Available online at http://scik.org
Adv. Fixed Point Theory, 2025, 15:19
https://doi.org/10.28919/afpt/9257
ISSN: 1927-6303

VISCOSITY APPROXIMATION OF A COMMON SOLUTION OF FIXED POINT, GENERALIZED EQUILIBRIUM AND CONSTRAINED CONVEX MINIMIZATION PROBLEMS

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Abstract. In this paper, we introduce a viscosity iterative algorithm that approximates a common solution of constrained convex minimization problem, a generalized equilibrium problem involving averaged mapping and fixed point problem of directed nonexpansive mapping. We prove the strong convergence of the proposed iterative algorithm to a common solution that satisfies a variational inequality under some suitable conditions on the parameters. It generalizes the familiar gradient-projection algorithm for convex minimization problem. This result improves and extends some recent results in the literature.

Keywords: generalized equilibrium problem; metric projection; $\alpha-$inverse strongly monotone mapping; directed nonexpansive mapping.

2020 AMS Subject Classification: 47H10, 47J25, 47H09, 65J15.

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Received March 21, 2025

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A mapping $T: C \to C$ is said to be $L-$Lipschitzian mapping if for some $L \geq 0$, $\|Tx - Ty\| \leq L\|x - y\|$, for all $x, y \in C$. If $L = 1$, we say that T is a nonexpansive mapping, and if $0 \leq L < 1$, we say that T is a contraction mapping. We say that T is firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2 \; \forall x, y \in C.$$

We say that T is α-averaged mapping for some $\alpha \in (0,1)$ (see Tian and Liu [24]), if there is a nonexpansive mapping $S: C \to C$ such that $T = (1-\alpha)I + \alpha S$. We denote the fixed point set of T by $F(T)$; that is, $Fix(T) = \{x \in C : Tx = x\}$.

Definition 1.1. *(see Xu* [29], *Browder and Petryshyn* [7]*) A mapping $T: C \to C$ is said to be* ***monotone*** *if*

$$\langle x - y, Tx - Ty \rangle \geq 0 \; for \; all \; x, y \in C;$$

and is called ν-inverse strongly monotone (for short, ν-ism) for some $\nu > 0$, if

$$\langle x - y, Tx - Ty \rangle \geq \nu \|Tx - Ty\|^2 \; for \; all \; x, y \in C.$$

The monotone operators have been widely used to solve practical problems in various fields such as optimization problems, traffic assignment problems, equilibrium problems, radiation therapy, and so on. See [6, 9, 12, 14, 17, 26, 27, 29, 30, 31, 32, 33, 34] and references therein.

Let us discuss about the problems that motivated us to develop approximation techniques. Let $\phi : C \times C \to \mathbb{R}$ be a bi-function. In 1994, Blum and Oettli [3] introduced an equilibrium problem (EP) as the problem of finding $u \in C$ such that

$$\phi(u, v) \geq 0 \text{ for all } v \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(\phi)$. An equilibrium problem theory has motivated the study of problems which arise from image restoration, computer tomography, radiation therapy treatment planning, economics, optimization, etc. In some systems, solutions of equilibrium problems are also solutions of the fixed point problems of a nonlinear mapping. Many researchers looked for common solutions to the equilibrium and fixed point problems of a system. Several authors, have studied existence and approximation of common solutions of equilibrium and fixed point problems based on different relaxed monotonicity notions and various compactness assumptions. To mention some, see Blum and Oettli[3], Bnouhachem [4], Byrne[8, 9], Censor and Elfving [11], Moudafi[18], Zegeye et al. [35], and the references therein.

Many researchers considered a generalized equilibrium problem (GEP) of finding $z \in C$ such that

$$\phi(z,y) + \langle Az, y-z\rangle \geq 0 \text{ for all } y \in C. \tag{1.2}$$

where $A : C \to H$ is a monotone mapping. The set of solutions of (1.2) is denoted by $EP(\phi, A)$; that is, $EP(\phi, A) = \{z \in C : \phi(z,y) + \langle Az, y-z\rangle \geq 0\ \forall y \in C\}$. In the case when $A \equiv 0$, GEP reduces to EP. Numerous problems in physics, variational inequalities, optimization, minimax problems, the Nash equilibrium problem in non cooperative games and economics reduce to finding a solution of the GEP (1.2). See Moudafi and Thèra [17], Moudafi [18, 19], Xu [29], Yazdi [30], Yazdi and Sababe [34], and the references therein.

The second problem of our interest is a constrained convex minimization problem

$$minimize\{g(x) : x \in C\}, \tag{1.3}$$

where $g : C \to \mathbb{R}$ is a convex function. We denote the set of solutions of the problem (1.3) by U. The widely considered approximation method to solve these problems is the gradient projection algorithm(GPA). If g is (Frechet) differentiable, then the GPA generates a sequence x_n via the following recursive formula:

$$x_{n+1} = P_C(x_n - \lambda_n \nabla g(x_n)) \text{ for all } n \geq 0, \tag{1.4}$$

where $x_0 \in C$ is an arbitrarily initial guess and the parameter λ_n are positive real numbers satisfying certain condition. The convergence of the algorithm in (1.4) depends on the behavior of the gradient ∇g.

In 2010, Xu [28] proved the following

Theorem 1.2. *If $g : C \to \mathbb{R}$ is a continuously differentiable convex function such that the gradient ∇g is Lipschitz continuous with Lipschitz constant $L > 0$, and if the constrained convex* *minimization problem is consistent, then for each* $\lambda \in (0, \frac{2}{L})$, *the sequence* $\{x_n\}$ *generated by the gradient-projection algorithm (1.4) converges weakly to a solution of (1.3).*

In 2011, Xu [29] proposed an explicit operator-oriented approach to the algorithm (1.4) using the concept of an averaged mapping. He gave his averaged mapping approach to the GPA (1.4) and the relaxed gradient-projection algorithm. Moreover, he constructed a counter example which showed that the algorithm (1.4) does not converge in norm in an infinite-dimensional space and also presented two modifications of GPA which were shown to have strong convergence (see Xu [26, 27, 29]).

Many mathematicians in the field discussed approximation of a common solution for the three problems: fixed point problem for nonlinear mappings, generalized equilibrium problem, and constrained convex minimization problems. Some of them considered approximation of a common solution for combination of any two of them. Also many authors tried to develop approximation techniques for individual problems by studying the characteristics of each problem. See Combettes and Hiristoaga [12], Jung [15], Peng and Yao [21], Plubtieg and Punpaeng [22], Razani and Yazdi [23], Wang et al. [25], Yazdi [30], and their citations.

Let us discuss some of these results that are in line with our point of interest in this paper. In 2007, Plibtieng and Punpaneng [22] introduced an iterative scheme for finding a common element of the set of solutions of (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$
\begin{cases} \phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S u_n, \quad n \geq 1, \end{cases} \tag{1.5}
$$

where $\phi : H \times H \to \mathbb{R}$ is a bi-function, A is strongly positive bounded linear operator on H, S is a nonexpansive self-mapping of H such that $Fix(S) \cap EP(\phi) \neq \emptyset$, f is a contraction, $\gamma > 0$ is a constant, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$. They proved that the sequence $\{x_n\}$ defined in (1.5) converges strongly to the unique solution of a certain variational inequality. In 2010, Wang et al. [25] introduced the following composite iterative Scheme:

$$
\begin{cases} \phi(u_n,y)+\dfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0 \quad \mathit{for\ all}\ y\in H, \\ y_n=\alpha_n\gamma f(x_n)+(I-\alpha_n A)T_n u_n, \\ x_{n+1}=(1-\beta_n)y_n+\beta_n T_n y_n \qquad n\geq 1, \end{cases} \tag{1.6}
$$

where $\phi : H\times H\to\mathbb{R}$ is a bi-function, A is strongly positive bounded linear operator on H, f is a contraction, $\{T_n\}$ is a countable family of nonexpansive self-mappings of H such that $\bigcap_{n=1}^{\infty} Fix(T_n)\cap EP(\phi)\neq\emptyset$, $\gamma>0$ is some constant, $x_1\in H$, $\{\alpha_n\},\{\beta_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$. By imposing some strict conditions on the parameters, they proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a point in $\bigcap_{n=1}^{\infty} Fix(T_n)\cap EP(\phi)\neq\emptyset$.

In 2012, Tian and Liu [24] studied the following explicit composite iterative scheme by the viscosity approximation method for finding the common solution of an equilibrium problem and a constrained convex minimization problem:

$$
\begin{cases} \phi(u_n,y)+\dfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0 \quad \mathit{for\ all}\ y\in C, \\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)T_n u_n, \qquad n\geq 1, \end{cases} \tag{1.7}
$$

where $\phi : C\times C\to\mathbb{R}$ is a bi-function, ∇g is an L-Lipschitzian mapping with $L\geq 0$ such that $U\cap EP(\phi)\neq\emptyset$, f is a contraction, $x_1\in C$, $\{\alpha_n\}\subset(0,1)$, $\{r_n\}\subset(0,\infty)$, $P_C(I-\lambda_n\nabla g)=s_nI+(1-s_n)T_n$, $s_n=\dfrac{2-\lambda_n L}{4}$ and $\{\lambda_n\}\subset(0,\dfrac{2}{L})$. They proved that the sequences $\{u_n\}$ and $\{x_n\}$ defined in (1.7) converge strongly to a point in $U\cap EP(\phi)$ under certain conditions on the parameters.

In 2020, Yazdi [31] introduced the following explicit composite iterative method for finding the common solution of a generalized equilibrium problem and a constrained convex minimization problem:

$$
\begin{cases} \phi(u_n,y)+\dfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle+\langle Ax_n,y-u_n\rangle \geq 0 \quad \text{for all } y\in C, \\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)T_n u_n, \qquad n\geq 1, \end{cases} \tag{1.8}
$$

where $\phi : C\times C\to\mathbb{R}$ is a bi-function, ∇g is an L-Lipschitzian mapping with $L\geq 0$ such that $U\cap EP(\phi,A)\neq\emptyset$, $f : C\to C$ is a contraction with the constant $k\in[0,1)$ and $A : C\to C$ is an

$\alpha-ism$ mapping, $x_1 \in C$, $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset [a,b] \subset (0,2\alpha)$, $P_C(I-\lambda_n \nabla g) = s_n I + (1-s_n)T_n$, $s_n = \frac{2-\lambda_n L}{4}$ and $\{\lambda_n\} \subset (0, \frac{2}{L})$. The author proved that the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.8) converge strongly to $q \in U \cap EP(\phi, A)$ under certain conditions, and showed that q solves certain variational inequality.

In 2024, Yazdi and Sababe [34] proposed the two-layer iteration process defined as

$$\begin{cases} x_1 \in C, \\ \phi(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0 & \text{for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tu_n, & n \geq 1, \end{cases} \tag{1.9}$$

where $T : C \to C$ is an $\alpha-$strongly quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero, $\phi : C \times C \to \mathbb{R}$ is a bi-function such that $Fix(T) \cap EP(\phi) \neq \emptyset$, $f : C \to C$ is a contraction with the constant $k \in [0,1,)$, $\{\alpha_n\} \subset [0,1]$, and $\{r_n\} \subset (0,\infty)$. The authors proved that the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.9) converge strongly to $q \in Fix(T) \cap EP(\phi)$ under certain conditions on the parameters.

Motivated and inspired by the above results, we propose a viscosity iterative scheme to approximate a common solution of fixed point problem of directed nonexpansive mappings, a generalized equilibrium problem and a constrained convex minimization problem. Then, we prove a strong convergence theorem which improves and extends recent results in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|\cdot\|$. Weak and strong convergences are denoted by $\rightharpoonup$ and $\to$, respectively. We have the following well known facts from the definition of norm and inner product on Hilbert spaces:

Lemma 2.1 (Khamsi and Kirk [16]). *Let H be a real Hilbert space. Then for every $x, y \in H$ and $\lambda \in (0,1)$*

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.3}$$

Let C be a nonempty closed convex subset of H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, satisfying

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}. \tag{2.4}$$

We say that P_C is a metric projection of H onto C. Some of useful properties of projections are gathered in the lemma below.

Lemma 2.2. *(Khamsi and Kirk* [16]*, Cai et al.* [10]*)*

(a) For $x \in H$ and $z \in C$,

$$z = P_C(x) \iff \langle x - z, z - y\rangle \geq 0 \text{ for all } y \in C. \tag{2.5}$$

(b) P_C is a firmly nonexpansive mapping; that is,

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle \text{ for all } x, y \in H. \tag{2.6}$$

(c) P_C satisfies,

$$\|x - P_C(x)\|^2 \leq \|x - y\|^2 - \|y - P_C(x)\|^2 \text{ for all } x \in H,\ y \in C. \tag{2.7}$$

The following lemmas are key in proving our main results.

Lemma 2.3 (Goebel and Kirk [13])**.** *Let H be a real Hilbert space, C be a closed convex subset of H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C that converges weakly to x, and if $\{x_n - Tx_n\}$ converges strongly to y, then $(I - T)x = y$.*

Definition 2.4 (Blum and Oettli [3])**.** *Let C be a nonempty closed convex subset of H. A bifunction $\phi : C \times C \to \mathbb{R}$ is said to satisfy "**Condition A**" if the following four conditions hold:*

(A_1) *$\phi(x,x) = 0$ for all $x \in C$;*

(A_2) *ϕ is monotone; that is, $\phi(x,y) + \phi(y,x) \leq 0$ for all $x, y \in C$;*

(A_3) *for each $x, y, z \in C$,*

$$\lim_{t \downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x,y);$$

(A_4) *for each $x \in C$, $y \mapsto \phi(x,y)$ is convex and weakly lower semi continuous.*

Lemma 2.5 (Blum and Oettli [3])**.** *Let C be a nonempty closed convex subset of H and ϕ : $C \times C \to \mathbb{R}$ be a bi-function satisfying **Condition A**. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \; for\ all\ y \in C. \tag{2.8}$$

Lemma 2.6 (Combettes and Hirstoaga[12])**.** *Assume that $\phi : C \times C \to \mathbb{R}$ satisfies **Condition A**. For $r > 0$ and $x \in H$, define a mapping $Q_r : H \to C$ as follows:*

$$Q_r x = \{z \in C : \phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \; for\ all\ y \ \in C\}.$$

Then, the following hold:

(i) *Q_r is single-valued;*

(ii) *Q_r is firmly nonexpansive; that is, for all $x, y \in H$*

$$\|Q_r x - Q_r y\|^2 \leq \langle Q_r x - Q_r y, x - y\rangle;$$

(iii) *$F(Q_r) = EP(\phi)$;*

(iv) *$EP(\phi)$ is closed and convex.*

Lemma 2.7 (Brezis [5], Byrne [9], Xu [29])**.** *Let $S, T, V, T_1, T_2 : H \to H$ be given mappings.*

(i) *T is nonexpansive if and only if $I - T$ is $\frac{1}{2}$-ism.*

(ii) *If T is ν-ism, then for every $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*

(iii) *T is averaged if and only if $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. In fact, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

(iv) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is averaged and V is nonexpansive , then T is averaged.*

(v) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(vi) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(vii) *If T_1 is α_1-averaged, and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite $T_1 T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

Lemma 2.8 (Xu [28])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $g : H \to \mathbb{R}$ is a convex function whose gradient ∇g is an L-Lipschitzian mapping with $L \geq 0$. Assume that the constrained convex minimization problem in (1.3) is consistent. Then*

(i) ∇g *is an* $\frac{1}{L}-$*inverse strongly monotone mapping (shortly* $\frac{1}{L}$*-ism).*

(ii) For $\lambda > 0$*, the mapping* $I - \lambda \nabla g$ *is* $\frac{\lambda L}{2}-$*averaged .*

(iii) The composite $P_C(I - \lambda \nabla g)$ *is* $(\frac{2+\lambda L}{4})$*-avereged for* $0 < \lambda < \frac{2}{L}$.

(iv) $x^* \in C$ *solves the minimization problem (1.3) if and only if* $x^* \in C$ *solves the fixed point equation*

$$x^* = P_C(I - \lambda \nabla g)x^*,$$

where $\lambda > 0$ *is any fixed positive number.*

Lemma 2.9 (Yazdi [31])**.** *Suppose C is a nonempty closed convex subset of a real Hilbert space H, A is an α-inverse-strongly monotone mapping on C and $0 < r < 2\alpha$. Then, $I - rA$ is a nonexpansive mapping.*

Lemma 2.10. *(Yazdi* [31]*) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\phi : C \times C \to \mathbb{R}$ be bi-function satisfying the conditions* ***Condition A*** *and B be an α-inverse-strongly monotone mapping. Suppose $\{x_n\}$ is a bounded sequence in C and $\{r_n\} \subset [a,b] \subset (0,2\alpha)$ is a real sequence. If $u_n = Q_{r_n}(x_n - r_n B x_n)$, then*

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n|,$$

where $M = sup\{\|Bx_n\| + \frac{1}{a}\|u_{n+1} - (x_{n+1} - r_{n+1}Bx_{n+1})\| : n \in \mathbb{N}\}$.

Lemma 2.11 (Aoyama et al. [1])**.** *Assume that $\{a_n\} \subseteq [0,\infty)$, $\{\gamma_n\} \subseteq [0,1]$, $\{\mu_n\} \subseteq [0,\infty)$ and $\{\upsilon_n\} \subseteq \mathbb{R}$ such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \upsilon_n + \mu_n.$$

Then the conditions $\sum_{n=1}^{\infty} \gamma_n = \infty$, $\limsup_{n\to\infty} \upsilon_n \leq 0$, *and* $\sum_{n=1}^{\infty} \mu_n < \infty$, *imply*

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 2.12 (Naidu and Sangago [20])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H, let Π_C denote the family of all contraction self mappings of C and suppose $T : C \to C$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$. Then there is a unique mapping $\triangle : \Pi_C \to Fix(T)$ such that*

$$\limsup_{n\to\infty} \langle (I-f)\triangle(f), \triangle(f) - x_n \rangle \leq 0,$$

for any given $f \in \Pi_C$ and a bounded approximate fixed point sequence $\{x_n\}$ of T in C.

3. Main Results

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. We introduce the following definition.

Definition 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be a* ***directed nonexpansive*** *mapping if it satisfies the following three conditions:*

(DNE1) *T is a nonexpansive mapping;*

(DNE2) *$Fix(T) \neq \emptyset$; and*

(DNE3) *$\|Tx - p\|^2 \leq \|x - p\|^2 - \|x - Tx\|^2$ for every $x \in C$, for every $p \in Fix(T)$.*

Obviously, every firmly nonexpansive mapping with nonempty fixed point set is a directed nonexpansive mapping. There are nonexpansive mappings that are not directed nonexpansive. For instance the mapping $T : B_H \to B_H$ (where $B_H = \{x \in H : \|x\| \leq 1\}$) defined by $Tx = -x$ is nonexpansive, but not directed nonexpansive mapping. Lemma 2.2 and Lemma 2.6 imply that the operators P_C and Q_r, respectively, are directed nonexpansive mappings.

Throughout this section we use the following assumptions.

(B1) Assume that $g : C \to \mathbb{R}$ is a real-valued convex function whose ∇g is a Lipschitzian mapping with Lipschitz constant $L > 0$. The solution set of the minimization problem $\min\{g(x) : x \in C\}$ is denoted by U; that is,

$$U = \{z \in C : \; g(z) = \min_{y\in C} g(y)\}. \tag{3.1}$$

Let $\{\lambda_n\}$ be a sequence of positive real numbers in $(0, \frac{2}{L})$ such that

$$P_C(I - \lambda_n \nabla g) = (\frac{2 - \lambda_n L}{4})I + (\frac{2 + \lambda_n L}{4})T_n = \gamma_n I + (1 - \gamma_n)T_n \tag{3.2}$$

where $T_n : C \to C$ is nonexpansive, and $\gamma_n = \frac{2 - \lambda_n L}{4}$. Assume that $\lim_{n\to\infty} \gamma_n = 0$ (or alternatively, $\lim_{n\to\infty} \lambda_n = \frac{2}{L}$) and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

(B2) Let $\phi : C \times C \to \mathbb{R}$ be a bi-function satisfying **Condition A**, $B : C \to C$ an α-ism mapping. The solution set of the generalized equilibrium problem is denoted by $EP(\phi, B)$; that is,

$$EP(\phi, B) = \{z \in C : \phi(z, y) + \langle y - z, Bz \rangle \geq 0 \; for \; all \; y \in C\}. \tag{3.3}$$

(B3) Let $f : C \to C$ be a contraction mapping with contraction constant $k \in [0, 1)$.

(B4) Let $S : C \to C$ be a directed nonexpansive mapping with fixed point set $Fix(S)$.

(B5) $\Sigma = U \cap EP(\phi, B) \cap Fix(S) \neq \emptyset$.

Now we introduce a viscosity approximation scheme for finding a common solution of the fixed point problem of directed nonexpansive mapping S, the generalized equilibrium problem (3.3) and the constrained convex minimization problem (3.1).

$$\begin{cases} x_1 \in C, \\ \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle + \langle y - u_n, Bx_n \rangle \geq 0 \text{ for all } y \in C, \\ v_n = \alpha_n x_n + (1 - \alpha_n)T_n u_n \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)Sv_n, \; n \geq 1. \end{cases} \tag{3.4}$$

Now we state and prove a strong convergence theorem with some conditions on the parameters.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let ϕ, g, ∇g, f, B, T_n and S be as defined in the assumptions (B1)-(B5). Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are real sequences satisfying the following conditions:*

$$\{\alpha_n\} \subset (0, 1], \lim_{n\to\infty} \alpha_n = 0, \; \sum_{n=1}^{\infty} \alpha_n = \infty \; \text{ and } \; \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty; \tag{3.5}$$

$$\{\beta_n\} \subset (0,1], \lim_{n\to\infty} \beta_n = 0, \ \sum_{n=1}^{\infty} \beta_n = \infty \ \text{ and } \ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty; \tag{3.6}$$

$$\{r_n\} \subset [a,b] \subset (0,2\alpha) \ \text{ and } \ \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \tag{3.7}$$

Then the sequences $\{x_n\}$, $\{v_n\}$ and $\{u_n\}$ defined by (3.4) converge strongly to $q = P_\Sigma f(q)$, which solves the variational inequality:

$$\langle q - f(q), y - q\rangle \geq 0, \ \text{for all } y \in C. \tag{3.8}$$

Proof. We note that $u_n = Q_{r_n}(x_n - r_n B x_n)$ for $n = 1,2,\cdots$. For $p \in \Sigma$, we note that

$$\phi(p,y) + \frac{1}{r_n}\langle y - p, p - (I - r_n B)(p)\rangle = \phi(p,y) + \langle y - p, Bp\rangle \geq 0, \ \forall y \in C;$$

so that $Q_{r_n}(p - r_n Bp) = p$ for each $n = 1,2,3,\cdots$. We also have $P_C(I - \lambda_n \nabla g)p = p$, $T_n p = p$, and $Sp = p$.

Now we prove step by step.

Step 1. The sequences $\{x_n\}$, $\{v_n\}$, and $\{u_n\}$ are bounded.

Let $p \in \Sigma$. It follows from Lemma 2.7, Lemma 2.9 and (3.4) that

$$\begin{aligned}
\|v_n - p\| &\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|T_n u_n - p\| \\
&\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|u_n - p\| \\
&\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|Q_{r_n}(I - r_n B)(x_n) - Q_{r_n}(I - r_n B)(p)\| \\
&\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\leq \|x_n - p\|. \qquad (3.9)
\end{aligned}$$

Thus from (3.4) and (3.9) we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n f(x_n) + (1-\beta_n) S v_n) - p\| \\
&\leq \beta_n \|f(x_n) - f(p)\| + \beta_n \|f(p) - p\| + (1-\beta_n)\|v_n - p\| \\
&\leq k\beta_n \|x_n - p\| + \beta_n \|f(p) - p\| + (1-\beta_n)\|x_n - p\| \\
&= (1 - (1-k)\beta_n)\|x_n - p\| + (1-k)\beta_n \left[\frac{1}{1-k}\|f(p) - p\|\right] \\
&\leq \max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-k}\right\} \qquad (3.10)
\end{aligned}$$

By induction it follows from (3.10) that

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1-k}\right\}. \tag{3.11}$$

Hence $\{x_n\}$ is bounded, and consequently so are $\{f(x_n)\}$, $\{Sx_n\}$, $\{v_n\}$, $\{u_n\}$, $\{Sv_n\}$, $\|P_C(I - \lambda_{n+1}\nabla g)u_n\|$, $\|P_C(I-\lambda_{n+1}\nabla g)x_n\|$ and $\{T_n u_n\}$.

Step 2. We prove that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
Put $M_1 = \sup\{\max\{\|x_n\|, \|v_n\|, \|u_n\|, \|f(x_n)\|, \|T_n u_n\|, \|Sx_n\|, \|Sv_n\|\} : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, it follows from (3.4) that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\beta_{n+1} f(x_{n+1}) + (1-\beta_{n+1})Sv_{n+1} - \beta_n f(x_n) - (1-\beta_n)Sv_n\| \\
&\le (1-\beta_{n+1})\|Sv_{n+1} - Sv_n\| + |\beta_{n+1} - \beta_n|\|Sv_n\| \\
&\quad +\beta_{n+1}\|f(x_{n+1}) - f(x_n)\| + |\beta_{n+1} - \beta_n|\|f(x_n)\| \\
&\le (1-\beta_{n+1})\|v_{n+1} - v_n\| + \beta_{n+1}k\|x_{n+1} - x_n\| + 2M_1|\beta_{n+1} - \beta_n| \qquad (3.12)
\end{aligned}$$

On the other hand, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \|\alpha_{n+1}x_{n+1} + (1-\alpha_{n+1})T_{n+1}u_{n+1} - \alpha_n x_n - (1-\alpha_n)T_n u_n\| \\
&\le \alpha_{n+1}\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|x_n\| \\
&\quad +(1-\alpha_{n+1})\|T_{n+1}u_{n+1} - T_n u_n\| + |\alpha_{n+1} - \alpha_n|\|T_n x_n\| \\
&\le (1-\alpha_{n+1})\|T_{n+1}u_{n+1} - T_n u_n\| + \alpha_{n+1}\|x_{n+1} - x_n\| + 2M|\alpha_{n+1} - \alpha_n| \\
&\le (1-\alpha_{n+1})\left[\|u_{n+1} - u_n\| + \|T_{n+1}u_n - T_n u_n\|\right] \\
&\quad +\alpha_{n+1}\|x_{n+1} - x_n\| + 2M_1|\alpha_{n+1} - \alpha_n| \qquad (3.13)
\end{aligned}$$

It follows from (3.2) that

$$\begin{aligned}
\|T_{n+1}u_n - T_n u_n\| &= \|\frac{P_C(I-\lambda_{n+1}\nabla g) - \gamma_{n+1}I}{1-\gamma_{n+1}}u_n - \frac{P_C(I-\lambda_n\nabla g) - \gamma_n I}{1-\gamma_n}u_n\| \\
&\le \|\frac{4P_C(I-\lambda_{n+1}\nabla g)}{2+\lambda_{n+1}L}u_n - \frac{4P_C(I-\lambda_n\nabla g)}{2+\lambda_n L}u_n\| \\
&\quad +\|\frac{2-\lambda_{n+1}L}{2+\lambda_{n+1}L}u_n - \frac{2-\lambda_n L}{2+\lambda_n L}u_n\|
\end{aligned}$$

$$\begin{aligned}
&= \left\|\frac{4(2+\lambda_n L)P_C(I-\lambda_{n+1}\nabla g)u_n - 4(2+\lambda_{n+1}L)P_C(I-\lambda_n\nabla g)u_n}{(2+\lambda_{n+1}L)(2+\lambda_n L)}\right\| \\
&\quad + \frac{4L|\lambda_{n+1}-\lambda_n|}{(2+\lambda_{n+1}L)(2+\lambda_n L)}\|u_n\| \\
&\leq \frac{4L|\lambda_{n+1}-\lambda_n|\|P_C(I-\lambda_{n+1}\nabla g)u_n\|}{(2+\lambda_{n+1}L)(2+\lambda_n L)} + \frac{4L|\lambda_{n+1}-\lambda_n|}{(2+\lambda_{n+1}L)(2+\lambda_n L)}\|u_n\| \\
&\quad + \frac{4(2+\lambda_{n+1}L)\|P_C(I-\lambda_{n+1}\nabla g)u_n - P_C(I-\lambda_n\nabla g)u_n\|}{(2+\lambda_{n+1}L)*(2+\lambda_n L)} \\
&\leq |\lambda_{n+1}-\lambda_n|[L\|P_C(I-\lambda_{n+1}\nabla g)u_n\| + 4\|\nabla g(u_n)\| + L\|u_n\|] \\
&\leq M_2|\lambda_{n+1}-\lambda_n|, \qquad (3.14)
\end{aligned}$$

where $M_2 = \sup_{n\in\mathbb{N}}[L\|P_C(I-\lambda_{n+1}\nabla g)u_n\| + 4\|\nabla g(u_n)\| + L\|u_n\|]$. From Lemma 2.10 we have

$$\|u_{n+1}-u_n\| \leq \|x_{n+1}-x_n\| + M_3|\gamma_{n+1}-\gamma_n| \qquad (3.15)$$

where $M_3 = \sup\{\|Bx_n\| + \frac{1}{a}\|u_{n+1}-x_{n+1}\| : n\in\mathbb{N}\}$. For $M = \max\{M_1, M_2, M_3\}$, it follows from (3.13), (3.14) and (3.15) that

$$\|v_{n+1}-v_n\| \leq \|x_{n+1}-x_n\| + 2M\left(|\alpha_{n+1}-\alpha_n| + |\lambda_{n+1}-\lambda_n| + |\gamma_{n+1}-\gamma_n|\right). \qquad (3.16)$$

From (3.12) and (3.16) we get

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leq (1-\beta_{n+1}(1-k))\|x_{n+1}-x_n\| + 2M|\beta_{n+1}-\beta_n| \\
&\quad +2M(1-\beta_{n+1})\left(|\alpha_{n+1}-\alpha_n| + (1-\alpha_{n+1})(|\lambda_{n+1}-\lambda_n| + |\gamma_{n+1}-\gamma_n|)\right) \\
&\leq (1-\beta_{n+1}(1-k))\|x_{n+1}-x_n\| + 2M|\beta_{n+1}-\beta_n| \\
&\quad +2M\left(|\alpha_{n+1}-\alpha_n| + |\lambda_{n+1}-\lambda_n| + |\gamma_{n+1}-\gamma_n|\right) \\
&\quad -2M\beta_{n+1}\left(|\alpha_{n+1}-\alpha_n| + (1-\alpha_{n+1})(|\lambda_{n+1}-\lambda_n| + |\gamma_{n+1}-\gamma_n|\right) \\
&\quad +2M\left[|\beta_{n+1}-\beta_n| + |\alpha_{n+1}-\alpha_n| + |\lambda_{n+1}-\lambda_n| + |\gamma_{n+1}-\gamma_n|\right] \\
&\leq (1-a_{n+1})\|x_{n+1}-x_n\| + a_{n+1}\mu_{n+1} + \chi_{n+1}, \qquad (3.17)
\end{aligned}$$

where

$$\begin{aligned}
a_{n+1} &= \beta_{n+1}(1-k), \\
\mu_{n+1} &= \frac{2M}{1-k}\left[|\beta_{n+1}-\beta_n| - |\alpha_{n+1}-\alpha_n| - |\lambda_{n+1}-\lambda_n| - |\gamma_{n+1}-\gamma_n|\right]
\end{aligned}$$

$$\chi_{n+1} = 2M\left[|\beta_{n+1}-\beta_n| + |\alpha_{n+1}-\alpha_n| + |\lambda_{n+1}-\lambda_n| + |\gamma_{n+1}-\gamma_n|\right].$$

Then it follows from Assumptions on (B1), (3.5), (3.6) and (3.7) that $\limsup_{n\to\infty} \mu_n \le 0$, $\sum_{n=1}^{\infty} a_n = (1-k)\sum_{n=1}^{\infty} \beta_n = \infty$, and $\sum_{n=1}^{\infty} \chi_n < \infty$. Thus it follows from (3.17) and Lemma 2.11 that

$$\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0. \tag{3.18}$$

<u>**Step 3**</u>. In this step we prove $\lim_{n\to\infty} \|x_n - u_n\| = 0$.

Let $p \in \Sigma$. Then it follows from Lemma 2.6 that

$$\begin{aligned}
\|u_n - p\|^2 &= \|Q_{r_n}(x_n - r_n Bx_n) - Q_{r_n}(p - r_n Bp)\|^2 \\
&\le \|x_n - r_n Bx_n - p + r_n Bp\|^2 \\
&= \|x_n - p\|^2 + r_n^2\|Bx_n - Bp\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle \\
&\le \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bp\|^2. \qquad (3.19)
\end{aligned}$$

Then it follows from Lemma 2.1, (3.4) and (3.19) that

$$\begin{aligned}
\|v_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)T_n u_n - p\|^2 = \|(T_n u_n - p) + \alpha_n(x_n - T_n u_n)\|^2 \\
&= \|T_n u_n - p\|^2 + \alpha_n^2\|x_n - T_n u_n\|^2 + 2\alpha_n\langle x_n - T_n u_n, T_n u_n - p\rangle \\
&\le \|u_n - p\|^2 + \alpha_n^2\|x_n - T_n u_n\|^2 + 2\alpha_n\|x_n - T_n u_n\|\|T_n u_n - p\| \\
&\le \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bp\|^2 + \alpha_n^2\|x_n - T_n u_n\|^2 \\
&\quad + 2\alpha_n\|x_n - T_n u_n\|\|u_n - p\|. \qquad (3.20)
\end{aligned}$$

We get from (3.4) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n f(x_n) + (1-\beta_n)Sv_n - p\|^2 = \|(Sv_n - p) + \beta_n(f(x_n) - Sv_n)\|^2 \\
&= \|Sv_n - p\|^2 + \beta_n^2\|f(x_n) - Sv_n\|^2 + 2\beta_n\langle Sv_n - p, f(x_n) - Sv_n\rangle \\
&\le \|v_n - p\|^2 + \beta_n^2\|f(x_n) - Sv_n\|^2 + 2\beta_n\|Sv_n - p\|\,\|f(x_n) - Sv_n\| \qquad (3.21)
\end{aligned}$$

It follows from (3.4), (3.20) and (3.21) that

$$r_n(2\alpha - r_n)\|Bx_n - Bp\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2$$

$$
\begin{aligned}
&+\beta_n^2\|f(x_n)-Sv_n\|^2+\alpha_n^2\|x_n-T_nu_n\|^2\\
&+2\beta_n\|T_nu_n-p\|\,\|f(x_n)-Sv_n\|\\
&+2\alpha_n\|x_n-T_nu_n\|\|u_n-p\|\\
\leq\;&(\|x_n-p\|+\|x_{n+1}-p\|)(\|x_{n+1}-x_n\|)\\
&+\beta_n^2\|f(x_n)-Sv_n\|^2+2\beta_n\|T_nu_n-p\|\,\|f(x_n)-Sv_n\|\\
&+\alpha_n^2\|x_n-T_nu_n\|^2+2\alpha_n\|x_n-T_nu_n\|\|u_n-p\|. \qquad (3.22)
\end{aligned}
$$

From (3.5), (3.6), (3.7), (3.18) and (3.22), we get

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0. \qquad (3.23)$$

By Lemma 2.1 and Lemma 2.6 we get

$$
\begin{aligned}
\|u_n-p\|^2 &= \|Q_{r_n}(x_n-r_nBx_n)-Q_{r_n}(p-r_nBp)\|^2\\
&\leq \langle (x_n-r_nBx_n)-(p-r_nBp),u_n-p\rangle\\
&= \frac{1}{2}\left(\|x_n-r_nBx_n-(p-r_nBp)\|^2+\|u_n-p\|^2\right)\\
&\quad -\frac{1}{2}\|(x_n-u_n)+r_n(Bx_n-Bp)\|^2\\
&\leq \frac{1}{2}\|x_n-p\|^2+\frac{1}{2}\|u_n-p\|^2-\frac{1}{2}\|x_n-u_n\|^2\\
&\quad +r_n\langle x_n-u_n,Bx_n-Bp\rangle-\frac{1}{2}r_n^2\|Bx_n-Bp\|^2\\
&\leq \frac{1}{2}\|x_n-p\|^2+\frac{1}{2}\|u_n-p\|^2-\frac{1}{2}\|x_n-u_n\|^2\\
&\quad +r_n\|x_n-u_n\|\,\|Bx_n-Bp\|-\frac{1}{2}r_n^2\|Bx_n-Bp\|^2. \qquad (3.24)
\end{aligned}
$$

It follows from (3.24) that

$$
\begin{aligned}
\|u_n-p\|^2 &\leq \|x_n-p\|^2-\|x_n-u_n\|^2-r_n^2\|Bx_n-Bp\|^2\\
&\quad +2r_n\|x_n-u_n\|\,\|Bx_n-Bp\|. \qquad (3.25)
\end{aligned}
$$

By similar argument we obtain

$$\|v_n-p\|^2 = \|T_nu_n-p\|^2+\alpha_n^2\|x_n-T_nu_n\|^2+2\alpha_n\langle x_n-T_nu_n,T_nu_n-p\rangle$$

$$\leq \|u_n - p\|^2 + \alpha_n^2 \|x_n - T_n u_n\|^2 + 2\alpha_n \|x_n - T_n u_n\| \|u_n - p\|. \tag{3.26}$$

We get from (3.25) and (3.26) that

$$\begin{aligned} \|v_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 - r_n^2 \|Bx_n - Bp\|^2 \\ &\quad + 2r_n \|x_n - u_n\| \|Bx_n - Bp\| + \alpha_n^2 \|x_n - T_n u_n\|^2 \\ &\quad + 2\alpha_n \|x_n - T_n u_n\| \|u_n - p\|. \end{aligned} \tag{3.27}$$

Finally from (3.4) we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\beta_n(f(x_n) - Sv_n) + (Sv_n - p)\|^2 \\ &\leq \|v_n - p\|^2 + \beta_n^2 \|f(x_n) - Sv_n\|^2 + 2\beta_n \|f(x_n) - Sv_n\| \|v_n - p\|. \end{aligned} \tag{3.28}$$

It follows from (3.27) and (3.28) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 - r_n^2 \|Bx_n - Bp\|^2 \\ &\quad + 2r_n \|x_n - u_n\| \|Bx_n - Bp\| + \alpha_n^2 \|x_n - T_n u_n\|^2 \\ &\quad + 2\alpha_n \|x_n - T_n u_n\| \|u_n - p\| + \beta_n^2 \|f(x_n) - Sv_n\|^2 \\ &\quad + 2\beta \|f(x_n) - Sv_n\| \|v_n - p\|. \end{aligned} \tag{3.29}$$

Now we get from (3.29) that

$$\begin{aligned} \|x_n - u_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - r_n^2 \|Bx_n - Bp\|^2 \\ &\quad + 2r_n \|x_n - u_n\| \|Bx_n - Bp\| + \alpha_n^2 \|x_n - T_n u_n\|^2 \\ &\quad + 2\alpha_n \|x_n - T_n u_n\| \|u_n - p\| + \beta_n^2 \|f(x_n) - Sv_n\|^2 \\ &\quad + 2\beta \|f(x_n) - Sv_n\| \|v_n - p\| \\ &\leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_{n+1} - x_n\| - r_n^2 \|Bx_n - Bp\|^2 \\ &\quad + 2r_n \|x_n - u_n\| \|Bx_n - Bp\| + \alpha_n^2 \|x_n - T_n u_n\|^2 \\ &\quad + 2\alpha_n \|x_n - T_n u_n\| \|u_n - p\| + \beta_n^2 \|f(x_n) - Sv_n\|^2 \\ &\quad + 2\beta_n \|f(x_n) - Sv_n\| \|v_n - p\|. \end{aligned} \tag{3.30}$$

From (3.5), (3.6), (3.7), (3.18), (3.23) and (3.30), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.31}$$

Step 4. In this step we prove $\lim_{n\to\infty} \|v_n - Sv_n\| = 0$.

By the $k-$contraction of f and directed nonexpansivity of S, we further derive that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n(f(x_n) - Sv_n) + (Sv_n - p)\|^2 \\
&\leq \|Sv_n - p\|^2 + \beta_n^2\|f(x_n) - Sv_n\|^2 + 2\beta_n \|f(x_n) - Sv_n\| \|v_n - p\| \\
&\leq \|v_n - p\|^2 - \|v_n - Sv_n\|^2 + \beta_n^2\|f(x_n) - Sv_n\|^2 \\
&\quad +2\beta_n \|f(x_n) - Sv_n\| \|v_n - p\|.
\end{aligned} \tag{3.32}$$

It follows from (3.27) and (3.32) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 - \|v_n - Sv_n\|^2 - \|x_n - u_n\|^2 \\
&\quad -r_n^2\|Ax_n - Ap\|^2 + 2r_n \|x_n - u_n\| \|Ax_n - Ap\| \\
&\quad +\alpha_n^2\|x_n - T_n u_n\|^2 + 2\alpha_n\|x_n - T_n u_n\| \|u_n - p\| \\
&\quad +\beta_n^2\|f(x_n) - Sv_n\|^2 + 2\beta_n \|f(x_n) - Sv_n\| \|v_n - p\|.
\end{aligned} \tag{3.33}$$

It follows from (3.33) that

$$\begin{aligned}
\|v_n - Sv_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - \|x_n - u_n\|^2 \\
&\quad -r_n^2\|Ax_n - Ap\|^2 + 2r_n \|x_n - u_n\| \|Ax_n - Ap\| \\
&\quad +\alpha_n^2\|x_n - T_n u_n\|^2 + 2\alpha_n\|x_n - T_n u_n\| \|u_n - p\| \\
&\quad +\beta_n^2\|f(x_n) - Sv_n\|^2 + 2\beta_n \|f(x_n) - Sv_n\| \|v_n - p\| \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_{n+1} - x_n\| - \|x_n - u_n\|^2 \\
&\quad -r_n^2\|Ax_n - Ap\|^2 + 2r_n \|x_n - u_n\| \|Ax_n - Ap\| \\
&\quad +\alpha_n^2\|x_n - T_n u_n\|^2 + 2\alpha_n\|x_n - T_n u_n\| \|u_n - p\| \\
&\quad +\beta_n^2\|f(x_n) - Sv_n\|^2 + 2\beta_n \|f(x_n) - Sv_n\| \|v_n - p\|.
\end{aligned} \tag{3.34}$$

We get from (3.5), (3.6), (3.7), (3.23), Step 2, Step 3 and (3.34) that

$$\lim_{n\to\infty} \|Sv_n - v_n\| = 0. \tag{3.35}$$

<u>**Step 5**</u>. In this step we prove $\lim_{n\to\infty} \|u_n - T_n u_n\| = 0$ and $\lim_{n\to\infty} \|P_C(I - \frac{2}{L}\nabla g)u_n - u_n\| = 0$. From the conditions (3.5), (3.6) and (3.7) on the parameters, we have

$$\lim_{n\to\infty} \|v_n - T_n u_n\| = \lim_{n\to\infty} \alpha_n \|x_n - T_n u_n\| = 0 \tag{3.36}$$

$$\lim_{n\to\infty} \|x_{n+1} - Sv_n\| = \lim_{n\to\infty} \beta_n \|f(x_n) - Sv_n\| = 0, \tag{3.37}$$

Moreover, by the triangle inequality we have

$$\begin{aligned} \|u_n - T_n u_n\| &\leq \|u_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - Sv_n\| \\ &\quad + \|Sv_n - v_n\| + \|v_n - T_n u_n\| \end{aligned} \tag{3.38}$$

Hence it follows from Step 2, Step 3, Step 4, (3.36), (3.37) and (3.38) that

$$\lim_{n\to\infty} \|u_n - T_n u_n\| = 0. \tag{3.39}$$

Now we observe that for $s_n = \dfrac{2 - \lambda_n L}{4}$,

$$\|P_C(I - \lambda_n \nabla g)u_n - u_n\| = \|s_n u_n + (1 - s_n)T_n u_n - u_n\| = (1 - s_n)\|T_n u_n - u_n\|;$$

so that it follows from (3.39) that

$$\lim_{n\to\infty} \|P_C(I - \lambda_n \nabla g)u_n - u_n\| = 0. \tag{3.40}$$

We also observe that

$$\begin{aligned} \|P_C(I - \frac{2}{L}\nabla g)u_n - u_n\| &\leq \|P_C(I - \frac{2}{L}\nabla g)u_n - P_C(I - \lambda_n \nabla g)u_n\| \\ &\quad + \|P_C(I - \lambda_n \nabla g)u_n - u_n\| \\ &\leq \|(I - \frac{2}{L}\nabla g)u_n - (I - \lambda_n \nabla g)u_n\| + \|P_C(I - \lambda_n \nabla g)u_n - u_n\| \\ &\leq (\frac{2}{L} - \lambda_n)\|\nabla g)(u_n)\| + \|T_n u_n - u_n\|. \end{aligned} \tag{3.41}$$

Since $\lim_{n\to\infty} \lambda_n = \dfrac{2}{L}$ and $\lim_{n\to\infty} \|T_n u_n - u_n\| = 0$, we conclude that

$$\lim_{n\to\infty} \|P_C(I - \frac{2}{L}\nabla g)u_n - u_n\| = 0. \tag{3.42}$$

Step 6. For $q = P_\Sigma f(q)$, we prove that

$$\limsup_{n\to\infty}\langle f(q)-q, x_n - q\rangle \le 0. \tag{3.43}$$

Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$ such that $u_{n_i} \rightharpoonup z$ for some $z \in C$ and

$$\limsup_{n\to\infty}\langle q - f(q), q - u_n\rangle = \lim_{i\to\infty}\langle q - f(q), q - u_{n_i}\rangle.$$

Since ∇g is $\frac{1}{L}$-ism, $P_C(I - \frac{2}{L}\nabla g)$ is a nonexpansive self mapping on C. Therefore, from (3.42) and Lemma 2.3, we obtain

$$z = P_C(I - \frac{2}{L}\nabla g)z,$$

and hence $z \in U$. Because $u_n = Q_{r_n}(x_n - r_n A x_n)$, we can write

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle + \langle A x_n, y - u_n\rangle \ge 0 \text{ for all } y \in C.$$

Because ϕ satisfies **Condition A**, we get

$$\langle B x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge \phi(y, u_n) \text{ for all } y \in C.$$

In particular for the subsequence $\{u_{n_i}\}$, we have

$$\langle B x_{n_i}, y - u_{n_i}\rangle + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \ge \phi(y, u_{n_i}) \text{ for all } y \in C. \tag{3.44}$$

For $t \in (0,1]$ and $y \in C$, set $y_t = ty + (1-t)z$. So, from (3.44) we obtain:

$$\begin{aligned}
\langle y_t - u_{n_i}, B y_t\rangle &\ge \langle y_t - u_{n_i}, B y_t\rangle - \langle B x_{n_i}, y_t - u_{n_i}\rangle \\
&\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + \phi(y_t, u_{n_i}) \\
&= \langle y_t - u_{n_i}, B y_t - B u_{n_i}\rangle + \langle y_t - u_{n_i}, B u_{n_i} - B x_{n_i}\rangle \\
&\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + \phi(y_t, u_{n_i}).
\end{aligned} \tag{3.45}$$

Because $\lim_{i\to\infty}\|u_{n_i} - x_{n_i}\| = 0$ and B is monotone, we have

$$\lim_{i\to\infty}\|B u_{n_i} - B x_{n_i}\| = 0. \tag{3.46}$$

$$\langle y_t - u_{n_i}, B y_t - B u_{n_i}\rangle \ge 0 \text{ for all } i = 1, 2, 3, \cdots \tag{3.47}$$

Using (3.46) and (3.47), and letting $i \to \infty$ in (3.45), we get

$$\langle y_t - z, A y_t\rangle \ge \phi(y_t, z). \tag{3.48}$$

By **Condition A** and (3.48) we have

$$
\begin{aligned}
0 &= \phi(y_t, y_t) \le t\phi(y_t, y) + (1-t)\phi(y_t, z) \\
&\le t\phi(y_t, y) + (1-t)\langle y_t - z, By_t\rangle \\
&= t\phi(y_t, y) + (1-t)t\langle y - z, By_t\rangle. \qquad (3.49)
\end{aligned}
$$

Hence we get from (3.49) that

$$0 \le \phi(y_t, y) + (1-t)\langle y - z, By_t\rangle. \qquad (3.50)$$

Letting $t \to 0$ in (3.50) and using **Condition A**, we have

$$0 \le \phi(z, y) + \langle y - z, Bz\rangle.$$

Since y is an arbitrary element of C, we conclude that

$$\phi(z, y) + \langle y - z, Bz\rangle \ge 0 \text{ for all } y \in C; \qquad (3.51)$$

and thus $z \in EP(\phi, B)$.

Since $u_{n_i} \rightharpoonup z$, $\lim_{n\to\infty} \|u_{n_i} - x_{n_i}\| = 0$, and $\lim_{n\to\infty} \|u_{n_i} - T_{n_i}u_{n_i}\| = 0$, we have $x_{n_i} \rightharpoonup z$ and $T_{n_i}u_{n_i} \rightharpoonup z$. So that $v_{n_i} \rightharpoonup z$ as well. By the demiclosedness of S at zero and from (3.35) we conclude that $z \in Fix(S)$. Therefore, $z \in \Sigma$. By Lemma 2.2

$$\langle f(q) - q, z - q\rangle \le 0. \qquad (3.52)$$

It follows from (3.52) that

$$\limsup_{n\to\infty}\langle f(q) - q, u_n - q\rangle = \lim_{i\to\infty}\langle f(q) - q, u_{n_i} - q\rangle = \langle f(q) - q, z - q\rangle \le 0. \qquad (3.53)$$

Step 7. The sequences $\{x_n\}$, $\{v_n\}$ and $\{u_n\}$ converge strongly to $q = P_C f(q)$.

It follows from (3.4) and Lemma 2.1 that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\beta_n f(x_n) + (1-\beta_n)Sv_n - q\|^2 \\
&= \|\beta_n(f(x_n) - f(q)) + \beta_n(f(q) - q) + (1-\beta_n)(Sv_n - q)\|^2 \\
&\le \beta_n^2\|f(x_n) - f(q)\|^2 + (1-\beta_n)^2\|Sv_n - q\|^2 \\
&\quad +2\beta_n\langle f(q) - q, x_{n+1} - q\rangle
\end{aligned}
$$

$$\leq \beta_n^2 k^2 \|x_n - q\|^2 + (1-\beta_n)^2 \|x_n - q\|^2 + 2\beta_n \langle f(q) - q, x_{n+1} - q \rangle$$

$$\leq (1 - \beta_n(1-k)) \|x_n - q\|^2 + 2\beta_n \langle f(q) - q, x_{n+1} - q \rangle \tag{3.54}$$

Put $a_n = \|x_n - q\|^2$, $b_n = \dfrac{2}{1-k}\langle f(q) - q, x_{n+1} - q \rangle$ and $\delta_n = \beta_n(1-k)$. Then for each $n = 1, 2, 3, \cdots$, we get from (3.54) that

$$a_{n+1} \leq (1-\delta_n)a_n + \delta_n b_n. \tag{3.55}$$

Then it follows from (3.6) and (3.53) that $\limsup\limits_{n\to\infty} b_n \leq 0$, $\lim\limits_{n\to\infty} \delta_n = 0$ and $\sum\limits_{n=1}^{\infty} \delta_n = \infty$. Thus by Lemma 2.11 and (3.55), we have

$$\lim_{n\to\infty} \|x_n - q\| = 0. \tag{3.56}$$

It follows from (3.56) that

$$\lim_{n\to\infty} \|u_n - q\| = 0. \tag{3.57}$$

$$\lim_{n\to\infty} \|v_n - q\| = 0. \tag{3.58}$$

This completes the proof. □

Remark 3.3. *$q = P_\Sigma f(q)$ is the unique fixed point of the mapping $P_\Sigma f$. Moreover, $q = \triangle(f)$ satisfies Lemma 2.12 for the nonexpansive mapping S and for the approximate fixed point sequence $\{x_n\}$.*

4. Conclusions

In our main theorem, we proposed an approximation scheme to find a common solution of a constrained convex minimization problem, a generalized equilibrium problem, and a fixed point problem of a directed nonexpansive mapping in a Hilbert space. The family of firmly nonexpnasive mappings is contained in the family of directed nonexpansive mappings, and the family of directed nonexpansive mappings is contained in the family of nonexpansive mapping. We introduced an approximation technique which is not a direct composite viscosity method. Many authors considered composite iterative schemes to approximate solutions of any two problems among the three or for both. Our methods of proof are mainly in line with the methodologies implemented by Cai et al. [10], Naidu and Sangago [20], Xu [26, 27, 28, 29], Yazdi [30, 31], Yazdi and Sababe [33, 34].

Conflict of Interests

The authors declare that there is no conflict of interests.

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