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NATURAL EXTENSION OF BANACH FIXED POINT THEOREM

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Abstract. This study aims to develop new versions of the Banach fixed-point theorem in generalized metric spaces endowed with a direct sum structure. Specifically, we assume a diagonal matrix A in $\mathbb{R}^{d\times d}$ and establish more appropriate contraction conditions to improve the applicability of fixed point results within this framework. Since the condition that the matrix A must converge to zero is unnecessary, our approach yields stronger results than the Perov one. As an application of our findings, we examine the existence and uniqueness of solutions for a system of matrix equations. This version is more powerful than the Perov version. We introduced some examples and applications to illustrate our result.

Keywords: fixed point; Banach contraction principle; generalized metric space; direct sum.
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1. Introduction

Let $\mathfrak{F}$ be a self-operator on a set $\mathscr{X}$. A fixed point ($\mathbb{FP}$) in $\mathscr{X}$ is defined as an element q such that $\mathfrak{F}q = q$, where $\mathfrak{F}$ is an operator on $\mathscr{X}$. The $\mathbb{FP}$ Theorem argues that under specific

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conditions (on the operator $\mathfrak{F}$ and the space $\mathscr{X}$), an operator $\mathfrak{F}$ of $\mathscr{X}$ in itself admits one or more $\mathbb{FP}$. Many results exist for different cases of the $\mathbb{FP}$ Theorem. The basic foundations upon which $\mathbb{FP}$ theory studies are built include the Banach contraction principle ($\mathbb{BCP}$), Brouwer's $\mathbb{FP}$ theorem, Schauder's $\mathbb{FP}$ theorem, and the Contraction operators Theorem.

The $\mathbb{FP}$ theorem, commonly recognized as the Banach contraction principle, first appeared in definitive form in Banach's thesis in 1922 [1], where it was used to establish the existence of a solution to an integral equation ($\mathbb{IE}$). Since then, its simplicity and usefulness have become a widely used tool to solve problems in many branches of mathematical analysis. This principle indicates that if $(\mathscr{X},d)$ is a complete metric space and $\mathfrak{F}\colon \mathscr{X} \to \mathscr{X}$ is a contraction operator (that is, $\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}\varpi,\mathfrak{F}\rho) \leq \lambda\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho)$ for all $\varpi,\rho \in \mathscr{X}$, where $\lambda \in (0,1)$ is a constant), then $\mathfrak{F}$ has a $\mathbb{FP}$.

The $\mathbb{BCP}$ has been generalized in various ways over the years. Some generalizations, such as those in [2, 3] and others, relax the contractive conditions of the operator, while others, including [4, 5, 6] and others, weaken the topological assumptions. In [7], Nadler extended the Banach $\mathbb{FP}$ theorem from single-valued operators to set-valued contractive operators. Additional $\mathbb{FP}$ results for set-valued operators can be found in [8] and the references therein.

2. PRELIMINARIES

The real vector space sum of the real vector spaces $\mathbb{R}_1,\ldots,\mathbb{R}_d$ in a nonempty finite ordered list is the real vector space whose underlying set is the Cartesian product $\mathbb{R}_1 \times \cdots \times \mathbb{R}_d = \prod_{i=1}^{d}\mathbb{R}_i$, with vector space operations defined by the following formulas:

$$(\varpi_1,\varpi_2,\cdots,\varpi_d)+(\rho_1,\rho_2,\cdots,\rho_d) = (\varpi_1+\rho_1,\varpi_2+\rho_2,\cdots,\varpi_d+\rho_d).$$

$$\alpha(\varpi_1,\varpi_2,\cdots,\varpi_d) = (\alpha\varpi_1,\alpha\varpi_2,\cdots,\alpha\varpi_d).$$

Definition 2.1. [9] *The direct sum, or external direct sum, of a family $(\mathbb{R}_i)_{i\in I}$ is the following sub-module of $\prod_{i\in I}\mathbb{R}_i$:*

$$\mathbb{R}^{\oplus^I} = \{(\varpi_i)_{i\in I} \in \prod_{i\in I}\mathbb{R}_i \mid \varpi_i = 0 \ \text{ for almost all } i \in I\}.$$

Immediately, this defines a sub-module. If $I=\emptyset$, then $\mathbb{R}^{\oplus^I}=0$.
If $I=\{1\}$, then $\mathbb{R}^{\oplus^I}=\mathbb{R}_1$. If $I=\{1,2,\cdots,d\}$, then $\mathbb{R}^{\oplus^I}$ is also denoted by $\mathbb{R}_1\oplus\mathbb{R}_2\oplus\mathbb{R}_3\oplus\cdots\oplus\mathbb{R}_d=\bigoplus\sum_{i=1}^{d}\mathbb{R}_i$, and coincides with $\mathbb{R}_1\times\mathbb{R}_2\times\cdots\times\mathbb{R}_d$.
The direct sum $\mathbb{R}^{\oplus^I}$ comes with an injection $\iota_j\colon\mathbb{R}_j\to\mathbb{R}^{\oplus^I}$ for every $j\in I$, defined by its components: for all $\varpi_j\in\mathbb{R}_j$,

$$\iota_j(\varpi)_j=\varpi\in\mathbb{R}_j,\ \ \iota_j(\varpi)_i=0\in\mathbb{R}_i\ \text{ for all } i\neq j.$$

every ι_j is an injective homomorphism.
If $\mathbb{R}_1,\ldots,\mathbb{R}_d$ are normed spaces, their vector space sum can be equipped with a norm inspired by the norm of the Euclidean n space. Specifically, a norm on the vector space sum can be defined in a way that resembles the standard Euclidean norm.

Definition 2.2. *Let $\mathbb{R}_1,\cdots,\mathbb{R}_d$ be a normed space with respective norms $|\cdot|_{\mathbb{R}_1},\cdots,|\cdot|_{\mathbb{R}_d}$ is the normed space whose underlying vector space is the vector space sum of $\mathbb{R}_1,\cdots,\mathbb{R}_d$ and whose norm is the direct sum norm given by the formula*

$$\|(\varpi_1,\cdots,\varpi_n)\|=\left(\sum_{j=1}^{n}|\ \varpi_j\ |_{\mathbb{R}_j}^{2}\right)^{\frac{1}{2}}.$$

This normed space is denoted by $\mathbb{R}_1\oplus\cdots\oplus\mathbb{R}_d$.

Remark 2.3. *For any two spaces A and B, the direct sum can be defined as a homomorphism $f\colon A\oplus B\to M_s$, where the matrix M_s of size s, given by:*

$$f(A,B)=A\oplus B=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix},$$

Where A and B are embedded as diagonal blocks in the matrix form, and 0 represents appropriately sized zero matrices.

Theorem 2.4. [10] *Let $\mathbb{R}_1,\mathbb{R}_2,\cdots,\mathbb{R}_d$ be a family of sets of all real numbers. Then the following are equivalent:*

(1) $\mathbb{R}^{\oplus^d}=\bigoplus\sum_{i=1}^{d}\mathbb{R}_i$ is a direct sum.

(2) If $0 = \sum_{i=1}^{d} \varpi_i$, $\varpi_i \in \mathbb{R}_i$, *then* $\varpi_i = 0$, $i = 1,2,\cdots,d$.

(3) $\mathbb{R}_i \cap \bigoplus \sum_{j=1,\ j\neq i}^{d} \mathbb{R}_j = \tilde{0}$, $i = 1,2,\cdots,d$.

Theorem 2.5. [10] *Let* $\mathbb{R}_1, \mathbb{R}_2, \cdots, \mathbb{R}_d$ *be a family of sets of all real numbers, and let* $\mathbb{R}_1 \times \cdots \times \mathbb{R}_d = \prod_{i=1}^{d} \mathbb{R}_i$ *be their direct product. Let* $\mathbb{R}_i^* = \{(,\cdots,0,\varpi_i,0,\cdots,0)|\ \varpi_i \in \mathbb{R}_i\}$. *Then* $\mathbb{R}^{\oplus^d} = \bigoplus \sum_{i=1}^{d} \mathbb{R}_i^*$ *is a direct sum of* $\mathbb{R}_i^*$.

We recall some notations and auxiliary results that will be used throughout this paper. Let $\mathbb{R}^{\oplus^d}$ be the direct sum set, and $\varpi, \rho \in \mathbb{R}^{\oplus^d}$, $\varpi = (\varpi_1, \varpi_2, \cdots, \varpi_d)$, $\rho = (\rho_1, \rho_2, \cdots, \rho_d)$ by $\varpi \precsim \rho$ (respectively, $\varpi_i < \rho_i$) for all $i = 1,2,\cdots d$. In addition, $\mathbb{R}_+^{\oplus^d} = \{\varpi \in \mathbb{R}^{\oplus^d} : \varpi_i \geq 0,\ \forall i = 1,2,\cdots,d\}$ is the set of positive elements in $\mathbb{R}^{\oplus^d}$, and we denote $\varpi \succsim \tilde{0}$ if $\varpi_i \geq 0$ for all $i = 1,2,\cdots,d$, where $\tilde{0}$ is the $d \times d$ zero matrix in $\mathbb{R}^{\oplus^d}$.

By taking the product of two square diagonal matrices, we can define $\varpi \cdot \rho$ as follows:

$$\varpi \cdot \rho = \begin{pmatrix} \varpi_1\rho_1 & 0 & \cdots & 0 \\ 0 & \varpi_2\rho_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varpi_d\rho_d \end{pmatrix} = \operatorname{diag}\Big(\varpi_1\rho_1, \quad \varpi_2\rho_2, \quad \cdots, \quad \varpi_d\rho_d\Big) \in \mathbb{R}^{\oplus^d},$$

and the direct sum unit over $\mathbb{R}^{\oplus^d}$ will be denoted by

$$I_+^{\mathbb{R}^{\oplus^d}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \operatorname{diag}\Big(1, \quad 1, \quad \cdots, \quad 1\Big).$$

An element $\varpi = (\varpi_1, \varpi_2, \cdots, \varpi_d) \in \mathbb{R}^{\oplus^d}$ has an invers if $\varpi_i \neq 0$, for all $i = 1,2,\cdots,d$ and is denoted by

$$\varpi^{-1} = \operatorname{diag}\Big(\varpi_1^{-1}, \quad \varpi_2^{-1}, \quad \cdots \quad \varpi_d^{-1}\Big) = \operatorname{diag}\Big(\tfrac{1}{\varpi_1}, \quad \tfrac{1}{\varpi_2}, \quad \cdots \quad \tfrac{1}{\varpi_d}\Big),$$

and ϖ is called invertible if it has an inverse.

Definition 2.6. *Let $\mathscr{X}$ be a non-empty set. A mapping $\delta_{\mathbb{R}^{\oplus d}}: \mathscr{X} \times \mathscr{X} \to \mathbb{R}^{\oplus^d}$ is called generalized metric space endowed with the direct sum if the following properties are satisfied:*

(1) *$\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho) \succsim \widetilde{0}$ and $\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho) = \widetilde{0}$ if and only if $\varpi = \rho$;*
(2) *$\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho) = \delta_{\mathbb{R}^{\oplus d}}(\rho,\varpi)$, for all $\varpi,\rho \in \mathscr{X}$;*
(3) *$\delta_{\mathbb{R}^{\oplus d}}(\varpi,\kappa) \precsim \delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho) + \delta_{\mathbb{R}^{\oplus d}}(\rho,\kappa)$, for all $\varpi,\rho,\kappa \in \mathscr{X}$.*

Then, we call $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ a generalized metric space on $\mathscr{X}$.

Definition 2.7. *Suppose that $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ is a generalized metric space endowed with the direct sum. Then a sequence $\{\varpi_n\}$ in $\mathscr{X}$ is called:*

(1) *convergent sequence with respect to $\mathbb{R}^{\oplus^d}$. If, for every $\widetilde{\varepsilon} \in \mathbb{R}^{\oplus^d}$, with $\widetilde{\varepsilon} \succ \widetilde{0}$, if there exists $\varpi \in \mathscr{X}$ and there is $N \in \mathbb{N}$ such that for all $n > N$, $\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi) \prec \widetilde{\varepsilon}$.*
(2) *Cauchy sequence with respect to $\mathbb{R}^{\oplus^d}$. If, for any $\widetilde{\varepsilon} \succ \widetilde{0}$ there exists $N \in \mathbb{N}$ such that for all $n,m > N$, $\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_m) \prec \widetilde{\varepsilon}$.*

Definition 2.8. *Suppose that $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ and $(\mathscr{Y}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ are two generalized metric spaces endowed with the direct sum.*

(1) *The space $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ is a **complete** if every Cauchy sequence with respect to $\mathbb{R}^d$ is converges.*
(2) *A function $\mathfrak{F}: \mathscr{X} \to \mathscr{Y}$ is said to be continuous at a point $\varpi \in \mathscr{X}$ if, for every sequence $\{\varpi_n\} \subseteq \mathscr{X}$ that converges to ϖ, the sequence $\mathfrak{F}(\varpi_n)$ converges to $\mathfrak{F}(\varpi)$.*

Example 2.9. *Let $\mathscr{X}$ be a set with a metric defined by $\delta_{\mathscr{X}}: \mathscr{X} \times \mathscr{X} \to \mathbb{R}_+$. We can generalize this metric as follows:*

$$\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho) = \operatorname{diag}\Big(\delta_{\mathscr{X}}(\varpi,\rho), \quad \delta_{\mathscr{X}}(\varpi,\rho), \quad \cdots, \quad \delta_{\mathscr{X}}(\varpi,\rho)\Big) \in \mathbb{R}^{\oplus^d}.$$

Thus, $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ forms a generalized metric space endowed with the direct sum.

Example 2.10. *Let $\mathscr{X}$ be a Banach space defined by $\|\cdot\|_{\mathscr{X}}: \mathscr{X} \times \mathscr{X} \to \mathbb{R}_+$. We can generalize this norm as follows:*

$$\|\varpi-\rho\|_{\mathbb{R}^{\oplus d}}=\operatorname{diag}\Big(\|\varpi-\rho\|_{\mathscr{X}},\quad \|\varpi-\rho\|_{\mathscr{X}},\quad \cdots,\quad \|\varpi-\rho\|_{\mathscr{X}}\Big)\in\mathbb{R}^{\oplus^d}.$$

Thus, $(\mathscr{X},\mathbb{R}^{\oplus^d},\|\cdot\|_{\mathbb{R}^{\oplus d}})$ forms a generalized complete metric space endowed with the direct sum.

3. Main Results

This section will present some Banach fixed point theorems combining the direct sum.

Definition 3.1. *Suppose that $(\mathscr{X},\mathbb{R}^{\oplus^d},\delta_{\mathbb{R}^{\oplus d}})$ is a complete generalized metric space endowed with the direct sum. A mapping $\mathfrak{F}\colon \mathscr{X}\to\mathscr{X}$ is said to be a contraction and endowed with the direct sum if there exists a diagonal matrix $\Lambda=\operatorname{diag}(\lambda_1,\lambda_2,\cdots,\lambda_d)\in\mathbb{R}^d_+$ with $\tilde{0}\precsim\Lambda\prec I^{\oplus^d}$ such that*

$$\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi),\,\mathfrak{F}(\rho))\precsim\Lambda\,\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho),$$

for all $x,\ y\in\mathscr{X}$.

Example 3.2. *Let $\mathscr{X}=\mathbb{R}$ and $\delta_{\mathbb{R}^{\oplus d}}:\ \mathscr{X}\times\mathscr{X}\to\mathbb{R}^{\oplus^d}$ be a generalized metric space endowed with the direct sum defined by $\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho)=\operatorname{diag}\Big(\mid\varpi-\rho\mid,\quad\mid\varpi-\rho\mid,\quad\cdots,\quad\mid\varpi-\rho\mid\Big)\in\mathbb{R}^{\oplus^d}$. and let $\Lambda=\operatorname{diag}\left(\frac{1}{2},\quad\frac{1}{2},\quad\cdots,\quad\frac{1}{2}\right)$ and $\mathfrak{F}\colon\mathscr{X}\to\mathscr{X}$ such that $\mathfrak{F}(\varpi)=\frac{\varpi}{2}$. Then,*

$$\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi),\mathfrak{F}(\rho))=\frac{1}{2}\operatorname{diag}\Big(\mid\varpi-\rho\mid,\quad\mid\varpi-\rho\mid,\quad\cdots,\quad\mid\varpi-\rho\mid\Big)\in\mathbb{R}^{\oplus^d}.$$

It is clear that

$$\begin{aligned}\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi),\mathfrak{F}(\rho))&\precsim\Lambda\cdot\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho)\\&=\operatorname{diag}\left(\frac{1}{2},\quad\frac{1}{2},\quad\cdots,\quad\frac{1}{2}\right)\cdot\operatorname{diag}\Big(\mid\varpi-\rho\mid,\quad\mid\varpi-\rho\mid,\quad\cdots,\quad\mid\varpi-\rho\mid\Big),\end{aligned}$$

where $\Lambda\prec I_+^{\mathbb{R}^{\oplus^d}}$.

Therefore, $\mathfrak{F}$ is a contraction and is endowed with the direct sum on the generalized metric space endowed with the direct sum $(\mathscr{X},\mathbb{R}^{\oplus^d},\delta_{\mathbb{R}^{\oplus d}})$.

Example 3.3. *(Numerical example)*

Let $\mathscr{X} = \mathbb{R}$ *and* $\delta_{\mathbb{R}^{\oplus d}} : \mathscr{X} \times \mathscr{X} \to \mathbb{R}^{\oplus^d}$ *be a generalized metric space endowed with the direct sum defined by* $\delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) = \operatorname{diag}\Big(\mid \varpi - \rho \mid, \ \mid \varpi - \rho \mid, \ \cdots, \ \mid \varpi - \rho \mid\Big) \in \mathbb{R}^{\oplus^d}$*. and let* $\Lambda = \operatorname{diag}\left(\frac{1}{4}, \ \frac{1}{4}, \ \cdots, \ \frac{1}{4}\right)$ *and* $\mathfrak{F} : \mathscr{X} \to \mathscr{X}$ *such that* $\mathfrak{F}(\varpi) = \frac{\varpi}{4}$*. Then,*

$$\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi), \mathfrak{F}(\rho)) = \frac{1}{4}\operatorname{diag}\Big(\mid \varpi - \rho \mid, \ \mid \varpi - \rho \mid, \ \cdots, \ \mid \varpi - \rho \mid\Big) \in \mathbb{R}^{\oplus^d}.$$

It is clear that

$$\begin{aligned} \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi), \mathfrak{F}(\rho)) &\precsim \Lambda \cdot \delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) \\ &= \operatorname{diag}\left(\frac{1}{4}, \ \frac{1}{4}, \ \cdots, \ \frac{1}{4}\right) \cdot \operatorname{diag}\Big(\mid \varpi - \rho \mid, \ \mid \varpi - \rho \mid, \ \cdots, \ \mid \varpi - \rho \mid\Big); \end{aligned}$$

where $\Lambda \prec I_+^{\mathbb{R}^{\oplus^d}}$.

Therefore, $\mathfrak{F}$ *is a contraction endowed the direct sum for the a generalized metric space endowed with the direct sum* $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \ \delta_{\mathbb{R}^{\oplus d}})$*.*

Theorem 3.4. ***(Banach contraction Type)*** *Suppose that* $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ *is a complete generalized metric space endowed with the direct sum, and let* $\mathfrak{F} : \mathscr{X} \to \mathscr{X}$ *be a contraction mapping endowed the direct sum. That is,*

$$\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi), \mathfrak{F}(\rho)) \precsim \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho),$$

for all $\varpi, \rho \in \mathscr{X}$*, where* $\lambda_i \in (0,1)$ *for* $i = 1, 2, \ldots, d$*. Then* $\mathfrak{F}$ *has a unique* $\mathbb{FP}$ *in* $\mathscr{X}$*.*

Proof. Let ϖ_0 be any point in $\mathscr{X}$, that is $\varpi_0 \in \mathscr{X}$. Let us define a sequence $\{\varpi_n\}$ in $\mathscr{X}$ as given below.

$$\varpi_{n+1} = \mathfrak{F}(\varpi_n) = \mathfrak{F}^{n+1}(\varpi_0) \ \forall \ n \geq 0.$$

By contraction condition we get

$$\begin{aligned} \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1}, \varpi_n) &= \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-2}), \mathfrak{F}(\varpi_{n-1})) \\ &\precsim \operatorname{diag}\Big(\lambda_1, \ \lambda_2, \ \cdots, \ \lambda_d\Big)\, [\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-2}, \varpi_{n-1})] \end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&\precsim \ \operatorname{diag}\left(\lambda_1^{n-1}, \ \ \lambda_2^{n-1}, \ \ \cdots, \lambda_d^{n-1}\right)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1).
\end{aligned}
$$

Let us prove that $\{\varpi_n\}$ is a Cauchy sequence.

Suppose that $n>m$ that from the contraction and triangle inequality property, we can write as:

$$
\begin{aligned}
&\delta_{\mathbb{R}\oplus^d}(\varpi_m,\varpi_n) \precsim \delta_{\mathbb{R}\oplus^d}(\varpi_m,\varpi_{m+1}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+1},\varpi_n)\\
\precsim\ & \delta_{\mathbb{R}\oplus^d}(\varpi_m,\varpi_{m+1}) + \left[\delta_{\mathbb{R}\oplus^d}(\varpi_{m+1},\varpi_{m+2}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+2},\varpi_n)\right]\\
\precsim\ & \delta_{\mathbb{R}\oplus^d}(\varpi_m,\varpi_{m+1}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+1},\varpi_{m+2}) + \left[\delta_{\mathbb{R}\oplus^d}(\varpi_{m+2},\varpi_{m+3}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+3},\varpi_n)\right]\\
\vdots\ &\\
\precsim\ & \delta_{\mathbb{R}\oplus^d}(\varpi_m,\varpi_{m+1}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+1},\varpi_{m+2}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+2},\varpi_{m+3}) + \cdots + \delta_{\mathbb{R}\oplus^d}(\varpi_{n-1},\varpi_n)\\
\precsim\ & \operatorname{diag}(\lambda_1^m,\lambda_2^m,\cdots,\lambda_d^m)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1) + \operatorname{diag}(\lambda_1^{m+1},\lambda_2^{m+1},\cdots,\lambda_d^{m+1})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
&\quad + \cdots + \operatorname{diag}(\lambda_1^{n-1},\lambda_2^{n-1},\cdots,\lambda_d^{n-1})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
=\ & \Big((\lambda_1^m + \lambda_1^{m+1} + \cdots + \lambda_1^{n-1})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1), (\lambda_2^m + \lambda_2^{m+1} + \cdots + \lambda_2^{n-1})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
&\quad ,\cdots,(\lambda_d^m + \lambda_d^{m+1} + \cdots + \lambda_d^{n-1})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\Big)\\
=\ & \operatorname{diag}\left(\sum_{j=m}^{n-1}\lambda_1^j\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1), \sum_{j=m}^{n-1}\lambda_2^j\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1),\cdots,\sum_{j=m}^{n-1}\lambda_d^j\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\right)\\
=\ & \operatorname{diag}\left(\sum_{j=m}^{n-1}\lambda_1^j, \sum_{j=m}^{n-1}\lambda_2^j,\cdots,\sum_{j=m}^{n-1}\lambda_d^j\right)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
=\ & \operatorname{diag}\left(\lambda_1^m\sum_{j=0}^{n-m-1}\lambda_1^j, \lambda_2^m\sum_{j=0}^{n-m-1}\lambda_2^j,\cdots,\lambda_1^m\sum_{j=0}^{n-m-1}\lambda_d^j\right)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
=\ & \operatorname{diag}\Big(\lambda_1^m\sum_{j=0}^{n-m-1}\lambda_1^j, \ \lambda_2^m\sum_{j=0}^{n-m-1}\lambda_2^j,\cdots, \ \lambda_1^m\sum_{j=0}^{n-m-1}\lambda_d^j\Big)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)
\end{aligned}
$$

$$\begin{aligned}
&\precsim \operatorname{diag}\left(\lambda_1^m \sum_{j=0}^{\infty} \lambda_1^j + \lambda_2^m \sum_{j=0}^{\infty} \lambda_2^j + \cdots + \lambda_1^m \sum_{j=0}^{\infty} \lambda_d^j\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
&= \operatorname{diag}\left(\lambda_1^m \left[\frac{1}{1-\lambda_1}\right], \lambda_2^m \left[\frac{1}{1-\lambda_2}\right], \cdots, \lambda_1^m \left[\frac{1}{1-\lambda_d}\right]\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
&\precsim \operatorname{diag}(\lambda_1^m, \lambda_2^m, \cdots, \lambda_d^m) \operatorname{diag}\left(\left[\frac{1}{1-\lambda_1}\right], \left[\frac{1}{1-\lambda_2}\right], \cdots, \left[\frac{1}{1-\lambda_d}\right]\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \to 0,
\end{aligned}$$

as $n, m \to \infty$. Since $\tilde{0} \prec \operatorname{diag}(\lambda_1^j, \lambda_2^j, \cdots, \lambda_d^j) \prec I_+^{\mathbb{R}^{\oplus d}}$ and $\delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1)$ are fixed, it is evident that by selecting m sufficiently large (with $n > m$), we can make $\delta_{\mathbb{R}^{\oplus d}}(\varpi_n, \varpi_m)$ arbitrarily small. This shows that $\{\varpi_n\}$ is a Cauchy sequence. Finally, because $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ is complete, there exists some $\kappa \in \mathscr{X}$ such that $\varpi_n \to \kappa$.

To show that κ is a $\mathbb{FP}$, we consider the distance $\delta_{\mathbb{R}^{\oplus d}}(\kappa, \mathfrak{F}(\kappa))$. From the triangle inequality and contraction condition, we get

$$\begin{aligned}
\delta_{\mathbb{R}^{\oplus d}}(\kappa, \mathfrak{F}(\kappa)) &\precsim \delta_{\mathbb{R}^{\oplus d}}(\kappa, \varpi_n) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_n, \mathfrak{F}(\kappa)) \\
&= \delta_{\mathbb{R}^{\oplus d}}(\kappa, \varpi_n) + \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}), \mathfrak{F}(\kappa)) \\
&\precsim \delta_{\mathbb{R}^{\oplus d}}(\kappa, \varpi_n) + \operatorname{diag}\Big(\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_d\Big) \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1}, \kappa),
\end{aligned}$$

and since $\varpi_n \to \kappa$ we can make this distance as small as we please by choosing m sufficiently large. We conclude that

$$\delta_{\mathbb{R}^{\oplus d}}(\kappa, \mathfrak{F}(\kappa)) = 0 \implies \mathfrak{F}(\kappa) = \kappa,$$

so $\kappa \in \mathscr{X}$ is a $\mathbb{FP}$.

Suppose there are two $\mathbb{FP}$s $\varpi = f(\varpi)$ and $\rho = \mathfrak{F}(\rho)$. Then, from the contraction condition, we have

$$\delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) = \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi), \mathfrak{F}(\rho)) \precsim \operatorname{diag}\Big(\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_d\Big) \ \delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho),$$

this implies $\delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) = 0$ since $\tilde{0} \prec \Lambda \prec I_+^{\mathbb{R}^{\oplus d}}$. Hence $\varpi = \rho$, and the $\mathbb{FP}$ $\mathscr{X}$ of $\mathfrak{F}$ is unique. □

Theorem 3.5. ***(Kannan Type)*** *Suppose that* $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ *is a complete generalized metric space endowed with the direct sum and* $\mathfrak{F}\colon \mathscr{X} \to \mathscr{X}$ *is a continuous mapping satisfying the following condition*

$$\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi),\mathfrak{F}(\rho)) \precsim \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\,[\delta_{\mathbb{R}^{\oplus d}}(\varpi,\mathfrak{F}(\varpi))+\delta_{\mathbb{R}^{\oplus d}}(\rho,\mathfrak{F}(\rho))], \tag{3.1}$$

where $\alpha_i \in (0,\frac{1}{2})$ *for all* $i = 1,2,\cdots,d$*. Then* $\mathfrak{F}$ *has a unique* $\mathbb{FP}$ *in* $\mathscr{X}$.

Proof. Let ϖ_0 be any point in $\mathscr{X}$, that is $\varpi_0 \in \mathscr{X}$. Let us define a sequence $\{\varpi_n\}$ in $\mathscr{X}$ as given below.

$$\varpi_{n+1} = \mathfrak{F}(\varpi_n) = \mathfrak{F}^{n+1}(\varpi_0) \ \forall \ n \geq 0.$$

We have

$$\begin{aligned}
&\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) = \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}),\mathfrak{F}(\varpi_n))\\
\precsim\ & \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\,\big[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\mathfrak{F}(\varpi_{n-1}))+\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\mathfrak{F}(\varpi_n))\big]\\
=\ & \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n) + \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}),
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{diag}\Big((1-\alpha_1), \quad (1-\alpha_2), \quad \cdots, \quad (1-\alpha_d)\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1})\\
\precsim\ & \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \cdots \quad \alpha_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n),
\end{aligned}$$

since the inverse of the diagonal matrix is $\operatorname{diag}\Big((1-\alpha_1)^{-1}, \quad (1-\alpha_2)^{-1}, \quad \cdots, \quad (1-\alpha_d)^{-1}\Big)$. Thus,

$$\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) \precsim \operatorname{diag}\left(\frac{\alpha_1}{(1-\alpha_1)},\frac{\alpha_2}{(1-\alpha_2)},\cdots\cdots, \ \frac{\alpha_d}{(1-\alpha_d)}\right)\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n).$$

To simplify, we put $\frac{\alpha_i}{(1-\alpha_i)} = \lambda_i$ and the inequality becomes

$$\begin{aligned}
\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) &\precsim \operatorname{diag}(\lambda_1, \ \lambda_2, \ \cdots\cdots, \lambda_d)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)\\
&\precsim \operatorname{diag}(\lambda_1^3, \ \lambda_2^3, \ \lambda_d^3)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-3},\varpi_{n-2})\\
&\ \ \vdots
\end{aligned}$$

$$\precsim \quad \operatorname{diag}(\lambda_1^n, \lambda_2^n \cdots\cdots,\ \lambda_d^n)\, \delta_{\mathbb{R}\oplus^d}(\varpi_0, \varpi_1).$$

Let us prove that $\{\varpi_n\}$ is a Cauchy sequence. Suppose that $n > m$, then from (3.1) and the triangle inequality property, we can write as:

$$\begin{aligned}
&\delta_{\mathbb{R}\oplus^d}(\varpi_m, \varpi_n) \precsim \Big[\delta_{\mathbb{R}\oplus^d}(\varpi_m, \varpi_{m+1}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+1}, \varpi_n)\Big] \\
\precsim \quad & \delta_{\mathbb{R}\oplus^d}(\varpi_m, \varpi_{m+1}) + \Big[\delta_{\mathbb{R}\oplus^d}(\varpi_{m+1}, \varpi_{m+2}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+2}, \varpi_n)\Big] \\
\precsim \quad & \delta_{\mathbb{R}\oplus^d}(\varpi_m, \varpi_{m+1}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+1}, \varpi_{m+2}) + \Big[\delta_{\mathbb{R}\oplus^d}(\varpi_{m+2}, \varpi_{m+3}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+3}, \varpi_n)\Big] \\
\vdots & \\
\precsim \quad & \delta_{\mathbb{R}\oplus^d}(\varpi_m, \varpi_{m+1}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+1}, \varpi_{m+2}) + \delta_{\mathbb{R}\oplus^d}(\varpi_{m+2}, \varpi_{m+3}) + \cdots + \delta_{\mathbb{R}\oplus^d}(\varpi_{n-1}, \varpi_n) \\
= \quad & \operatorname{diag}(\lambda_1^m, \lambda_2^m \cdots\cdots,\ \lambda_d^m)\, \delta_{\mathbb{R}\oplus^d}(\varpi_0, \varpi_1) \\
& + \operatorname{diag}\big(\lambda_1^{m+1}, \lambda_2^{m+1}, \cdots\cdots,\ \lambda_d^{m+1}\big)\, \delta_{\mathbb{R}\oplus^d}(\varpi_0, \varpi_1) \\
& + \operatorname{diag}\big(\lambda_1^{m+2}, \lambda_2^{m+2}, \cdots\cdots,\ \lambda_d^{m+2}\big)\, \delta_{\mathbb{R}\oplus^d}(\varpi_0, \varpi_1) \\
& + \cdots + \operatorname{diag}\big(\lambda_1^{n-1}, \lambda_2^{n-1}, \cdots\cdots,\ \lambda_d^{n-1}\big)\, \delta_{\mathbb{R}\oplus^d}(\varpi_0, \varpi_1) \\
= \quad & \operatorname{diag}\Bigg(\big[\lambda_1^m + \lambda_1^{m+1} + \cdots + \lambda_1^{n-1}\big], \big[\lambda_2^m + \lambda_2^{m+1} + \cdots + \lambda_2^{n-1}\big], \\
& \quad \big[\lambda_d^m + \lambda_d^{m+1} + \cdots + \lambda_d^{n-1}\big]\Bigg)\delta_{\mathbb{R}\oplus^d}(\varpi_0, \varpi_1) \\
= \quad & \operatorname{diag}\left(\sum_{j=m}^{n-1} \lambda_1^i, \sum_{j=m}^{n-1} \lambda_2^i, \cdots, \sum_{j=m}^{n-1} \lambda_d^i\right) \delta_{\mathbb{R}\oplus^d}(\varpi_0, \varpi_1) \\
= \quad & \operatorname{diag}\left(\lambda_1^m \sum_{j=0}^{n-m-1} \lambda_1^j,\ \lambda_2^m \sum_{j=0}^{n-m-1} \lambda_2^j, \cdots, \lambda_d^m \sum_{j=0}^{n-m-1} \lambda_d^j\right) \delta_{\mathbb{R}\oplus^d}(\varpi_0, \varpi_1) \\
\precsim \quad & \operatorname{diag}\left(\lambda_1^m \sum_{j=0}^{\infty} \lambda_1^j,\ \lambda_2^m \sum_{j=0}^{\infty} \lambda_2^j, \cdots,\ \lambda_d^m \sum_{j=0}^{\infty} \lambda_d^j\right) \delta_{\mathbb{R}\oplus^d}(\varpi_0, \varpi_1) \\
= \quad & \operatorname{diag}\left(\lambda_1^m,\ \lambda_2^m, \cdots, \lambda_d^m\right) \operatorname{diag}\left(\sum_{j=0}^{\infty} \lambda_1^j,\ \sum_{j=0}^{\infty} \lambda_2^j,\ \sum_{j=0}^{\infty} \lambda_d^j\right) \delta_{\mathbb{R}\oplus^d}(\varpi_0, \varpi_1)
\end{aligned}$$

$$= \operatorname{diag}\left(\lambda_1^m,\ \lambda_2^m, \cdots, \lambda_d^m\right) \operatorname{diag}\left(\left[\frac{1}{1-(\lambda_1)}\right], \left[\frac{1}{1-(\lambda_2)}\right], \left[\frac{1}{1-(\lambda_1)}\right]\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1).$$

Since $[\frac{\alpha_i}{(1-\alpha_i)}] = \lambda_i \in (0,1)$ for all $i = 1,2,\cdots,d$ and $\delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1)$ are fixed, it is evident that by selecting m sufficiently large (with $n > m$), we can make $\delta_{\mathbb{R}^{\oplus d}}(\varpi_n, \varpi_m)$ arbitrarily small. This shows that $\{\varpi_n\}$ is a Cauchy sequence. Finally, because $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ is complete, there exists some $\kappa \in \mathscr{X}$ such that $\varpi_n \to \kappa$.

To show that κ is a $\mathbb{FP}$, we consider the distance $\delta_{\mathbb{R}^{\oplus d}}(\kappa, \mathfrak{F}(\kappa))$. From the triangle inequality and contraction condition, we get

$$\begin{aligned}
&\delta_{\mathbb{R}^{\oplus d}}(\kappa, \mathfrak{F}(\kappa)) \precsim \delta_{\mathbb{R}^{\oplus d}}(\kappa, \varpi_n) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_n, \mathfrak{F}(\kappa)) \\
= \;& \delta_{\mathbb{R}^{\oplus d}}(\kappa, \varpi_n) + \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}), \mathfrak{F}(\kappa)) \\
\precsim \;& \delta_{\mathbb{R}^{\oplus d}}(\kappa, \varpi_n) + \operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1}, \mathfrak{F}(\varpi_{n-1})) + \delta_{\mathbb{R}^{\oplus d}}(\kappa, \mathfrak{F}(\kappa)) \\
\precsim \;& \operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \delta_{\mathbb{R}^{\oplus d}}(\kappa, \mathfrak{F}(\kappa)).
\end{aligned}$$

It is clear that we can make $\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1}, \mathfrak{F}(\varpi_{n-1}))$ as small as we wish by choosing n sufficiently large. Since $\alpha_i < 1$ for all $i = 1, \ldots, d$, we arrive at a contradiction.

$$\delta_{\mathbb{R}^{\oplus d}}(\kappa, \mathfrak{F}(\kappa)) = 0 \implies \mathfrak{F}(\kappa) = \kappa,$$

so $\kappa \in \mathscr{X}$ is a $\mathbb{FP}$ of $\mathfrak{F}$.

Suppose there are two $\mathbb{FP}$s $\varpi = f(\varpi)$ and $\rho = \mathfrak{F}(\rho)$. Then, from the contraction condition, we have

$$\begin{aligned}
\delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) &= \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi), \mathfrak{F}(\rho)) \\
&\precsim \operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \; [\delta_{\mathbb{R}^{\oplus d}}(\varpi, \mathfrak{F}(\varpi)) + \delta_{\mathbb{R}^{\oplus d}}(\rho, \mathfrak{F}(\rho))],
\end{aligned}$$

this implies $\delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) = 0$. Hence $\varpi = \rho$, and the $\mathbb{FP}$ κ of $\mathfrak{F}$ is unique. □

Theorem 3.6. *(* ***Chatterjee Type****) Suppose that $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ is a complete generalized metric space endowed with the direct sum and $\mathfrak{F}\colon \mathscr{X} \to \mathscr{X}$ is a mapping satisfying the* *following condition*

(3.2)

$$\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi),\mathfrak{F}(\rho)) \precsim \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\,[\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi),\rho)+\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\rho),\varpi)],$$

for $\alpha_i \in (0,\frac{1}{2})$ *for all* $i=1,2,\cdots,d$, *and for each* $\varpi,\rho \in \mathscr{X}$. *Then* $\mathfrak{F}$ *has a unique* $\mathbb{FP}$ *in* $\mathscr{X}$.

Proof. Let ϖ_0 be any point in $\mathscr{X}$, that is $\varpi_0 \in \mathscr{X}$. Let us define a sequence $\{\varpi_n\}$ in $\mathscr{X}$ as given below.

$$\varpi_{n+1} = \mathfrak{F}(\varpi_n) = \mathfrak{F}^{n+1}(\varpi_0) \;\; \forall \;\; n \geq 0.$$

We have

$$\begin{aligned}
&\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) = \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}),\mathfrak{F}(\varpi_n))\\
\precsim\;& \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\Big[\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}),\varpi_n)+\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_n),\varpi_{n-1})\Big]\\
=\;& \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)[\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_n)+\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n+1},\varpi_{n-1})]\\
=\;& \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_n),\mathfrak{F}(\varpi_{n-2}))\\
\precsim\;& \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\Big[\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_n),\mathfrak{F}(\varpi_{n-1}))+\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}),\mathfrak{F}(\varpi_{n-2}))\Big]\\
=\;& \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\Big[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n+1},\varpi_n)+\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n-1})\Big].
\end{aligned}$$

Thus,

$$\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) \quad \precsim \quad \operatorname{diag}\Big(\tfrac{\alpha_1}{(1-\alpha_1)}, \quad \tfrac{\alpha_2}{(1-\alpha_2)}, \quad \cdots, \quad \tfrac{\alpha_d}{(1-\alpha_d)}\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n),$$

To simplify, we put $\frac{\alpha_i}{(1-\alpha_i)} = \lambda_i$ and the inequality becomes

$$\begin{aligned}
&\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) = \operatorname{diag}(\lambda_1,\lambda_2,\cdots,\lambda_d)\;\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)\\
\precsim\;& \operatorname{diag}(\lambda_1^2,\lambda_2^2,\cdots,\lambda_d^2)\;\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-2},\varpi_{n-1})\\
\precsim\;& \operatorname{diag}(\lambda_1^3,\lambda_2^3,\cdots,\lambda_d^3)\;\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-3},\varpi_{n-2})\\
&\vdots\\
\precsim\;& \operatorname{diag}(\lambda_1^n,\lambda_2^n,\cdots,\lambda_d^n)\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1).
\end{aligned}$$

Let us prove that $\{\varpi_n\}$ is a Cauchy sequence. Suppose that $n > m$ that from (3.2) and triangle inequality property, we can write as:

$$
\begin{aligned}
&\delta_{\mathbb{R}\oplus^d}(\varpi_m,\varpi_n) \precsim \left[\delta_{\mathbb{R}\oplus^d}(\varpi_m,\varpi_{m+1})+\delta_{\mathbb{R}\oplus^d}(\varpi_{m+1},\varpi_n)\right]\\
\precsim\ & \delta_{\mathbb{R}\oplus^d}(\varpi_m,\varpi_{m+1})+\left[\delta_{\mathbb{R}\oplus^d}(\varpi_{m+1},\varpi_{m+2})+\delta_{\mathbb{R}\oplus^d}(\varpi_{m+2},\varpi_n)\right]\\
\precsim\ & \delta_{\mathbb{R}\oplus^d}(\varpi_m,\varpi_{m+1})+\delta_{\mathbb{R}\oplus^d}(\varpi_{m+1},\varpi_{m+2})+\left[\delta_{\mathbb{R}\oplus^d}(\varpi_{m+2},\varpi_{m+3})+\delta_{\mathbb{R}\oplus^d}(\varpi_{m+3},\varpi_n)\right]\\
\vdots\ & \\
\precsim\ & \delta_{\mathbb{R}\oplus^d}(\varpi_m,\varpi_{m+1})+\delta_{\mathbb{R}\oplus^d}(\varpi_{m+1},\varpi_{m+2})+\delta_{\mathbb{R}\oplus^d}(\varpi_{m+2},\varpi_{m+3})+\cdots+\delta_{\mathbb{R}\oplus^d}(\varpi_{n-1},\varpi_n)\\
=\ & \operatorname{diag}(\lambda_1^m,\lambda_2^m,\cdots,\lambda_d^m)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)+\operatorname{diag}(\lambda_1^{m+1},\lambda_2^{m+1},\cdots,\lambda_d^{m+1})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
&\quad+\operatorname{diag}(\lambda_1^{m+2},\lambda_2^{m+2},\cdots,\lambda_d^{m+2})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)+\cdots\\
&+\operatorname{diag}(\lambda_1^{n-1},\lambda_2^{n-1},\cdots,\lambda_d^{n-1})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
=\ & \Big((\lambda_1^m+\lambda_1^{m+1}+\cdots+\lambda_1^{n-1})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1),(\lambda_2^m+\lambda_2^{m+1}+\cdots+\lambda_2^{n-1})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
&\quad,\cdots,(\lambda_d^m+\lambda_d^{m+1}+\cdots+\lambda_d^{n-1})\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\Big)\\
=\ & \operatorname{diag}\left(\sum_{j=m}^{n-1}\lambda_1^j\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1),\sum_{j=m}^{n-1}\lambda_2^j\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1),\cdots,\sum_{j=m}^{n-1}\lambda_d^j\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\right)\\
=\ & \operatorname{diag}\left(\sum_{j=m}^{n-1}\lambda_1^j,\sum_{j=m}^{n-1}\lambda_2^j,\cdots,\sum_{j=m}^{n-1}\lambda_d^j\right)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
=\ & \operatorname{diag}\left(\lambda_1^m\sum_{j=0}^{n-m-1}\lambda_1^j,\lambda_2^m\sum_{j=0}^{n-m-1}\lambda_2^j,\cdots,\lambda_1^m\sum_{j=0}^{n-m-1}\lambda_d^j\right)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
\precsim\ & \operatorname{diag}\Big(\lambda_1^m\sum_{j=0}^{\infty}\lambda_1^j,\ \lambda_2^m\sum_{j=0}^{\infty}\lambda_2^j,\ \cdots,\ \lambda_1^m\sum_{j=0}^{\infty}\lambda_d^j\Big)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
=\ & \operatorname{diag}\left(\lambda_1^m\left[\frac{1}{1-\lambda_1}\right],\lambda_2^m\left[\frac{1}{1-\lambda_2}\right],\cdots,\lambda_1^m\left[\frac{1}{1-\lambda_d}\right]\right)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\\
\precsim\ & \operatorname{diag}(\lambda_1^m,\lambda_2^m,\cdots,\lambda_d^m)\operatorname{diag}\left(\left[\frac{1}{1-\lambda_1}\right],\left[\frac{1}{1-\lambda_2}\right],\cdots,\left[\frac{1}{1-\lambda_d}\right]\right)\delta_{\mathbb{R}\oplus^d}(\varpi_0,\varpi_1)\to 0,
\end{aligned}
$$

as $n,m\to\infty$. Since $\tilde{0}\prec \mathrm{diag}(\lambda_1^j,\lambda_2^j,\cdots,\lambda_d^j)\prec I_+^{\mathbb{R}^{\oplus^d}}$ and $\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)$ are fixed, it is evident that by selecting m sufficiently large (with $n>m$), we can make $\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_m)$ arbitrarily small. This shows that $\{\varpi_n\}$ is a Cauchy sequence. Finally, because $(\mathscr{X},\mathbb{R}^{\oplus^d},\delta_{\mathbb{R}^{\oplus d}})$ is complete, there exists some $\kappa\in\mathscr{X}$ such that $\varpi_n\to\kappa$.

To show that κ is a $\mathbb{FP}$, we consider the distance $\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa))$. From the triangle inequality and contraction condition, we get

$$\begin{aligned}
&\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa))\precsim \delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n)+\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\mathfrak{F}(\kappa))\\
=\;& \delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n)+\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}),\mathfrak{F}(\kappa))\\
\precsim\;& \delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n)+\mathrm{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix}(\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}),\kappa)+\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\kappa),\varpi_{n-1}))\\
=\;& \mathrm{diag}\begin{pmatrix}1+\alpha_1, & 1+\alpha_2, & \cdots, & 1+\alpha_d\end{pmatrix}\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\kappa)\\
&+\mathrm{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix}\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\kappa),\varpi_{n-1}),
\end{aligned}$$

and since $\varpi_n\to\kappa$ it is clear that

$$\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa))=0 \implies \mathfrak{F}(\kappa)=\kappa,$$

so $\varpi\in\mathscr{X}$ is a $\mathbb{FP}$ of $\mathfrak{F}$.

Suppose there are two $\mathbb{FP}$s $\varpi=f(\varpi)$ and $\rho=\mathfrak{F}(\rho)$. Then from contraction condition we have

$$\begin{aligned}
&\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho)=\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi),\mathfrak{F}(\rho))\\
\precsim\;& \mathrm{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix}[\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi),\varpi)+\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho)\\
&+\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\rho),\rho)+\delta_{\mathbb{R}^{\oplus d}}(\rho,\varpi)]\\
=\;& 2\,\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_d)\;\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho)\prec\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho),
\end{aligned}$$

this implies $\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho)=0$. Hence $\varpi=\rho$, and the $\mathbb{FP}$ $\mathscr{X}$ of $\mathfrak{F}$ is unique. □

Theorem 3.7. *Suppose that $(\mathscr{X},\mathbb{R}^{\oplus^d},\delta_{\mathbb{R}^{\oplus d}})$ is a complete generalized metric space endowed with the direct sum and $\mathfrak{F}\colon\mathscr{X}\to\mathscr{X}$ is a mapping satisfying the following condition*

$$\begin{aligned}\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}x,\mathfrak{F}\rho) &\precsim \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho)\\ &\quad +\operatorname{diag}\Big(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\Big)[\delta_{\mathbb{R}^{\oplus d}}(\varpi,\mathfrak{F}(\varpi))+\delta_{\mathbb{R}^{\oplus d}}(\rho,\mathfrak{F}(\rho))],\end{aligned}$$

for $\alpha_i,\ \beta_i\in\mathbb{R}^d_+$ *and* $\alpha_i+2\beta_i\in[0,1)$, $i=1,2,\cdots,d$ *for each* $\varpi,\rho\in\mathscr{X}$. *Then* $\mathfrak{F}$ *has a unique* $\mathbb{FP}$ *in* $\mathscr{X}$.

Proof. Let ϖ_0 be any point in X, that is $\varpi_0\in\mathscr{X}$. Let us define a sequence $\{\varpi_n\}$ in X as given below.

$$\varpi_{n+1}=\mathfrak{F}(\varpi_n)=\mathfrak{F}^{n+1}(\varpi_0)\ \forall\ n\geq 0.$$

We have

$$\begin{aligned}&\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1})=\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}),\mathfrak{F}(\varpi_n))\\ \precsim\ &\operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)\\ &\quad +\operatorname{diag}\Big(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\Big)[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\mathfrak{F}(\varpi_{n-1}))+\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\mathfrak{F}(\varpi_n))]\\ =\ &\operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)\\ &\quad +\operatorname{diag}\Big(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\Big)[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)+\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1})]\\ =\ &\operatorname{diag}\Big((\alpha_1+\beta_1), \quad (\alpha_2+\beta_2), \quad \cdots, \quad (\alpha_d+\beta_d)\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-2}),\mathfrak{F}(\varpi_{n-1}))\\ &\quad +\operatorname{diag}\Big(\beta_1 \quad \beta_2, \quad \cdots, \quad \beta_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}).\end{aligned}$$

Thus,

$$\begin{aligned}&\operatorname{diag}\Big((1-\beta_1), \quad (1-\beta_2), \quad \cdots, \quad (1-\beta_d)\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1})\\ \precsim\ &\operatorname{diag}\Big((\alpha_1+\beta_1), \quad (\alpha_2+\beta_2), \quad \cdots, \quad (\alpha_d+\beta_d)\Big)\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-2}),\mathfrak{F}(\varpi_{n-1})).\end{aligned}$$

This implies that

$$\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) \precsim \operatorname{diag}\left(\frac{(\alpha_1+\beta_1)}{(1-\beta_1)}, \quad \frac{(\alpha_2+\beta_2)}{(1-\beta_2)}, \quad \cdots, \quad \frac{(\alpha_d+\beta_d)}{(1-\beta_d)}\right)\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-2}),\mathfrak{F}(\varpi_{n-1})).$$

we use $\lambda_i = \dfrac{\alpha_i + \beta_i}{1-\beta_i} \in (0,1)$, $i = 1, 2, \cdots, d$ we obtain

$$\delta_{\mathbb{R}^{\oplus d}}(\varpi_n, \varpi_{n+1}) \precsim \operatorname{diag}\Big(\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_d\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi_n, \varpi_{n-1}).$$

So

$$\begin{aligned}
\delta_{\mathbb{R}^{\oplus d}}(\varpi_n, \varpi_{n+1}) &\precsim \operatorname{diag}\Big(\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1}, \varpi_n) \\
&\precsim \operatorname{diag}\Big(\lambda_1^2, \quad \lambda_2^2, \quad \cdots, \quad \lambda_d^2\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-2}, \varpi_{n-1}) \\
&\precsim \operatorname{diag}\Big(\lambda_1^3, \quad \lambda_2^3, \quad \cdots, \quad \lambda_d^3\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-3}, \varpi_{n-2}) \\
&\ \vdots \\
&\precsim \operatorname{diag}\Big(\lambda_1^n, \quad \lambda_2^n, \quad \cdots, \quad \lambda_d^n\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1). \qquad (3.3)
\end{aligned}$$

Let us prove that $\{\varpi_n\}$ is a Cauchy sequence. Suppose that $m < n$, then from (3.3) and the triangle inequality property, we can write as:

$$\begin{aligned}
&\delta_{\mathbb{R}^{\oplus d}}(\varpi_m, \varpi_n) \precsim \delta_{\mathbb{R}^{\oplus d}}(\varpi_m, \varpi_{m+1}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+1}, \varpi_n) \\
\precsim\ & \delta_{\mathbb{R}^{\oplus d}}(\varpi_m, \varpi_{m+1}) + \Big[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+1}, \varpi_{m+2}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+2}, \varpi_n)\Big] \\
\precsim\ & \delta_{\mathbb{R}^{\oplus d}}(\varpi_m, \varpi_{m+1}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+1}, \varpi_{m+2}) + \Big[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+2}, \varpi_{m+3}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+3}, \varpi_n)\Big] \\
\vdots\ & \\
\precsim\ & \delta_{\mathbb{R}^{\oplus d}}(\varpi_m, \varpi_{m+1}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+1}, \varpi_{m+2}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+2}, \varpi_{m+3}) + \cdots + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1}, \varpi_n) \\
=\ & \operatorname{diag}\Big(\lambda_1^m, \quad \lambda_2^m, \quad \cdots, \quad \lambda_d^m\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
& + \operatorname{diag}\Big(\lambda_1^{m+1}, \quad \lambda_2^{m+1}, \quad \cdots, \quad \lambda_d^{m+1}\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
& + \operatorname{diag}\Big(\lambda_1^{m+2}, \quad \lambda_2^{m+2}, \quad \cdots, \quad \lambda_d^{m+2}\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) + \cdots \\
& + \operatorname{diag}\Big(\lambda_1^{n-1}, \quad \lambda_2^{n-1}, \quad \cdots, \quad \lambda_d^{n-1}\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
=\ & \Big((\lambda_1^m + \lambda_1^{m+1} + \cdots + \lambda_1^{n-1})\delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1), (\lambda_2^m + \lambda_2^{m+1} + \cdots + \lambda_2^{n-1})\delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1)
\end{aligned}$$

$$
\begin{aligned}
&\quad ,\cdots,(\lambda_d^m+\lambda_d^{m+1}+\cdots+\lambda_d^{n-1})\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)\Big)\\
&= \operatorname{diag}\Big(\textstyle\sum_{j=m}^{n-1}\lambda_1^j\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1),\ \ \sum_{j=m}^{n-1}\lambda_2^j\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1),\ \ \cdots,\ \ \sum_{j=m}^{n-1}\lambda_d^j\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)\Big)\\
&= \operatorname{diag}\Big(\textstyle\sum_{j=m}^{n-1}\lambda_1^j,\ \ \sum_{j=m}^{n-1}\lambda_2^j,\ \ \cdots,\ \ \sum_{j=m}^{n-1}\lambda_d^j\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)\\
&= \operatorname{diag}\Big(\lambda_1^m\textstyle\sum_{j=0}^{n-m-1}\lambda_1^j,\lambda_2^m\sum_{j=0}^{n-m-1}\lambda_2^j,\cdots,\lambda_1^m\sum_{j=0}^{n-m-1}\lambda_d^j\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)\\
&\precsim \operatorname{diag}\Big(\lambda_1^m\textstyle\sum_{j=0}^{\infty}\lambda_1^j,\ \lambda_2^m\sum_{j=0}^{\infty}\lambda_2^j,\ \cdots,\ \lambda_1^m\sum_{j=0}^{\infty}\lambda_d^j\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)\\
&= \operatorname{diag}\left(\lambda_1^m\left[\frac{1}{1-\lambda_1}\right],\lambda_2^m\left[\frac{1}{1-\lambda_2}\right],\cdots,\lambda_d^m\left[\frac{1}{1-\lambda_d}\right]\right)\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)\\
&\precsim \operatorname{diag}\Big(\lambda_1^m,\lambda_2^m,\cdots,\lambda_d^m\Big)\operatorname{diag}\left(\left[\frac{1}{1-\lambda_1}\right],\left[\frac{1}{1-\lambda_2}\right],\cdots,\left[\frac{1}{1-\lambda_d}\right]\right)\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)\to 0,
\end{aligned}
$$

as $n,m\to\infty$. Since $\lambda_i\in(0,1)$ for all $i=1,2,\cdots,d$ and $\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)$ are fixed, it is evident that by selecting m sufficiently large (with $n>m$), we can make $\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_m)$ arbitrarily small. This shows that $\{\varpi_n\}$ is a Cauchy sequence. Finally, because $(\mathscr{X},\mathbb{R}^{\oplus^d},\delta_{\mathbb{R}^{\oplus d}})$ is complete, there exists some $\kappa\in\mathscr{X}$ such that $\varpi_n\to\kappa$.

To show that κ is a $\mathbb{FP}$, we consider the distance $\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa))$. From the triangle inequality and contraction condition, we get

$$
\begin{aligned}
\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa)) &\precsim \delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n)+\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\mathfrak{F}(\kappa))\\
&= \delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n)+\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}),\mathfrak{F}(\kappa))\\
&\precsim \delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n)+\operatorname{diag}\Big(\alpha_1,\ \ \cdots,\ \ \alpha_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\kappa)\\
&\quad+\operatorname{diag}\Big(\beta_1,\ \ \cdots,\ \ \beta_d\Big)\Big[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\mathfrak{F}(\varpi_{n-1}))+\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa))\Big],
\end{aligned}
$$

therefore

$$
\begin{aligned}
\Big(1-\beta_1,\ \ \cdots,\ \ 1-\beta_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa)) &\precsim \delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n)+\operatorname{diag}\Big(\alpha_1,\ \ \cdots,\ \ \alpha_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\kappa)\\
&\quad+\operatorname{diag}\Big(\beta_1,\ \ \cdots,\ \ \beta_d\Big)\Big[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\mathfrak{F}(\varpi_{n-1}))\Big]\to 0,
\end{aligned}
$$

since $\varpi_n\to\kappa$ it is clear that

$$
\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa))=0\ \Longrightarrow \mathfrak{F}(\kappa)=\kappa,
$$

so $\kappa \in \mathscr{X}$ is a $\mathbb{FP}$ of $\mathfrak{F}$.

Suppose there are two $\mathbb{FP}$s $\varpi = f(\varpi)$ and $\rho = \mathfrak{F}(\rho)$. Then from contraction condition we have

$$\begin{aligned}
\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho) &= \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi),\mathfrak{F}(\rho)) \\
&\precsim \operatorname{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix} \delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho) \\
&\quad + \operatorname{diag}\begin{pmatrix}\beta_1, & \beta_2, & \cdots, & \beta_d\end{pmatrix} [\delta_{\mathbb{R}^{\oplus d}}(\varpi,\mathfrak{F}(\varpi)) + \delta_{\mathbb{R}^{\oplus d}}(\rho,\mathfrak{F}(\rho))] \\
&= \operatorname{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix} \delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho) \prec \delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho),
\end{aligned}$$

this implies $\delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho) = 0$. Hence $\varpi = \rho$, and the $\mathbb{FP}$ x of $\mathfrak{F}$ is unique. $\square$

Theorem 3.8. *Suppose that $(\mathscr{X}, \mathbb{R}^{\oplus^d}, \delta_{\mathbb{R}^{\oplus d}})$ is a complete generalized metric space endowed with the direct sum and $\mathfrak{F}\colon \mathscr{X} \to \mathscr{X}$ is mapping satisfying the following condition*

$$\begin{aligned}
\delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}x,\mathfrak{F}\rho) &\precsim \operatorname{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix} \delta_{\mathbb{R}^{\oplus d}}(\varpi,\rho) \\
&\quad + \operatorname{diag}\begin{pmatrix}\beta_1, & \beta_2, & \cdots, & \beta_d\end{pmatrix} [\delta_{\mathbb{R}^{\oplus d}}(\varpi,\mathfrak{F}(\varpi)) + \delta_{\mathbb{R}^{\oplus d}}(\rho,\mathfrak{F}(\rho))] \\
&\quad + \operatorname{diag}\begin{pmatrix}\gamma_1, & \gamma_2, & \cdots, & \gamma_d\end{pmatrix} [\delta_{\mathbb{R}^{\oplus d}}(\varpi,\mathfrak{F}(\rho)) + \delta_{\mathbb{R}^{\oplus d}}(\rho,\mathfrak{F}(\varpi))], \qquad (3.4)
\end{aligned}$$

for $\alpha_i,\ \beta_i,\ \gamma_i \in \mathbb{R}_+\ \ \forall\, i = 1,2,3,\cdots,d$ and $\alpha_i + 2(\beta_i+\gamma_i) \in [0,1)$, for each $\varpi,\rho \in \mathscr{X}$. Then $\mathfrak{F}$ has a unique $\mathbb{FP}$ in $\mathscr{X}$.

Proof. Let ϖ_0 be any point in $\mathscr{X}$, that is $\varpi_0 \in \mathscr{X}$. Let us define a sequence $\{\varpi_n\}$ in $\mathscr{X}$ as given below.

$$\varpi_{n+1} = \mathfrak{F}(\varpi_n) = \mathfrak{F}^{n+1}(\varpi_0)\ \ \forall\ \ n \geq 0.$$

We have

$$\begin{aligned}
&\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) = \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}),\mathfrak{F}(\varpi_n)) \\
\precsim\ & \begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix} \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n) \\
&+ \operatorname{diag}\begin{pmatrix}\beta_1, & \beta_2, & \cdots, & \beta_d\end{pmatrix} [\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\mathfrak{F}(\varpi_{n-1})) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\mathfrak{F}(\varpi_n))]
\end{aligned}$$

$$
\begin{aligned}
&+\operatorname{diag}\Big(\gamma_1, \quad \gamma_2, \quad \cdots, \quad \gamma_d\Big)\ [\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\mathfrak{F}(\varpi_n))+\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\mathfrak{F}(\varpi_{n-1}))]\\
=\ & \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)\\
&+\operatorname{diag}\Big(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\Big)\ [\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)+\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1})]\\
&+\operatorname{diag}\Big(\gamma_1, \quad \gamma_2, \quad \cdots, \quad \gamma_d\Big)\,[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_{n+1}))]\\
\precsim\ & \operatorname{diag}\Big(\alpha_1+\beta_1, \quad \alpha_2+\beta_2, \quad \cdots, \quad \alpha_d+\beta_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)\\
&+\operatorname{diag}\Big(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1})\\
&+\operatorname{diag}\Big(\gamma_1, \quad \gamma_2, \quad \cdots, \quad \gamma_d\Big)\,[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)+\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}))]\\
=\ & \operatorname{diag}\Big(\alpha_1+\beta_1+\gamma_1, \quad \alpha_2+\beta_2+\gamma_2, \quad \cdots, \quad \alpha_d+\beta_d+\gamma_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)\\
&+\operatorname{diag}\Big(\beta_1+\gamma_1, \quad \beta_2+\gamma_2, \quad \cdots, \quad \beta_d+\gamma_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\operatorname{diag}\Big((1-(\beta_1+\gamma_1)), \quad (1-(\beta_2+\gamma_2)), \quad \cdots, \quad (1-(\beta_d+\gamma_d))\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1})\\
\precsim\ & \operatorname{diag}\Big(\alpha_1+\beta_1+\gamma_1, \quad \alpha_2+\beta_2+\gamma_2, \quad \cdots, \quad \alpha_d+\beta_d+\gamma_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n).
\end{aligned}
$$

This implies that

$$
\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) \precsim \operatorname{diag}\left(\frac{(\alpha_1+\beta_1+\gamma_1)}{(1-(\beta_1+\gamma_1))}, \quad \frac{(\alpha_2+\beta_2+\gamma_2)}{(1-(\beta_2+\gamma_2))}, \quad \cdots, \quad \frac{(\alpha_d+\beta_d+\gamma_d)}{(1-(\beta_d+\gamma_d))}\right)\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n).
$$

we use $\lambda_i = \dfrac{\alpha_i+\beta_i+\gamma_i}{1-(\beta_i+\gamma_i)} \in (0,1)$ for all $i=1,2,\cdots,d$, we obtain

$$
\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) \ \precsim\ \operatorname{diag}\Big(\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n).
$$

So

$$
\begin{aligned}
\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_{n+1}) \ &\precsim\ \operatorname{diag}\Big(\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n)\\
&\precsim\ \operatorname{diag}\Big(\lambda_1^2, \quad \lambda_2^2, \quad \cdots, \quad \lambda_d^2\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-2},\varpi_{n-1})\\
&\precsim\ \operatorname{diag}\Big(\lambda_1^3, \quad \lambda_2^3, \quad \cdots, \quad \lambda_d^3\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-3},\varpi_{n-2})
\end{aligned}
$$

$$\vdots$$

$$\precsim \ \operatorname{diag}\left(\lambda_1^n, \quad \lambda_2^n, \quad \cdots, \quad \lambda_d^n\right) \ \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1). \tag{3.5}$$

Let us prove that $\{\varpi_n\}$ is a Cauchy sequence. Suppose that $m < n$ then from (3.5) and the triangle inequality property, we can write as:

$$\begin{aligned}
& \delta_{\mathbb{R}^{\oplus d}}(\varpi_m, \varpi_n) \precsim \left[\delta_{\mathbb{R}^{\oplus d}}(\varpi_m, \varpi_{m+1}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+1}, \varpi_n)\right] \\
\precsim \quad & \delta_{\mathbb{R}^{\oplus d}}(\varpi_m, \varpi_{m+1}) + \left[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+1}, \varpi_{m+2}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+2}, \varpi_n)\right] \\
\precsim \quad & \delta_{\mathbb{R}^{\oplus d}}(\varpi_m, \varpi_{m+1}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+1}, \varpi_{m+2}) + \left[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+2}, \varpi_{m+3}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+3}, \varpi_n)\right] \\
\vdots \quad & \\
\precsim \quad & \delta_{\mathbb{R}^{\oplus d}}(\varpi_m, \varpi_{m+1}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+1}, \varpi_{m+2}) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{m+2}, \varpi_{m+3}) + \cdots + \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1}, \varpi_n) \\
= \quad & \operatorname{diag}\left(\lambda_1^m, \quad \lambda_2^m, \quad \cdots, \quad \lambda_d^m\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
& + \operatorname{diag}\left(\lambda_1^{m+1}, \quad \lambda_2^{m+1}, \quad \cdots, \quad \lambda_d^{m+1}\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
& + \operatorname{diag}\left(\lambda_1^{m+2}, \quad \lambda_2^{m+2}, \quad \cdots, \quad \lambda_d^{m+2}\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
& \quad + \cdots + \operatorname{diag}\left(\lambda_1^{n-1}, \quad \lambda_2^{n-1}, \quad \cdots, \quad \lambda_d^{n-1}\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
= \quad & \operatorname{diag}\Bigg((\lambda_1^m + \lambda_1^{m+1} + \cdots + \lambda_1^{n-1}), (\lambda_2^m + \lambda_2^{m+1} + \cdots + \lambda_2^{n-1}), \\
& \quad \cdots, (\lambda_d^m + \lambda_d^{m+1} + \cdots + \lambda_d^{n-1})\Bigg) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
= \quad & \operatorname{diag}\left(\sum_{j=m}^{n-1} \lambda_1^j, \ \sum_{j=m}^{n-1} \lambda_2^j, \ \cdots, \ \sum_{j=m}^{n-1} \lambda_d^j\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1) \\
= \quad & \operatorname{diag}\left(\lambda_1^m \sum_{j=0}^{n-m-1} \lambda_1^j, \ \lambda_2^m \sum_{j=0}^{n-m-1} \lambda_2^j, \ \cdots, \ \lambda_1^m \sum_{j=0}^{n-m-1} \lambda_d^j\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0, \varpi_1)
\end{aligned}$$

$$\begin{aligned}
&\precsim \operatorname{diag}\Big(\lambda_1^m \sum_{j=0}^{\infty} \lambda_1^j,\ \lambda_2^m \sum_{j=0}^{\infty} \lambda_2^j,\ \cdots,\ \lambda_1^m \sum_{j=0}^{\infty} \lambda_d^j\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)\\
&= \operatorname{diag}\Big(\lambda_1^m \Big[\frac{1}{1-\lambda_1}\Big],\ \lambda_2^m \Big[\frac{1}{1-\lambda_2}\Big],\ \cdots,\ \lambda_d^m \Big[\frac{1}{1-\lambda_d}\Big]\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)\\
&\precsim \operatorname{diag}(\lambda_1^m,\ \lambda_2^m,\ \cdots,\ \lambda_d^m)\operatorname{diag}\left(\Big[\frac{1}{1-\lambda_1}\Big],\ \ \Big[\frac{1}{1-\lambda_2}\Big],\ \ \cdots,\ \ \Big[\frac{1}{1-\lambda_d}\Big]\right) \delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)\\
&\to 0,
\end{aligned}$$

as $n,m \to \infty$. Since $\lambda_i \in (0,1)\ \forall i$ and $\delta_{\mathbb{R}^{\oplus d}}(\varpi_0,\varpi_1)$ are fixed, it is evident that by selecting m sufficiently large (with $n > m$), we can make $\delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\varpi_m)$ arbitrarily small. This shows that $\{\varpi_n\}$ is a Cauchy sequence. Finally, because $(\mathscr{X},\mathbb{R}^{\oplus^d},\delta_{\mathbb{R}^{\oplus d}})$ is complete, there exists some $\kappa \in \mathscr{X}$ such that $\varpi_n \to \kappa$.

To show that κ is a $\mathbb{FP}$, we consider the distance $\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa))$. From the triangle inequality and contraction condition, we get

$$\begin{aligned}
&\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa)) \precsim \delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n) + \delta_{\mathbb{R}^{\oplus d}}(\varpi_n,\mathfrak{F}(\kappa))\\
=\ & \delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n) + \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-1}),\mathfrak{F}(\kappa))\\
\precsim\ & \delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n) + \operatorname{diag}\Big(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\kappa)\\
&\quad + \operatorname{diag}\Big(\beta_1,\ \beta_2,\ \cdots,\ \beta_d\Big)\,[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\mathfrak{F}(\varpi_{n-1})) + \delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa))]\\
&\quad + \operatorname{diag}\Big(\gamma_1,\ \gamma_2,\ \cdots,\ \gamma_d\Big)\ [\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\mathfrak{F}(\kappa)) + \delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\varpi_{n-1}))]\\
=\ & \operatorname{diag}\Big(1+\gamma_1,\ 1+\gamma_2,\ \cdots,\ 1+\gamma_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\kappa,\varpi_n) + \operatorname{diag}\Big(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_d\Big)\\
& \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\kappa) + \operatorname{diag}\Big(\beta_1,\ \beta_2,\ \cdots,\ \beta_d\Big)\,[\delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\varpi_n) + \delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa))]\\
& + \operatorname{diag}\Big(\gamma_1,\ \gamma_2,\ \cdots,\ \gamma_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1},\mathfrak{F}(\kappa)),
\end{aligned}$$

hence

$$\operatorname{diag}\Big(1+\beta_1,\ 1+\beta_2,\ \cdots,\ 1+\beta_d\Big)\,\delta_{\mathbb{R}^{\oplus d}}(\kappa,\mathfrak{F}(\kappa))$$

$$\precsim \operatorname{diag}\Big(1+\gamma_1, \quad 1+\gamma_2, \quad \cdots, \quad 1+\gamma_d\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\kappa, \varpi_n)$$
$$+\operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1}, \kappa)$$
$$+\operatorname{diag}\Big(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi_{n-1}, \varpi_n)$$
$$+\operatorname{diag}\Big(\gamma_1, \quad \gamma_2, \quad \cdots, \quad \gamma_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi_{n-2}), \mathfrak{F}(\kappa)),$$

and since $\varpi_n \to \kappa$ it is clear that

$$\delta_{\mathbb{R}^{\oplus d}}(\kappa, \mathfrak{F}(\kappa)) = 0 \implies \mathfrak{F}(\kappa) = \kappa,$$

so $\kappa \in \mathscr{X}$ is a $\mathbb{FP}$ of $\mathfrak{F}$.

Suppose there are two $\mathbb{FP}$s $\varpi = \mathfrak{F}(\varpi)$ and $\rho = \mathfrak{F}(\rho)$. Then from contraction condition we have

$$\begin{aligned}
\delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) = \delta_{\mathbb{R}^{\oplus d}}(\mathfrak{F}(\varpi), \mathfrak{F}(\rho)) &\precsim \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) \\
&+\operatorname{diag}\Big(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\Big)\ [\delta_{\mathbb{R}^{\oplus d}}(\varpi, \mathfrak{F}(\varpi)) + \delta_{\mathbb{R}^{\oplus d}}(\rho, \mathfrak{F}(\rho))] \\
&+\operatorname{diag}\Big(\gamma_1, \quad \gamma_2, \quad \cdots, \quad \gamma_d\Big)\ [\delta_{\mathbb{R}^{\oplus d}}(\varpi, \mathfrak{F}(\rho)) + \delta_{\mathbb{R}^{\oplus d}}(\rho, \mathfrak{F}(\varpi))] \\
= \quad & \operatorname{diag}\Big(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\Big)\ \delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) \\
&+\operatorname{diag}\Big(2\gamma_1, \quad 2\gamma_2, \quad \cdots, \quad 2\gamma_d\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) \\
= \quad & \operatorname{diag}\Big(\alpha_1 + 2\gamma_1, \quad \alpha_2 + 2\gamma_2, \quad \cdots, \quad \alpha_d + 2\gamma_d\Big)\, \delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) \\
\prec \quad & \delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho),
\end{aligned}$$

this implies $\delta_{\mathbb{R}^{\oplus d}}(\varpi, \rho) = 0$. Hence $\varpi = \rho$, and the $\mathbb{FP}$ x of $\mathfrak{F}$ is unique. □

CONCLUSION

In this work, we have designed new extensions of the Banach fixed-point theorem within the framework of generalized metric spaces equipped with a direct sum structure. By using a diagonal matrix $A \in \mathbb{R}^{d\times d}$ and showing developed contraction conditions, we have improved the applicability and power of $\mathbb{FP}$ results in this setting. Notably, our method removes the restrictive condition that the matrix A must converge to zero, thereby showing a considerable improvement over classical results such as Perov's theorem. The creative refinements were further supported by exploring the existence and uniqueness of solutions to a system of matrix equations, displaying the broader utility of the suggested method. Through illustrative examples and applications, we have confirmed the usefulness of our results.

Authors' Contributions

Conceptualization, Writing original draft (G. Albeladi); Methodology, Data curation (S. Omran); Formal analysis, Revisions (S. Omran); Project administration (G. Albeladi); Supervision (S. Omran).

Conflict of Interests

The authors declare that there is no conflict of interests.

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