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APPLICATION OF MINMAX THEORY TO CYBERSECURITY MODELING

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Abstract. Cybersecurity is a critical area where adversaries continuously attempt to exploit system vulnerabilities, while defenders aim to minimize risk and damage. This paper explores how MinMax theory, a fundamental concept in game theory, can be applied to model cybersecurity problems. We present a mathematical formulation and an example illustrating the effectiveness of MinMax strategies in cybersecurity defense.

Keywords: fixed point; multivalued mapping; game theory; cybersecurity.

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1. Introduction and Preliminaries

Cybersecurity can be modeled as a game between an attacker and a defender, where the attacker aims to maximize damage while the defender aims to minimize potential losses. MinMax theory provides a framework to optimize defensive strategies against worst-case attacks.[10, 14, 16]

Mathematical formulation, consider a two-player zero-sum game:

- The attacker has a set of possible attack strategies $A = \{a_1, a_2, \ldots, a_n\}$.

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- The defender has a set of defensive strategies $D = \{d_1, d_2, \ldots, d_m\}$.
- The loss function $L(a_i, d_j)$ represents the damage caused by attack a_i when defense d_j is applied.

The defender's goal is to choose a strategy d_j that minimizes the maximum possible loss:

$$\min_{d_j \in D} \max_{a_i \in A} L(a_i, d_j) \tag{1.1}$$

Conversely, the attacker's goal is to choose an attack a_i that maximizes the minimum possible damage the defender can suffer:

$$\max_{a_i \in A} \min_{d_j \in D} L(a_i, d_j) \tag{1.2}$$

If there exists an optimal mixed strategy (p^*, q^*), then the Nash equilibrium is found by solving:

$$\max_{p} \min_{q} p^T L q \tag{1.3}$$

where p and q are probability distributions over attack and defense strategies, respectively.

For example, Intrusion Detection System (IDS), consider a simple scenario where an attacker can choose between a *brute-force attack* (a_1) and a *phishing attack* (a_2), while the defender can enable a *firewall* (d_1) or a *multi-factor authentication* (d_2). A possible loss matrix L (where lower values indicate better defense) is:

$$L = \begin{bmatrix} 7 & 3 \\ 4 & 6 \end{bmatrix} \tag{1.4}$$

The defender applies the MinMax strategy to minimize the worst-case damage.

2. Main Results

In this section, we establish a new version of the the minimax theorem using the existence of a common fixed point between a upper semi-continuous multivalued of a nonvoid compact convex into itself with nonempty closed values, which satisfies the two conditions in the Hausdorff locally convex spaces.[see, [12, 15, 17]]

A locally convex space is a topological vector space whose topology is generated by a family of seminorms, ensuring that every neighborhood of the origin contains a convex open set.[6, 9, 11, 13]

Theorem 2.1. [15, Theorem 2.3]

Let E be a locally convex Hausdorff topological vector space and K a nonvoid compact convex subset of E. Suppose that u, v are upper semicontinuous multivalued maps of K into itself with nonempty closed values, which satisfies the following two conditions:

(1) both u, v are convex multivalued maps.

(2) u and v are subcommuting in the sense that either $u(v(x)) \subseteq v(u(x))$ for all $\in K$ or $u(v(x)) \supseteq v(u(x))$ for all $\in K$

Then there exists at least one point $x_0 \in K$ such that $x_0 \in u(x_0)$ and $x \in v(x_0)$.

Remark 1. *If $u : K \to 2^K$ have the closed graph be an upper semicontinuous multimap with closed values, then, u have the closed graph in the regular.* [2, 7]

Remark 2. *The gain function f is then characterized by its values $f(x_i, y_j)$. The game (A, B, f) is then a matrix where the stratigies are identified whith rows and the negotiation stratigies whith columns.*

For maximizing his gain, the first player must choose $a_0 \in A$ such that

$$\min_{b\in B} f(a_0, b) = \max_{a\in A} \min_{b\in B} f(a, b); \tag{2.1}$$

and for minimizing his loss, the second player must choose $b_0 \in B$ such that

$$\max_{a\in A} f(a, b_0) = \min_{b\in B} \max_{a\in A} f(a, b). \tag{2.2}$$

Let us consider the function ϕ defined for all $a \in A$ by

$$\phi(a) := \min_{b\in B} f(a, b) = \min f(a \times B),$$

and the function ψ defined for all $b \in B$ by

$$\psi(b) := \max_{a\in A} f(a, b) = \max f(A \times b);$$

and assume that for each $(a, b) \in A \times B$, the subsets

$$N_b := \{a' \in A : f(a', b) = \psi(b)\} \text{ and } M_a := \{b' \in B : f(a, b') = \phi(a)\}$$

are nonempty. Obviousely, $\forall A' \times B' \subseteq A \times B$, we have

$$N_{B'} = \bigcup_{b \in B'} N_b \text{ and } M_{A'} = \bigcup_{a \in A'} M_a$$

Now, we establish our new version of the minmax theorem.

Theorem 2.2. *Let E be a Hausdorff locally convex space, and X, Y be a nonempty convex and compact subsets of E. Let f is a continuous function such that $f : X \times Y \to \mathbb{R}$.*
Suppose that:

1. *(1) $\forall x \in X,\ f(x,.)$ is convex,*
2. *(2) $\forall y \in Y,\ f(.,y)$ is concave.*
3. *(3) $\bigcup_{y \in M_x} N_y \times \{b\} \subseteq \{a\} \times \bigcup_{x \in N_y} M_x\ \forall (x,y) \in X \times Y$.*

Consider a two-player, zero-sum game with sets of strategies X and Y for Player 1 and Player 2, respectively. Let $f : X \times Y \to \mathbb{R}$ be the payoff function. Then, there exist optimal strategies $x^* \in X$ for Player 1 and $y^* \in Y$ for Player 2 such that:

$$\max_{y \in Y} \min_{x \in X} f(x, y^*) = \min_{x \in X} \max_{y \in Y} f(x^*, y)$$

Proof. Let us consider the closed product subset $C = X \times Y$ whose elements are $c = (a,b)$ and the following two multivalued mappings:

$$S : C \to 2^C \quad \text{and} \quad T : C \to 2^C$$
$$c \mapsto N_b \times \{b\} \qquad\qquad c \mapsto \{a\} \times M_a$$

the product set C is compact,(product of two compact) and the sets M_x and N_y are nonempty, closed because the functions f, ϕ and ψ are continuous.
By the hypothese (1) and (2), it is easy to see that the sets M_x and N_y are convex too.
T is closed: Let $\{(a_i, b_i)\}$ be a net in C such that $(a_i, b_i) \to (a,b) \in C$ and let $(u_i, v_i) \in T(a_i, b_i)$ such that $(u_i, v_i) \to (u, v)$; we have:

$$\begin{aligned}(u_i, v_i) \in T(a_i, b_i) \quad &\Leftrightarrow \quad (u_i, v_i) \in \{a_i\} \times M_{a_i} \\ &\Leftrightarrow \quad u_i = a_i \text{ and } f(a_i, v_i) = \phi(a_i).\end{aligned}$$

By continuity of f and ϕ we obtain $u = a$ and $fa, v) = \phi(a)$, and so $(u, v) \in T(a, b)$.
Using hypothesis (3), it is clear that $S(T(c)) \subseteq T(S(c))$ for all $c \in C$.

Thus, by theorem 2.1, there exists $c^\star = (a^\star, b^\star)$ a common fixed point of S and T. Ultimately,

$$(a^\star, b^\star) \in N_{b^\star} \times \{b^\star\} \Leftrightarrow f(a^\star, b^\star) = \max_{a\in A} f(a, b^\star) \geq \min_{b\in B} \max_{a\in A} f(a,b),$$

$$(a^\star, b^\star) \in \{a^\star\} \times M_{a^\star} \Leftrightarrow f(a^\star, b^\star) = \min_{b\in B} f(a^\star, b) \leq \max_{a\in A} \min_{b\in B} f(a,b).$$

By combining these last two inequalities with (4.3), we get

$$f(a^\star, b^\star) \leq \max_{a\in A} \min_{b\in B} f(a,b) \leq \min_{b\in B} \max_{a\in A} f(a,b) \leq f(a^\star, b^\star).$$

The result is then required and the proof is complete. □

3. Cybersecurity Applications

3.1. Intrusion Detection Systems (IDS). In IDS design, the system is modeled as a game between an attacker choosing an attack vector and the defender choosing a detection strategy. The payoff function captures detection cost, false positives, and undetected attacks. Minimax optimization is used to derive robust IDS configurations.

3.2. Adversarial Machine Learning. In adversarial learning, we formulate a minimax problem:

$$\min_{\theta} \max_{\delta\in\mathscr{D}} \mathscr{L}(\theta, \delta)$$

where θ are model parameters and δ are adversarial perturbations. The goal is to find model parameters that are robust against worst-case attacks.

3.3. Resource Allocation. Defenders must optimally distribute resources (e.g., firewalls, monitoring) across network nodes. Minimax theory helps find the allocation that minimizes the impact of the worst-case attack. [1, 4, 8, 5]

3.4. Example: Zero-Sum Security Game. Let the defender choose between monitoring targets A, B, and the attacker chooses one to attack. The utility matrix is:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving this minimax game yields the optimal mixed strategy for both players, minimizing the expected loss.

4. Conclusion

Minimax theory provides powerful tools for analyzing adversarial interactions in cybersecurity. By modeling systems as strategic games, defenders can design robust strategies to mitigate threats even under worst-case scenarios.

Conflict of Interests

The authors declare that there is no conflict of interests.

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