quad + \frac{1}{\Gamma(\nu)}\left[\int_0^\tau (\tau-z)^{\nu-1}\,|\mathbf{f}(z,\vartheta_1(z)) - \mathbf{f}(z,\vartheta_2(z))|\,dz\right] \\
&\quad + \lambda(\tau)\,[||\vartheta_1(\tau)|\,\mathbf{g}(\tau,\vartheta_1(\tau)) - |\vartheta_2(\tau)|\,\mathbf{g}(\tau,\vartheta_2(\tau))|] \\
&\quad + \frac{\lambda(\tau)}{\Gamma(\nu)}\left[\int_0^\tau (\tau-z)^{\nu-1}||\vartheta_1(\tau)\,|\mathbf{f}(z,\vartheta_1(z)) - |\vartheta_2(\tau)|\,\mathbf{f}(z,\vartheta_2(z))|\,dz\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\lambda(\tau)\left|\delta^*-\gamma^*\right|+\eta_{\mathbf{g}}\right)\left|\vartheta_1(\tau)-\vartheta_2(\tau)\right|+\left[\eta_{\mathbf{f}}\frac{T^\nu}{\Gamma(\nu+1)}\right]\left\|\vartheta_1(\tau)-\vartheta_2(\tau)\right\| \\
&+\lambda(\tau)\left[\left|\left|\vartheta_1(\tau)\right|\mathbf{g}(\tau,\vartheta_1(\tau))-\left|\vartheta_1(\tau)\right|\mathbf{g}(\tau,\vartheta_2(\tau))\right|\right. \\
&+\left.\left|\left|\vartheta_1(\tau)\right|\mathbf{g}(\tau,\vartheta_2(\tau))-\left|\vartheta_2(\tau)\right|\mathbf{g}(\tau,\vartheta_2(\tau))\right|\right] \\
&+\frac{\lambda(\tau)}{\Gamma(\nu)}\int_0^\tau(\tau-z)^{\nu-1}\left[\left|\left|\vartheta_1(\tau)\right|\mathbf{f}(z,\vartheta_1(z))-\left|\vartheta_1(\tau)\right|\mathbf{f}(z,\vartheta_2(z))\right|\right]dz \\
&+\frac{\lambda(\tau)}{\Gamma(\nu)}\int_0^\tau(\tau-z)^{\nu-1}\left[\left|\left|\vartheta_1(\tau)\right|\mathbf{f}(z,\vartheta_2(z))-\left|\vartheta_2(\tau)\right|\mathbf{f}(z,\vartheta_2(z))\right|\right]dz \\
&\leq\left(\lambda(\tau)\left|\delta^*-\gamma^*\right|+\eta_{\mathbf{g}}+\eta_{\mathbf{f}}\frac{T^\nu}{\Gamma(\nu+1)}\right)\left\|\vartheta_1-\vartheta_2\right\| \\
&+\lambda(\tau)\left[\eta_{\mathbf{g}}\left|\vartheta_1(\tau)\right|+\left|\mathbf{g}(\tau,\vartheta_2(\tau))-\mathbf{g}(\tau,0)\right|+\left|\mathbf{g}(\tau,0)\right|\right]\left|\vartheta_1(\tau)-\vartheta_2(\tau)\right| \\
&+\eta_{\mathbf{f}}\left|\vartheta_1(\tau)\right|\frac{\lambda(\tau)}{\Gamma(\nu)}\int_0^\tau(\tau-z)^{\nu-1}\left|\vartheta_1(z)-\vartheta_2(z)\right|dz \\
&+\left|\vartheta_1(\tau)-\vartheta_2(\tau)\right|\frac{\lambda(\tau)}{\Gamma(\nu)}\int_0^\tau(\tau-z)^{\nu-1}\left|\mathbf{f}(z,\vartheta_2(z))\right|dz \\
&\leq\left(\lambda^*\left|\delta^*-\gamma^*\right|+\eta_{\mathbf{g}}+\eta_{\mathbf{f}}\frac{T^\nu}{\Gamma(\nu+1)}\right)\left\|\vartheta_1(\tau)-\vartheta_2(\tau)\right\| \\
&+\lambda^*\left[\eta_{\mathbf{g}}\left|\vartheta_1(\tau)\right|+\eta_{\mathbf{g}}\left|\vartheta_2(\tau)\right|+\kappa_{\mathbf{g}}\right]\left|\vartheta_1(\tau)-\vartheta_2(\tau)\right| \\
&+\eta_{\mathbf{f}}\left|\vartheta_1(\tau)\right|\frac{\lambda^*T^\nu}{\Gamma(\nu+1)}\left\|\vartheta_1-\vartheta_2\right\| \\
&+\left|\vartheta_1(\tau)-\vartheta_2(\tau)\right|\frac{\lambda^*}{\Gamma(\nu)}\int_0^\tau(\tau-z)^{\nu-1}\left(\left|\mathbf{f}(z,\vartheta_2(z))-\mathbf{f}(z,0)\right|+\left|\mathbf{f}(z,0)\right|\right)dz \\
&\leq\left(\lambda^*\left(\left|\delta^*-\gamma^*\right|+2\eta_{\mathbf{g}}\Re+\kappa_{\mathbf{g}}+\frac{T^\nu}{\Gamma(\nu+1)}(2\eta_{\mathbf{f}}\Re+\kappa_{\mathbf{f}})\right)+\eta_{\mathbf{g}}+\eta_{\mathbf{f}}\frac{T^\nu}{\Gamma(\nu+1)}\right)\left\|\vartheta_1-\vartheta_2\right\| \\
&\leq\left(\lambda^*(L_1+2K_1\Re)+K_1\right)\left\|\vartheta_1-\vartheta_2\right\|
\end{aligned}
$$

Hence $\mathscr{T}$ is a contraction, then by Banach Principe contraction $\mathscr{T}$ has unique fixed point. So, we conclude (3) has unique solution. □

Next, we investigate the existence criteria of solutions via Krasnoselskii's fixed point theorem. To prove that the fractional integral equation (4) has at least one solution $\vartheta\in\mathscr{C}$, we define two operators $\mathscr{A}$ and $\mathscr{B}$ from $\mathscr{C}$ to $\mathscr{C}$, as follows:

$$\mathscr{A}\vartheta(\tau)=(1+\lambda(\tau)|\vartheta(\tau)|)\left(\delta^*-\gamma^*+\mathbf{g}(\tau,\vartheta(\tau))\right)$$

$$\mathscr{B}\vartheta(\tau)=\frac{(1+\lambda(\tau)|\vartheta(\tau)|)}{\Gamma(\nu)}\int_0^\tau(\tau-z)^{\nu-1}(\mathbf{f}(z,\vartheta(z)))dz$$

and the ball $\mathscr{M}_\rho=\{\vartheta\in\mathscr{C}/\|\vartheta\|\leq\rho\}$

Lemma 3.4. *Assuming $\mathscr{H}1)$–$\mathscr{H}3)$, in addition, only one of the following inequalities holds:*

$$\sqrt{\lambda L_2}+\sqrt{K_2}\leq 1 \quad or \quad \left|\sqrt{\lambda L_2}-\sqrt{K_2}\right|\geq 1 \tag{5}$$

and the inequality

$$\sqrt{\Delta_2}>\lambda^* L_2+K_2-1 \tag{6}$$

Given these, we have $\mathscr{A}\vartheta_1+\mathscr{B}\vartheta_2\in\mathscr{M}_\rho$ $\forall\vartheta_1,\vartheta_2\in\mathscr{M}_\rho$, where the constants K_2, L_2, and Δ_2 are defined below.

Proof. Let $\vartheta_1,\vartheta_2\in\mathscr{M}_\rho$, we show that, $\mathscr{B}\vartheta_1+\mathscr{A}\vartheta_2\in\mathscr{M}_\rho$, with considerations to $\mathscr{H}_1)$ and $\mathscr{H}_3)$, we get

$$\begin{aligned}
|\mathscr{B}\vartheta_1(\tau)+\mathscr{A}\vartheta_2(\tau)| &\leq (1+\lambda(\tau)|\vartheta_1(\tau)|)(|\delta^*-\gamma^*|+|\mathbf{g}(\tau,\vartheta_1(\tau))|)\\
&\quad+(1+\lambda(\tau)|\vartheta_2(\tau)|)\left[\frac{1}{\Gamma(\nu)}\int_0^\tau(\tau-s)^{\nu-1}|\mathbf{f}(s,\vartheta_2(s))|ds\right]\\
&\leq(1+\lambda^*\rho)[|\delta^*-\gamma^*|+|\mathbf{g}(\tau,\vartheta(\tau))-\mathbf{g}(\tau,0)|+|\mathbf{g}(\tau,0)|]\\
&\quad+(1+\lambda^*\rho)\left[\frac{1}{\Gamma(\nu)}\int_0^\tau(\tau-s)^{\nu-1}(\mu_1(s)|\vartheta_2(\tau)|+\mu_2(s))ds\right]\\
&\leq(1+\lambda^*\rho)\left(|\delta^*-\gamma^*|+\eta_{\mathbf{g}}\rho+\kappa_{\mathbf{g}}+(\mu_1^*\rho+\mu_2^*)\frac{T^\nu}{\Gamma(\nu+1)}\right)\\
&\leq\Psi(\rho)+\rho,
\end{aligned}$$

where

$$\Psi(\rho)=\rho^2\lambda^*K_2+\rho(\lambda^*L_2+K_2-1)+L_2,$$

with

$$K_2=\eta_{\mathbf{g}}+\mu_1^*\frac{T^\nu}{\Gamma(\nu+1)} \text{ and } L_2=|\delta^*-\gamma^*|+\kappa_{\mathbf{g}}+\mu_2^*\frac{T^\nu}{\Gamma(\nu+1)}$$

Taking the supremum over τ, in the last estimate we obtain

$$\|\mathscr{B}\vartheta_1+\mathscr{A}\vartheta_2\|\leq\Psi(\rho)+\rho.$$

The discriminant Δ_2 of the equation $\Psi(\rho)=0$ is

$$\begin{aligned}\Delta_2 &= (\lambda^* L_2 + K_2 - 1)^2 - 4\lambda^* K_2 L_2 \\ &= \left(\lambda^* L_2 + K_2 - 1 + 2\sqrt{\lambda^* K_2 L_2}\right)\left(\lambda^* L_2 + K_2 - 1 - 2\sqrt{\lambda^* K_2 L_2}\right) \\ &= \left(\left(\sqrt{\lambda^* L_2} + \sqrt{K_2}\right)^2 - 1\right)\left(\left(\sqrt{\lambda^* L_2} - \sqrt{K_2}\right)^2 - 1\right).\end{aligned}$$

If only one of the inequalities in (5) holds, then Ψ has two roots. $\rho_1 = \frac{1-\lambda^* L_2 - K_2 - \sqrt{\Delta_2}}{2\lambda^* K_2}$ and $\rho_2 = \frac{1-\lambda^* L_2 - K_2 + \sqrt{\Delta_2}}{2\lambda^* K_2}$ with $\rho_1 \le \rho_2$ and from the inequality (6) we get $\rho_2 > 0$, we deduce that the set $[\rho_1, \rho_2] \cap [0, +\infty[$ is nonempty.

We take $\rho \in [\rho_1, \rho_2] \cap [0, +\infty)$, and we get $|\mathscr{B}\vartheta_1 + \mathscr{A}\vartheta_2| \le \rho$. We can now conclude the result. □

Lemma 3.5. *If* $1 - \eta_{\mathbf{g}} - \lambda^* \left(|\delta^* - \gamma^*| + \kappa_{\mathbf{g}}\right) > 0$ *and* $\rho \in]0, \Upsilon^*[$. *Then, the operator* $\mathscr{A}$ *is a contraction on* $\mathscr{M}_\rho$, *where*

$$\Upsilon^* = \frac{1 - \eta_{\mathbf{g}} - \lambda^* \left(|\delta^* - \gamma^*| + \kappa_{\mathbf{g}}\right)}{\lambda^* \eta_{\mathbf{g}}}.$$

Proof. Let $\vartheta_1, \vartheta_2 \in \mathscr{M}_\rho$ and $\tau \in J$,

$$\begin{aligned}&|\mathscr{A}\vartheta_1(\tau) - \mathscr{A}\vartheta_2(\tau)| \,|\le \lambda(\tau)\,|(\delta^* - \gamma^*)\,(|\vartheta_1(\tau)| - |\vartheta_2(\tau)|)| \\ &\quad + |\mathbf{g}(\tau, \vartheta_1(\tau)) - \mathbf{g}(\tau, \vartheta_2(\tau))| + \lambda(\tau)||\vartheta_1(\tau)\,|\mathbf{g}(\tau, \vartheta_1(\tau)) - |\vartheta_2(\tau)|\,\mathbf{g}(\tau, \vartheta_2(\tau))| \\ &\quad \le (\eta_{\mathbf{g}} + \lambda(\tau)\,|\delta^* - \gamma^*|)\,|\vartheta_1(\tau) - \vartheta_2(\tau)| + \lambda(\tau)||\vartheta_1(\tau)\,|\mathbf{g}(\tau, \vartheta_1(\tau)) - |\vartheta_2(\tau)|\,\mathbf{g}(\tau, \vartheta_1(\tau))| \\ &\quad + \lambda(\tau)||\vartheta_2(\tau)\,|\mathbf{g}(\tau, \vartheta_1(\tau)) - |\vartheta_2(\tau)|\,\mathbf{g}(\tau, \vartheta_2(\tau))| \\ &\quad \le (\eta_{\mathbf{g}} + \lambda(\tau)\,|\delta^* - \gamma^*| + |\mathbf{g}(\tau, \vartheta_1(\tau))| + \eta_{\mathbf{g}}\,|\vartheta_2(\tau)|))\,|\vartheta_1(\tau) - \vartheta_2(\tau)| \\ &\quad \le (\eta_{\mathbf{g}} + \lambda^*\,(|\delta^* - \gamma^*| + \kappa_{\mathbf{g}} + \eta_{\mathbf{g}}\rho))\,\|\vartheta_1 - \vartheta_2\|.\end{aligned}$$

Hence

$$\|\mathscr{A}\vartheta_1 - \mathscr{A}\vartheta_2\| \le (\eta_{\mathbf{g}} + \lambda^*(|\delta^* - \gamma^*| + \kappa_{\mathbf{g}} + \eta_{\mathbf{g}}\rho))\|\vartheta_1 - \vartheta_2\|.$$

Thus, $\mathscr{A}$ is a contraction mapping. □

Theorem 3.6. *Assume that the hypotheses of Lemma 3.5 and Lemma 3.4 are satisfied. Then, the problem (3) has at least one solution, and the set of solutions is bounded.*

Proof. The fractional integral equation (4) can be written as follows:

$$\vartheta(\tau)=\mathscr{B}\vartheta(\tau)+\mathscr{A}\vartheta(\tau)$$

According to Lemma 3.5 and Lemma 3.4, the conditions (1) and (3) of Theorem 2.3 are satisfied. To complete the proof of this theorem, it remains to show that $\mathscr{B}$ is continuous and compact. For this reason, we take a sequence (ϑ_n) such that $\vartheta_n \longrightarrow \vartheta$ in $\mathscr{M}_\rho$. So, we have

$$\begin{aligned}&|\mathscr{B}\vartheta_n(\tau)-\mathscr{B}\vartheta(\tau)|\\ &\leq \frac{1}{\Gamma(\nu)}\int_0^\tau(\tau-s)^{\nu-1}|(1+\lambda(\tau)|\vartheta_n(\tau)|)\mathbf{f}(s,\vartheta_n(s))-(1+\lambda(\tau)|\vartheta(\tau)|)\mathbf{f}(s,\vartheta(s))\mid ds\end{aligned}$$

Since $\vartheta_n \longrightarrow \vartheta$ as $n\longrightarrow\infty$ and $\mathbf{f}$ is continuous, then according to Lebesgue dominated convergence theorem proves that

$$\int_0^\tau(\tau-s)^{\nu-1}|(1+\lambda(\tau)|\vartheta_n(\tau)|)\mathbf{f}(s,\vartheta_n(s))-(1+\lambda(\tau)|\vartheta(\tau)|)\mathbf{f}(s,\vartheta(s))|ds\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty$$

Finally, $\mathscr{B}$ is continuous.

Now, we show that $\mathscr{B}$ is equi-continuous. For each $\vartheta\in\mathscr{M}_\rho$, and $\tau_1,\tau_2\in J$ with $\tau_1<\tau_2$, let's prove that

$$\lim_{\tau_1\longrightarrow\tau_2}|\mathscr{B}\vartheta(\tau_1)-\mathscr{B}\vartheta(\tau_2)|=0$$

we have

$$\begin{aligned}&|\mathscr{B}\vartheta(\tau_2)-\mathscr{B}\vartheta(\tau_1)|\leq\frac{1}{\Gamma(\nu)}\Bigg|\int_0^{\tau_2}(\tau_2-s)^{\nu-1}(1+\lambda(\tau_2)|\vartheta(\tau_2)|)\mathbf{f}(s,\vartheta(s))ds\\ &-\int_0^{\tau_1}(\tau_1-s)^{\nu-1}(1+\lambda(\tau_1)|\vartheta(\tau_1)|)\mathbf{f}(s,\vartheta(s))ds\Bigg|\\ &\leq\frac{1}{\Gamma(\nu)}\int_0^{\tau_1}|(\tau_2-s)^{\nu-1}(1+\lambda(\tau_2)|\vartheta(\tau_2)|)-(\tau_1-s)^{\nu-1}(1+\lambda(\tau_1)|\vartheta(\tau_1)|)||\mathbf{f}(s,\vartheta(s))|ds\\ &+\frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\nu-1}(1+\lambda(\tau_2)|\vartheta(\tau_2)|)|\mathbf{f}(s,\vartheta(s))|ds\\ &\leq\frac{(\rho||\mu_1||+||\mu_2||)}{\Gamma(\nu)}\int_0^{\tau_1}|(\tau_2-s)^{\nu-1}(1+\lambda(\tau_2)|\vartheta(\tau_2)|)-(\tau_1-s)^{\nu-1}(1+\lambda(\tau_1)|\vartheta(\tau_1)|)|ds\\ &+\frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\nu-1}(1+\lambda(\tau_2)|\vartheta(\tau_2)|)|\mathbf{f}(s,\vartheta(s))|ds\end{aligned}$$

On the other hand,

$$\lim_{\tau_1\longrightarrow\tau_2}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\nu-1}(1+\lambda(\tau_2)|\vartheta(\tau_2)|)|\mathbf{f}(s,\vartheta(s))|ds=0,$$

and

$$\lim_{\tau_1 \longrightarrow \tau_2} \int_0^{\tau_1} \left| (\tau_2 - s)^{\nu-1} (1 + \lambda(\tau_2) |\vartheta(\tau_2)|) - (\tau_1 - s)^{\nu-1} (1 + \lambda(\tau_1) |\vartheta(\tau_1)|) \right| ds = 0.$$

So

$$\lim_{\tau_1 \longrightarrow \tau_2} |\mathscr{B}\vartheta(\tau_1) - \mathscr{B}\vartheta(\tau_2)| = 0$$

We conclude that $\mathscr{B}$ is equi-continuous.

For arbitrary $\vartheta \in \mathscr{M}_\rho$, by using Lemma 3.4, we get $\|\mathscr{B}\vartheta\| \leq \|\mathscr{B}\vartheta\| + \|\mathscr{A}\vartheta\| \leq \rho$. we deduce that $\mathscr{B}\mathscr{M}_\rho$ is uniformly bounded on $\mathscr{M}_\rho$. Since it's equi-continuous, The Arzela-Ascoli theorem yields that the mapping $\mathscr{B}$ is compact. We now conclude the result of the theorem based on the Krasnoselkii's theorem above. □

4. EXAMPLES

Here, we give a nontrivial example to illustrate our main results.

Example 1. Consider the following Nonlinear fractional problem

$$\begin{cases} {}^{C}D^{\frac{1}{2}} \left(\frac{\vartheta(\tau)}{1+\exp(-\tau-5)|\vartheta(\tau)|} - \sin\left(\tau + \frac{1}{7}\vartheta(\tau)\right) \right) = \tau + \frac{2}{15}|\vartheta(\tau)|, \tau \in [0,5], \\ \vartheta(0) = \pi, \end{cases} \tag{7}$$

Notice that $\mathbf{g}(0, \vartheta(0)) = 1, \kappa_{\mathbf{f}} = 2, \kappa_{\mathbf{g}} = 1$ and $\lambda^* = e^{-5}$. Since, $\mathbf{g}(\tau, \vartheta(\tau)) = \sin(\tau + \frac{1}{7}\vartheta(\tau))$, $\mathbf{f}(\tau, \vartheta(\tau)) = \tau + \frac{2}{15}|\vartheta(\tau)|, \lambda(\tau) = \exp(-\tau - 5)$ we have

$$\begin{aligned} |\mathbf{f}(\tau, \vartheta_1(\tau)) - \mathbf{f}(\tau, \vartheta_2(\tau))| &| \leq \left| \frac{2}{15} \right| \vartheta_1(\tau) \left| - \frac{2}{15} \right| \vartheta_2(\tau)|| \\ &\leq \frac{2}{15} |\vartheta_1(\tau) - \vartheta_2(\tau)| \\ |\mathbf{g}(\tau, \vartheta_1(\tau)) - \mathbf{g}(\tau, \vartheta_2(\tau))| &| \leq \left| \sin\left(\tau + \frac{1}{7}\vartheta(\tau)\right) - \sin\left(\tau + \frac{1}{7}\vartheta_2(\tau)\right) \right| \\ &\leq \frac{1}{7} |\vartheta_1(\tau) - \vartheta_2(\tau)| \end{aligned}$$

Thus $\eta_{\mathbf{f}} = 2/15, \eta_{\mathbf{g}} = 1/7$. Now $\frac{1}{2K_1}\left(\frac{1-K_1}{\lambda^*} - L_1\right) = 72.1503, \mathfrak{R}_1 = 20.28154$ and $\mathfrak{R}_2 = 124.019247$, Let $\mathfrak{R} \in \left[\mathfrak{R}_1, \frac{1}{2K_1}\left(\frac{1-K_1}{\lambda^*} - L_1\right)\right]$ according to Lemma 3.2 and by Theorem 3.3, the problem (7) has unique solution $\vartheta \in \mathscr{C}$ such that $\|\vartheta\| \leq \mathfrak{R}$.

Example 2.

$$(5.2)\qquad \begin{cases} {}^{C}D^{\frac{3}{4}}\left(\frac{\vartheta(\tau)}{1+\exp(-\tau-10)|\vartheta(\tau)|}-\frac{\tau}{35}+\frac{|\vartheta(\tau)|e^{-\tau}}{\tau+4}\right)=\frac{\cos\left(3\tau^2+1\right)}{\tau^2+6\tau+4}+\frac{|\vartheta(\tau)|}{|\vartheta(\tau)|+\tau+7}, \tau\in[0,7] \\ \vartheta(0)=0 \end{cases}$$

In this example $\mathbf{g}(\tau,\vartheta(\tau))=\frac{\tau}{35}-\frac{|\vartheta(\tau)|e^{-\tau}}{\tau+4}, \mathbf{f}(\tau,\vartheta(\tau))=\frac{\cos\left(3\tau^2+1\right)}{\tau^2+6\tau+4}+\frac{|\vartheta(\tau)|}{|\vartheta(\tau)|+\tau+7}, \lambda(\tau)=\exp(-\tau-10)$ and we set $\nu=\frac{3}{4}$, $\lambda^*=e^{-10}$, $\delta^*=\gamma^*=0$ and $\kappa_{\mathbf{g}}=1/5$. Thus, we find

$$\begin{aligned} |\,\mathbf{g}(\tau,\vartheta_1(\tau))-\mathbf{g}(\tau,\vartheta_2(\tau))\| &\leq \left|\frac{|\vartheta_1(\tau)|\,e^{-\tau}}{\tau+4}-\frac{|\vartheta_2(\tau)|\,e^{-\tau}}{\tau+4}\right| \\ &\leq \frac{e^{-\tau}}{\tau+4}||\vartheta_1(\tau)|-|\vartheta_2(\tau)|| \\ &\leq \frac{1}{4}|\vartheta_1(\tau)-\vartheta_2(\tau)| \end{aligned}$$

and

$$|\mathbf{f}(\tau,\vartheta(\tau))|\leq\frac{|\vartheta(\tau)|}{|\vartheta(\tau)|+\tau+7}+\frac{\left|\cos\left(3\tau^2\right)\right|}{\tau^2+6\tau+4}\leq\frac{|\vartheta(\tau)|}{7}+\frac{1}{4}.$$

Hence, $\mu_1^*=1/7, \mu_2^*=1/4$ and $\eta_{\mathbf{g}}=1/4$. So, we get $1-\eta_{\mathbf{g}}-\lambda^*\left(|\delta^*-\gamma^*|+\kappa_{\mathbf{g}}\right)=0.7499909, \rho_1=17.069675$ and $\rho_2=1924.671137$.

Then, by Theorem 3.6 the problem (5.2) has at least one solution.

Conflict of Interests

The authors declare that there is no conflict of interests.

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