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FIXED POINT RESULTS IN SYMMETRIC G-METRIC SPACE

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Abstract. In the present paper, we generalize the results of Gaba [2] by using the concept of weakly compatible mappings in symmetric G-complete G-metric space. Further, we give examples to support our results.

Keywords: G-metric space; fixed point; contraction; weakly compatible maps.

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1. Introduction

Banach fixed point theorem guarantees the existance and uniqueness of fixed point for contraction self-maps of metric space. But a contraction map is a continuous map, so it is limitation of this theorem. Kannan [6] established a fixed point theorem where continuity of the function is relaxed. Then Sessa [8] defined the notion of weakly commuting. After that Jungck generalized this idea to compatible mappings [3] and to weakly compatible mappings [4]. Numerous examples are provided to show that each of these generalizations of commutativity is a proper extension of the previous definition. Mustafa and Sims [7] introduced the notion of G-metric space which was a generalization of metric space. In the present paper, we prove some fixed point theorems involving weakly compatible maps in the setting of symmetric G-metric space

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and generalize the results of Gaba [2]. Let us recall some basic rudiments of G-metric space before we start our main results.

Definition 1.1. [7] Let X be a nonempty set and let the function $G : X \times X \times X \to [0,\infty)$ satisfy the following properties:

(G_1) $G(x,y,z) = 0$ if $x = y = z$ whenever $x,y,z \in X$;

(G_2) $G(x,x,y) > 0$ whenever $x,y \in X with\, x \neq y$;

(G_3) $G(x,x,y) \leq G(x,y,z)$ whenever $x,y,z \in X with\, z \neq y$;

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = ...,(symmetry\ in\ all\ three\ variables)$;

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for any points $x,y,z,a \in X$.

Then the function G is called a generalized metric, or, more specifically a G-metric on X and the pair (X,G) is called a G-metric space.

Proposition 1.2. [7] *A G-metric space (X,G) is said to be symmetric if*

$$G(x,y,y) = G(y,x,x) \ for\ all\ x,y \in X.$$

Definition 1.3. [7]. Let (X,G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X, we say that $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n,m\to\infty} G(x,x_n,x_m) = 0,$$

that is,

for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$G(x,x_n,x_m) < \varepsilon \quad for\ all\ \ n,m \geq N.$$

We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 1.4. [7] Let (X,G) be a G-metric space. We say that $\{x_n\}$ is

(i) a G-Cauchy sequence if, for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$G(x_n,x_m,x_l) < \varepsilon \in X, \ \ for\ all\ n,m,l \geq N.$$

(ii) a G-Convergent sequence to $x \in X$ if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n,m \geq N$, $G(x,x_n,x_m) < \varepsilon$.

A G-metric space (X,G) is said to complete if every G-Cauchy sequence in X is G-Convergent in X.

Proposition 1.5. [7] *Let (X,G) be a G-metric space. The following are equivalent;*

(i) $\{x_n\}$ *is G-convergent to* $x \in X$.

(ii) $\lim_{n,m\to\infty} G(x_n,x_m,x) = 0.$

(iii) $\lim_{n\to\infty} G(x_n,x,x) = 0.$

(iv) $\lim_{n\to\infty} G(x,x,x_n) = 0.$

Proposition 1.6. [5] *A G-metric on a G-metric space (X,G) is continuous on its three variables.*

Definition 1.7. [2] A self mapping f defined on a G-metric space (X,G) is said to be orbitally continuous iff

$$\lim_{i\to\infty} f^{n_i}x = x^* \in X \implies fx^* = \lim_{i\to\infty} ff^{n_i}x.$$

Definition 1.8. [5] A pair (f,g) of self mappings of metric space (X,d) is said to be weakly compatible if the mappings commute at all of their coincidence points, that is, $fx = gx$ for some $x \in X$implies $fgx = gfx$.

Definition 1.9. [1] Let f and g be self-maps of a set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Proposition 1.10. [1] *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

2. Main Results

Theorem 2.1. *Let (X,G) be a symmetric G-complete G-metric space and $f,g\colon X \to X$ satisfy*

$$G(fx,fy,fz) \le \left(\frac{G(fx,gy,gz)+G(gx,fy,gz)+G(gx,gy,fz)}{2G(gx,fx,fx)+G(gy,fy,fy)+G(gz,fz,fz)+1}\right) G(gx,gy,gz) \tag{1}$$

for all $x,y,z \in X$. If $f(X) \subset g(X)$ and g(X) is a complete, then

(i) *f and g have atleast one coincident point $p \in X$;*

(ii) *for any $x \in X$, the sequence $\{f^n x\}$ G-converges to a coincidence point.*

(iii) *if $p, p^{'} \in X$ are two distinct coincident points, then $G(gp, gp', gp') = G(gp', gp', gp) \geq \frac{1}{3}$.*

Proof. Let f and g satisfy the condition (1) and let x_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, there is $x_1 \in X$ such that $gx_1 = fx_0$. Continuing the same process, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence. Let $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, by using (G_5) we obtain that

$$G(gx_n, gx_{n+1}, gx_{n+1}) = G(fx_{n-1}, fx_n, fx_n)$$

$$\begin{aligned}
&\leq \left(\frac{G(fx_{n-1}, gx_n, gx_n) + G(gx_{n-1}, fx_n, gx_n) + G(gx_{n-1}, gx_n, fx_n)}{2G(gx_{n-1}, fx_{n-1}, fx_{n-1}) + G(gx_n, fx_n, fx_n) + G(gx_n, fx_n, fx_n) + 1}\right) G(gx_{n-1}, gx_n, gx_n) \\
&= \left(\frac{G(gx_n, gx_n, gx_n) + G(gx_{n-1}, gx_{n+1}, gx_n) + G(gx_{n-1}, gx_n, gx_{n+1})}{2G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_n, gx_{n+1}, gx_{n+1}) + 1}\right) G(gx_{n-1}, gx_n, gx_n) \\
&= \left(\frac{2G(gx_{n-1}, gx_n, gx_{n+1})}{2G(gx_{n-1}, gx_n, gx_n) + 2G(gx_n, gx_{n+1}, gx_{n+1}) + 1}\right) G(gx_{n-1}, gx_n, gx_n) \\
&\leq \left(\frac{2G(gx_{n-1}, gx_n, gx_n) + 2G(gx_n, gx_{n+1}, gx_{n+1})}{2G(gx_{n-1}, gx_n, gx_n) + 2G(gx_n, gx_{n+1}, gx_{n+1}) + 1}\right) G(gx_{n-1}, gx_n, gx_n).
\end{aligned}$$

Put $\dfrac{2G(gx_{n-1}, gx_n, gx_n) + 2G(gx_n, gx_{n+1}, gx_{n+1})}{2G(gx_{n-1}, gx_n, gx_n) + 2G(gx_n, gx_{n+1}, gx_{n+1}) + 1} = \rho$, then $o \leq \rho < 1$ and

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \rho \, G(gx_{n-1}, gx_n, gx_n).$$

That is for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
G(gx_n, gx_{n+1}, gx_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\
&\leq \rho \, G(gx_{n-1}, gx_n, gx_n) \\
&\leq \rho^2 G(gx_{n-2}, gx_{n-1}, gx_{n-1}) \\
&\vdots \\
&\leq \rho^n G(gx_0, gx_1, gx_1).
\end{aligned}$$

Moreover, for all $n, m \in \mathbb{N}$; $n < m$, we have by rectangle inequality that

$$\begin{aligned}
G(gx_n, gx_m, gx_m) \leq\ & G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{n+2}, gx_{n+2}) + G(gx_{n+2}, gx_{n+3}, gx_{n+3}) \\
&+ \dots + G(gx_{m-1}, gx_m, gx_m)
\end{aligned}$$

$$\leq \left(\rho^n+\rho^{n+1}+\rho^{n+2}\ldots\ldots+\rho^{m-1}\right)G(gx_0,gx_1,gx_1)$$
$$\leq \frac{\rho^n}{1-\rho}G(gx_0,gx_1,gx_1),$$

and so $\lim G\left(gx_n,gx_m,gx_m\right)=0$, as $n,m\to\infty$.

Thus $\{gx_n\}$ is a G-Cauchy sequence. Since (X,G) is complete, there exists $q\in X$ such that $\{gx_n\}$ is G-Convergent to $q\in g(X)$. So there exists $p\in X$ such that $gp=q$. We will show that $gp=fp$. Let $gp\neq fp$. By (1), we have

$$G(gx_n,fp,fp)=G(fx_{n-1},fp,fp)$$
$$\leq\left(\frac{G(fx_{n-1},gp,gp)+G(gx_{n-1},fp,gp)+G(gx_{n-1},gp,fp)}{2G(gx_{n-1},fx_{n-1},fx_{n-1})+G(gp,fp,fp)+G(gp,fp,fp)+1}\right)G(gx_{n-1},gp,gp).$$

Taking limit as $n\to\infty$, and since G is continuous, we have $G(gp,fp,fp)=0$ and $gp=fp$. If p' is another coincidence point of f and g, then

$$\begin{aligned}G(gp,gp,gp')&=G(fp,fp,fp')\\&\leq\left(\frac{G(fp,gp,gp')+G(gp,fp,gp')+G(gp,gp,fp')}{2G(fp,fp,fp')+G(gp,fp,fp)+G(gp',fp',fp')+1}\right)G(gp,gp,gp')\\&\leq\left[G(gp,gp,gp')+G(gp,gp,gp')+G(gp,gp,gp')\right]G(gp,gp,gp')\\&=3G(gp,gp,gp')^2\end{aligned}$$

giving

$$G(gp,gp,gp')\geq\frac{1}{3}.$$

□

Remark 2.2. The maps f and g defined in Theorem 2.1 belong to the category of so called weakly Picard operators, as the uniqueness of coincidence point is not guaranteed. Further Theorem 2.1 can also be proved for non symmetric G-complete G-metric space.

Example 2.3. Let $X=[0,1]$ and define $f,g:X\to X$ by

$$fx=\begin{cases}\frac{1}{2}, & x=1,\\ 0, & otherwise\end{cases}\qquad\text{and}\qquad g(x)=\tfrac{x}{2}.$$

Then $g(x)\in\left[0,\frac{1}{2}\right]$ and $f(X)\subset g(X)$ and g(X) is a complete.

Further x = 0 and x = 1 are two coincidence points of f and g. Define $G : X^3 \to [0,\infty)$ as

$$G\left(0,0,\frac{1}{2}\right) = 2 = G\left(o,\frac{1}{2},\frac{1}{2}\right),$$

$$G(1,0,0) = G(0,1,1) = 3,$$

$$G(x,x,x) = 0 \quad for\ all\ x \in X.$$

We will consider only two cases $G(f0, f0, f1)$ and $G(f0, f1, f1)$ as other cases are straight forward.

Case I. Consider

$$\begin{aligned} 2 = G(0,0,\frac{1}{2}) = G(f0,f0,f1) &\leq \left(\frac{G(0,0,\frac{1}{2})+G(0,0,\frac{1}{2})+G(0,0,\frac{1}{2})}{2G(0,0,0)+G(0,0,0)+G(\frac{1}{2},\frac{1}{2},\frac{1}{2})+1}\right)G(0,0,\frac{1}{2}) \\ &= 3G(0,0,\frac{1}{2})G(0,0,\frac{1}{2}) \\ &= 12. \end{aligned}$$

Case II.

$$\begin{aligned} 2 = G(0,\frac{1}{2},\frac{1}{2}) = G(f0,f1,f1) &\leq \left(\frac{G(0,\frac{1}{2},\frac{1}{2})+G(0,\frac{1}{2},\frac{1}{2})+G(0,\frac{1}{2},\frac{1}{2})}{2G(0,0,0)+G(\frac{1}{2},\frac{1}{2},\frac{1}{2})+G(\frac{1}{2},\frac{1}{2},\frac{1}{2})+1}\right)G(0,\frac{1}{2},\frac{1}{2}) \\ &= 3[G(0,0,\frac{1}{2})]^2 \\ &= 12. \end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied. Further $\{0,1\}$ are two distinct coincident points of f and g, and $G(1,0,0) = G(0,1,1) = 3 \geq \frac{1}{3}$.

By setting g to be an identity function in Theorem 2.1, we get immediately the following:

Corollary 2.4. [2, Theorem 2.1] *Let (X,G) be a symmetric G-complete G-metric space and $f : X \to X$ satisfying*

$$G(fx, fy, fz) \leq \left(\frac{G(fx,y,z)+G(x,fy,z)+G(x,y,fz)}{2G(x,fx,fx)+G(y,fy,fy)+G(z,fz,fz)+1}\right)G(x,y,z)$$

for all $x,y,z \in X$. Then

(i) *f has atleast one fixed point $p \in X$;*

(ii) *for any $x \in X$, the sequence $\{f^n x\}$ G-converges to a fixed point.*

(iii) *if $p, p^{'} \in X$ are two distinct fixed points, then $G(p, p', p') = G(p', p', p) \geq \frac{1}{3}$.*

It is to be noted that we have derived Corollary 2.4 with a different approach than Gaba [2].

Remark 2.5. In order to validate Corollary 2.4, Gaba [2] considered the space $X = \{0, \frac{1}{2}, 1\}$ and $G : X^3 \to [0, \infty)$. The space (X, G) is not a G-complete G-metric space and thus the purpose of [2, Example 2.2] is forfeited.

The following result guarantees the existence of unique fixed point for weakly compatible mappings.

Theorem 2.6. *Let (X, G) be a symmetric G-complete G-metric space and $f, g \colon X \to X$ satisfying*

(2)
$$G(fx, fy, fz) \leq \alpha \left[\frac{\min\{G(gy, fy, fy), G(gz, fz, fz)\}[1 + G(gx, fx, fx)]}{[1 + G(gx, gy, gz)]} \right] + \beta G(gx, gy, gz)$$

for all $x, y, z \in X$ where α and β are non negative reals with $\alpha + \beta < 1$. If $f(X) \subset g(X)$ and $g(X)$ is complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let f and g satisfy the condition (2) and let x_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, there is $x_1 \in X$ such that $gx_1 = fx_0$. Continuing the same process, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence. Let $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
&G(gx_n, gx_{n+1}, gx_{n+1}) = G(fx_{n-1}, fx_n, fx_n) \\
&\leq \alpha \left[\frac{\min\{G(gx_n, fx_n, fx_n), G(gx_n, fx_n, fx_n)\}[1 + G(gx_{n-1}, fx_{n-1}, fx_{n-1})]}{1 + G(gx_{n-1}, gx_n, gx_n)} \right] \\
&+ \beta G(gx_{n-1}, gx_n, gx_n) \\
&= \alpha \left[\frac{\min\{G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_n, gx_{n+1}, gx_{n+1})\}[1 + G(gx_{n-1}, gx_n, gx_n)]}{1 + G(gx_{n-1}, gx_n, gx_n)} \right] \\
&+ \beta G(gx_{n-1}, gx_n, gx_n)
\end{aligned}$$

$$= \alpha \left[\frac{G(gx_n, gx_{n+1}, gx_{n+1})[1 + G(gx_{n-1}, gx_n, gx_n)]}{1 + G(gx_{n-1}, gx_n, gx_n)} \right] + \beta G(gx_{n-1}, gx_n, gx_n)$$

$$= \alpha\ G(gx_n, gx_{n+1}, gx_{n+1}) + \beta G(gx_{n-1}, gx_n, gx_n),$$

and so

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &\leq \left(\frac{\beta}{1-\alpha}\right) G(gx_{n-1}, gx_n, gx_n) \\ &= \rho\ G(gx_{n-1}, gx_n, gx_n), \end{aligned}$$

where

$$\rho = \frac{\beta}{1-\alpha} < 1.$$

The similar arguments as of Theorem 2.1 yield $gp = fp$.

Claim: f and g have a unique point of coincidence. Let $fq = gq$ for some $q \in X$. Using (2), it follows that

$$\begin{aligned} G(gp, gp, gq) &= G(fp, fp, fq) \\ &\leq \alpha \left[\frac{\min\{G(gp, fp, fp), G(gq, fq, fq)\}[1 + G(gp, fp, fp)]}{1 + G(gp, gp, gq)} \right] + \beta G(gp, gp, gq) \\ &= \alpha \left[\frac{\min\{G(gp, gp, gp), G(gq, fq, fq)\}[1 + G(gp, gp, gp)}{1 + G(gp, gp, gq)} \right] + \beta G(gp, gp, gq) \\ &= \beta G(gp, gp, gq), \end{aligned}$$

which is a contradiction since $\beta < 1$, proving our claim. Therefore $gp = gq$. This gives that f and g have a unique point of coincidence and Proposition 1.10 makes us to go through. □

Example 2.7. Let $X = [0, 2]$, $G(x, y, z) = \max\{|x-y|, |y-z|, |x-z|\}$.
Define $f, g \colon X \to X$ by

$$fx = 1 \quad \textit{and} \quad gx = 2 - x.$$

The use of (2) makes

$$\begin{aligned} 0 &= G(fx, fy, fz) \\ &\leq \alpha \left[\frac{\min\{G(gy, fy, fy), G(gz, fz, fz)\}[1 + G(gx, fx, fx)]}{1 + G(gx, gy, gz)} \right] + \beta G(gx, gy, gz) \end{aligned}$$

$$\leq \alpha\,[\min\{|gy-1|,|gz-1|\}[1+|gx-1|]]+\beta\,[max\{|gx-gy|,|gy-gz|,|gz-gx|\}]$$

$$\leq 2\alpha\ +\beta$$

$$<\alpha+1$$

which is always true. Thus all the conditions of Theorem 2.6 are satisfied and consequently f and g have a unique common fixed point, indeed, $x=1$.

By setting g to be an identity function in Theorem 2.6, we immediately have

Corollary 2.8. [2, Theorem 2.4] *Let (X,G) be a symmetric G-complete G-metric space and $f\colon X\to X$ satisfying*

$$G(fx,fy,fz)\leq \alpha\left[\frac{\min\{G(y,fy,fy),G(z,fz,fz)\}1+G(x,fx,fx)}{[1+G(x,y,z)]}\right]+\beta G(x,y,z)$$

for all $x,y,z\in X$ where α and β are non negative reals with

$$\alpha+\beta<1.$$

Then f has a fixed point in X.

The following two results are generalization of Theorem 2.6.

Theorem 2.9. *Let (X,G) be a symmetric G-complete G-metric space and and $f,g\colon X\to X$ be mappings satisfying*

$$\begin{aligned}G(fx,fy,fz)\leq a_1&\left[\frac{G(gy,fy,fy)[1+G(gx,fx,fx)]}{1+G(gx,gy,gz)}\right]\\&+a_2\left[\frac{G(gz,fz,fz)[1+G(gx,fx,fx)]}{1+G(gx,gy,gz)}\right]\\&+a_3[G(gx,gy,gz)]\end{aligned}\tag{3}$$

for all $x,y,z\in X$ where $a_i=a_i\,(x,y,z)\,,i=1,2,3$ are non-negative functions such that for arbitrary $0<\lambda_1<1$:

$$a_1\,(x,y,z)+a_2\,(x,y,z)+a_3\,(x,y,z)=\sum_{i=1}^{3}a_i\,(x,y,z)\leq\lambda_1.$$

Further if $f(X)\subset g(X)$ and g(X) is complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let f and g satisfy condition (3) and let x_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, there is $x_1 \in X$ such that $gx_1 = fx_0$. Continuing the same process, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence. Let $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
G(gx_n, gx_{n+1}, gx_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\
&\leq a_1 \left[\frac{G(gx_n, fx_n, fx_n)[1 + G(gx_{n-1}, fx_{n-1}, fx_{n-1})]}{1 + G(gx_{n-1}, gx_n, gx_n)} \right] \\
&+ a_2 \left[\frac{G(gx_n, fx_n, fx_n)[1 + G(gx_{n-1}, fx_{n-1}, fx_{n-1})]}{1 + G(gx_{n-1}, gx_n, gx_n)} \right] \\
&+ a_3[G(gx_{n-1}, gx_n, gx_n)] \\
&= a_1 \left[\frac{G(gx_n, gx_{n+1}, gx_{n+1})[1 + G(gx_{n-1}, gx_n, gx_n)]}{1 + G(gx_{n-1}, gx_n, gx_n)} \right] \\
&+ a_2 \left[\frac{G(gx_n, gx_{n+1}, gx_{n+1})[1 + G(gx_{n-1}, gx_n, gx_n)]}{1 + G(gx_{n-1}, gx_n, gx_n)} \right] \\
&+ a_3[G(gx_{n-1}, gx_n, gx_n)] \\
&\leq a_1 G(gx_n, gx_{n+1}, gx_{n+1}) + a_2 G(gx_n, gx_{n+1}, gx_{n+1}) \\
&+ a_3 G(gx_{n-1}, gx_n, gx_n).
\end{aligned}$$

Therefore

$$\begin{aligned}
G(gx_n, gx_{n+1}, gx_{n+1}) &\leq \frac{a_3}{1 - (a_1 + a_2)} G(gx_{n-1}, gx_n, gx_n) \\
&= \rho G(gx_{n-1}, gx_n, gx_n),
\end{aligned}$$

where

$$\frac{a_3}{1 - (a_1 + a_2)} = \rho < 1,$$

since $a_1 + a_2 + a_3 < 1$. As usual procedure, uniqueness of fixed point can be established. □

Letting g be an identity function in Theorem 2.9, we get the following

Corollary 2.10. [2, Theorem 2.5] *Let (X,G) be a symmetric G-complete G-metric space and and $f : X \to X$ be mapping satisfying*

$$G(fx, fy, fz) \le a_1 \left[\frac{G(y, fy, fy)[1 + G(x, fx, fx)]}{1 + G(x, y, z)} \right] + a_2 \left[\frac{G(z, fz, fz)[1 + G(gx, fx, fx)]}{1 + G(x, y, z)} \right] + a_3[G(x, y, z)]$$

for all $x, y, z \in X$ where $a_i = a_i(x, y, z), i = 1, 2, 3$ are non-negative functions such that for arbitrary $0 < \lambda_1 < 1$:

$$a_1(x, y, z) + a_2(x, y, z) + a_3(x, y, z) = \sum_{i=1}^{3} a_i(x, y, z) \le \lambda_1.$$

Then f has a fixed point in X.

Theorem 2.11. *Let (X, G) be a symmetric G-complete G-metric space. Suppose f and g satisfy the following condition:*

$$\begin{aligned} G(fx, fy, fz) \le\ & a_1 G(gx, gy, gz) \\ & + a_2[G(gx, fx, fx) + G(gy, fy, fy) + G(gz, fz, fz)] \\ & + a_3[G(fx, gy, gz) + G(gx, fy, gz) + G(gx, gy, fz)] \\ & + a_4 \min\{G(gy, fy, fy), G(gz, fz, fz)\}[1 + G(gx, fx, fx)][1 + G(gx, gy, gz)]^{-1} \\ & + a_5 G(fx, gy, gz)[1 + G(gx, fy, gz) + G(gx, gy, fz)][1 + G(gx, gy, gz)]^{-1} \\ & + a_6 G(gx, gy, gz)[1 + G(gx, fx, fx) + G(fx, gy, gz)][1 + G(gx, gy, gz)]^{-1} \\ & + a_7 G(fx, gy, gz) \end{aligned} \tag{4}$$

for all $x, y, z \in X$ where $a_i = a_i(x, y, z), i = 1, ..., 7$, are non-negative functions such that for arbitrary $0 < \lambda_1 < \frac{1}{2}$

$$a_1(x, y, z) + 3a_2(x, y, z) + 4a_3(x, y, z) + a_4(x, y, z) + a_6(x, y, z) \le \lambda_1.$$

Further if $f(X) \subset g(X)$ and g(X) is complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible then f and g have a unique common fixed point.

Proof. Let f and g satisfy the condition (4) and let x_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, there is $x_1 \in X$ such that $gx_1 = fx_0$. Continuing the same process, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence. Let $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, we obtain that

$$G(gx_n, gx_{n+1}, gx_{n+1}) = G(fx_{n-1}, fx_n, fx_n)$$

$$\leq a_1 G(gx_{n-1}, gx_n, gx_n)$$

$$+ a_2[G(gx_{n-1}, fx_{n-1}, fx_{n-1}) + G(gx_n, fx_n, fx_n) + G(gx_n, fx_n, fx_n)]$$

$$+ a_3[G(fx_{n-1}, gx_n, gx_n) + G(gx_{n-1}, fx_n, gx_n) + G(gx_{n-1}, gx_n, fx_n)]$$

$$+ a_4 \frac{\min\{G(gx_n, fx_n, fx_n), G(gx_n, fx_n, fx_n)\}[1 + G(gx_{n-1}, fx_{n-1}, fx_{n-1})]}{1 + G(gx_{n-1}, gx_n, gx_n)}$$

$$+ a_5 \frac{G(fx_{n-1}, gx_n, gx_n)[1 + G(x_{n-1}, fx_n, gx_n) + G(gx_{n-1}, gx_n, fx_n)]}{[1 + G(gx_{n-1}, gx_n, gx_n)]}$$

$$+ a_6 \frac{G(gx_{n-1}, gx_n, gx_n)[1 + G(gx_{n-1}, fx_{n-1}, fx_{n-1}) + G(fx_{n-1}, gx_n, gx_n)]}{[1 + G(gx_{n-1}, gx_n, gx_n)]}$$

$$+ a_7 G(fx_{n-1}, gx_n, gx_n)$$

$$= a_1[G(gx_{n-1}, gx_n, gx_n)]$$

$$+ a_2[G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_n, gx_{n+1}, gx_{n+1})]$$

$$+ a_3[G(gx_n, gx_n, gx_n) + G(gx_{n-1}, gx_{n+1}, gx_n) + G(gx_{n-1}, gx_n, gx_{n+1})]$$

$$+ a_4 G(gx_n, gx_{n+1}, gx_{n+1}) \left(\frac{1 + G(gx_{n-1}, gx_n, gx_n)}{1 + G(gx_{n-1}, gx_n, gx_n)} \right)$$

$$+ a_6 \frac{G(gx_{n-1}, gx_n, gx_n)[1 + G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_n, fx_n)]}{[1 + G(gx_{n-1}, gx_n, gx_n)]} + a_7 G(gx_n, gx_n, gx_n)$$

$$= a_1\ [G(gx_{n-1}, gx_n, gx_n)] + a_2[G(gx_{n-1}, gx_n, gx_n) + 2G(gx_n, gx_{n+1}, gx_{n+1})]$$

$$+ a_3[2G(gx_{n-1}, gx_{n+1}, gx_n)] + a_4[G(gx_n, gx_{n+1}, gx_{n+1}) + a_6 G(gx_{n-1}, gx_n, gx_n)$$

$$= a_1[G(gx_{n-1}, gx_n, gx_n)] + a_2[G(gx_{n-1}, gx_n, gx_n)] + 2a_2[G(gx_n, gx_{n+1}, gx_{n+1})]$$

$$+ 2a_3[G(gx_{n-1}, gx_{n+1}, gx_n)] + a_4[G(gx_n, gx_{n+1}, gx_{n+1})] + a_6[G(gx_{n-1}, gx_n, gx_n)]$$

$$= (a_1 + a_2 + 2a_3 + a_6)\ G(gx_{n-1}, gx_n, gx_n) + (2a_2 + 2a_3 + a_4)\ G(gx_n, gx_{n+1}, gx_{n+1})$$

which gives

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \lambda_1 G(gx_{n-1}, gx_n, gx_n) + \lambda_1 G(gx_n, gx_{n+1}, gx_{n+1})$$

and so

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \frac{\lambda_1}{1-\lambda_1} \, [G(gx_{n-1}, gx_n, gx_n)].$$

Since, $0 < \lambda_1 < \frac{1}{2}$, so $0 < \rho = \frac{\lambda_1}{1-\lambda_1} < 1$, we have

$$G\left(gx_n, gx_{n+1}, gx_{n+1}\right) \leq \rho G\left(gx_{n-1}, gx_n, gx_n\right).$$

Now by similar arguments of our previous Theorems, uniqueness of fixed point can be established. □

Example 2.12. Let $X = [0,2]$, $G(x,y,z) = \max\{|x-y|, |y-z|, |x-z|\}$.
The mappings $f, g \colon X \to X$ defined by

$$fx = 1 \quad and \quad gx = 2-x,$$

along with $a_5 = 0 = a_7$ will satisfy the conditions of Theorem 2.11 and $x = 1$ is the unique common fixed point.

Letting g to be an identity function in Theorem 2.11, we obtain

Corollary 2.13. [2, Theorem 2.6] *Let (X,G) be a symmetric G-complete G-metric space where f is an orbitally continuous mapping from X to itself. If it is the case that f satisfies the following condition:*

$$\begin{aligned} G(fx, fy, fz) \leq\; & a_1 G(x,y,z) \\ & + a_2[G(x,fx,fx) + G(y,fy,fy) + G(z,fz,fz)] \\ & + a_3[G(fx,y,z) + G(x,fy,z) + G(x,y,fz)] \\ & + a_4 \min\{(G(y,fy,fy), G(z,fz,fz))\} \frac{[1+G(x,fx,fx)]}{1+G(x,y,z)} \\ & + a_5 G(fx,y,z)[1+G(x,fy,z)+G(x,y,fz)][1+G(x,y,z)]^{-1} \\ & + a_6 G(x,y,z)[1+G(x,fx,fx)+G(fx,y,z)][1+G(x,y,z)]^{-1} \end{aligned}$$

$$+a_7G(fx,y,z) \tag{5}$$

for all $x,y,z \in X$ *where* $a_i := a_i(x,y,z), i = 1,......,7$, *are non-negative functions such that for arbitrary* $0 < \lambda_1 < 1$:

$$a_1(x,y,z)+3a_2(x,y,z)+4a_3(x,y,z)+a_4(x,y,z)+a_6(x,y,z) \leq \lambda_1.$$

Then f *has a fixed point in* X.

Remark 2.14. Please note that we have not imposed any condition of f in Theorem 2.11 where as in Corollary 2.13, f is orbitally continuous.

Remark 2.15. Taking g as an identity map and imposing restrictions on a_i, where $a_i's$ are non-nagative real numbers less than 1, in Theorem 2.11, we can generalize and extend many more results present in the literature of fixed point theory. Further, our results can be proved for non-symmetric G-complete G-metric space.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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