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TRIPLED FIXED POINT THEOREMS IN COMPLETE $E-$FUZZY METRIC SPACES

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Abstract. Tripled fixed point theorems have significant role in the study of nonlinear differential equations and nonlinear integral equations. In this article, we establish some tripled fixed point theorems in complete $E-$fuzzy metric spaces and find some application of obtained results in integral equations.

Keywords: E-fuzzy metric space; complete E-fuzzy metric space; tripled fixed point; ϕ map.

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1. Introduction

L A Zadeh [10] introduced the concept of fuzzy set in 1965. The concept of fuzzy set is extended to many fields such as topology, functional analysis, algebra, analysis, graph theory and probability theory and the known results are generalized and applied to various fields. In 1994, A George and P Veeramani[2] introduced fuzzy metric space and studied some of its properties. In 2005, Mustafa and Sims[4] generalized the concept of metric space is called G-metric space and developed some results in this space.

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Following the concept of $G-$metric space, as a generalization of Fuzzy metric space, Sukanya K P and Sr. Magie Jose[8] introduced E-fuzzy metric space. They developed Hausdorff topology of this space. Using this concept we developed some fixed point theorems and coupled fixed point theorems in complete E-fuzzy metric spaces[7] and shown application of these theorems in existence and uniqueness of integral equations.

As an extension of coupled fixed point Samet B and Vetro C[5] developed the concept of fixed point of n order and when $n = 3$, it is the definition of tripled fixed point. Also Berinde V and Borcut M[1]modifying this concept of tripled fixed point for nonlinear mapping in partially ordered metric spaces.

In this paper, we prove tripled fixed point theorems for the mapping under contractive conditions and extend these results to prove tripled common fixed point theorems for two mappings satisfying ϕ contraction in complete $E-$fuzzy metric spaces. Finally as an application, The existence and uniqueness for the solutions of system of nonlinear non homogeneous fredholm integral equations of the second kind are presented.

2. Preliminaries

Definition 2.1. [6]: An operation $* : [0,1] \times [0,1] \to [0,1]$ which is binary, is a continuous $t-$norm if it has the following properties:

- commutativity and associativity,
- continuity,
- for all $a \in [0,1]$, $a * 1 = a$,
- for all $a,b,c,d \in [0,1]$ and $a \leq c$ and $b \leq d$, $a * b \leq c * d$.

Definition 2.2. [8] A 3$-$tuple $(X,E,*)$ is called a $E-$fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$norm and E is a fuzzy set on $X^3 \times (0,\infty)$ satisfying the following conditions for each $x,y,z,a \in X$ and $t,s > 0$,

- $E(x,y,z,t) > 0$ and $E(x,x,y,t) \geq E(x,y,z,t)$ for all $x,y,z \in X$ with $z \neq y$,
- $E(x,y,z,t) = 1$, for all $t > 0$ if and only if $x = y = z$,
- $E(x,y,z,t) = E(p(x,y,z),t)$ (symmetry), where p is a permutation function,
- $E(x,a,z,t) * E(a,y,z,s) \leq E(x,y,z,t+s)$,

- $E(x,y,z,.) : (0,\infty) \to [0,1]$ is continuous,

Example 2.1. [9] Let $X = R$ and G is a $G-$metric on X. The $t-$ norm is $a * b = ab$ for all $a,b \in [0,1]$. For each $t > 0$ and $E(x,y,z,t) = [exp(\frac{G(x,y,z)}{t})]^{-1}$. Then $(X,E,*)$ is a $E-$fuzzy metric space.

Lemma 2.1. [8] If $(X,E,*)$ be a $E-$fuzzy metric space, then $E(x,y,z,t)$ is non decreasing with respect to t, $\forall x,y,z \in X$.

Definition 2.3. [9] $(X,E,*)$ be a $E-$fuzzy metric space. An open ball center x_0 and radius r is given by $B_E(x_0,r,t) = \{x \in X; E(x_0,x,x,t) > 1-r\}$.

Definition 2.4. [9] $(X,E,*)$ be a $E-$fuzzy metric space. A sequence (x_n) in X converges to a point $x \in X$ if $E(x_n,x,x,t) \to 1$ as $n \to \infty$.

Proposition 2.1. [9] If (x_n) is a sequence in $(X,E,*)$ converges to $x \in X$, then

- $E(x_n,x_n,x,t) \to 1$ as $n \to \infty$
- $E(x_n,x_m,x,t) \to 1$ as $n \to \infty$
- $E(x_n,x_m,x,t) > 1-\varepsilon$ for $\varepsilon > 0$ and $m,n \geq n_0$.

Definition 2.5. [9] A sequence (x_n) in $(X,E,*)$ is said to be a cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$ there exists $n_0 \in N$ such that $E(x_m,x_n,x_l,t) > 1-\varepsilon$ for each $l,m,n \geq n_0$.

Definition 2.6. [9] A $E-$fuzzy metric space in which every cauchy sequence is convergent is said to be a complete E-fuzzy metric space.

Definition 2.7. [9] Let $(X_1,E_1,*)$ and $(X_2,E_2,*)$ be two generalized $E-$fuzzy metric spaces. A function $f : X_1 \to X_2$ is said to be continuous at a point $a \in X_1$ if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $E_2(f(x), f(a), f(a), s > 1-\varepsilon$ whenever $E_1(x,a,a,t) > 1-\delta$, $t,s > 0$.

Definition 2.8. [5] An element $(x,y,z) \in X^3$ is called a tripled fixed point of a mapping $F : X^3 \to X$ if $x = F(x,y,z), y = F(y,z,x)$ and $z = F(z,x,y)$.

Definition 2.9. [3] An element $(x,y,z) \in X^3$ is called

(1) a tripled coincidence point of the mappings $F : X^3 \to X$ and $g : x \to X$ if $gx = F(x,y,z), gy = F(y,z,x)$ and $gz = F(z,x,y)$. In this case (gx, gy, gz) is called a tripled point of coincidence.

(2) a common tripled fixed point of the mappings $F : X^3 \to X$ and $g : x \to X$ if $x = gx = F(x,y,z), y = gy = F(y,z,x)$ and $z = gz = F(z,x,y)$.

Example 2.2. [3] Let $X = R$. Define $F : X^3 \to X$ and $g : X \to X$ as follows $F(x,y,z) = sin(x)sin(y)sin(z)$ and $gx = 1 + x - \frac{\pi}{2}\ \forall x,y,z \in X$. Then $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ is a tripled coincidence point of F and g.

3. Main Results

This section proves the tripled fixed point theorems as an extension of the coupled fixed point theorems.

Theorem 3.1. Let $(X, E, *)$ be a complete $E-$fuzzy metric space and $F : X \times X \times X \to X$ be a mapping such that $\forall\ x,y,z,a,b,c,u,v,w \in X$, $t > 0$ and $0 < k < 1$,

$$E(F(x,y,z), F(a,b,c), F(u,v,w), kt) \geq E(x,a,u,t) * E(y,b,v,t) * E(z,c,w,t).$$

Then there exists unique $x \in X$ such that $x = F(x,x,x)$.

Proof. Let x_0, y_0 and z_0 be three arbitrary points in X. The sequences $(x_n), (y_n)$ and (z_n) in X can be constructed such that for $n \geq 0$

$$\begin{aligned} x_{n+1} &= F(x_n, y_n, z_n) \\ y_{n+1} &= F(y_n, z_n, x_n) \\ z_{n+1} &= F(z_n, x_n, y_n). \end{aligned}$$

Step 1: To prove the sequences $(x_n), (y_n)$ and (z_n) are cauchy sequences in $(X, E, *)$.

$$\begin{aligned} E(x_n, x_{n+1}, x_{n+2}, kt) &= E(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1}), kt) \\ &\geq E(x_{n-1}, x_n, x_{n+1}, t) * E(y_{n-1}, y_n, y_{n+1}, t) * E(z_{n-1}, z_n, z_{n+1}, t) \end{aligned}$$

Similarly,

$$E(y_n,y_{n+1},y_{n+2},kt)\geq E(y_{n-1},y_n,y_{n+1},t)*E(z_{n-1},z_n,z_{n+1},t)*E(x_{n-1},x_n,x_{n+1},t)$$

and

$$E(z_n,z_{n+1},z_{n+2},kt)\geq E(z_{n-1},z_n,z_{n+1},t)*E(x_{n-1},x_n,x_{n+1},t)*E(y_{n-1},y_n,y_{n+1},t)$$

Thus

$E(x_n,x_{n+1},x_{n+2},kt)$

$$\begin{aligned}
&\geq E(x_{n-1},x_n,x_{n+1},t)*E(y_{n-1},y_n,y_{n+1},t)*E(z_{n-1},z_n,z_{n+1},t)\\
&\geq E(x_{n-2},x_{n-1},x_n,\frac{t}{k})*E(y_{n-2},y_{n-1},y_n,\frac{t}{k})*E(z_{n-2},z_{n-1},z_n,\frac{t}{k})\\
&\quad *E(y_{n-2},y_{n-1},y_n,\frac{t}{k})*E(z_{n-2},z_{n-1},z_n,\frac{t}{k})*E(x_{n-1},x_n,x_{n+1},\frac{t}{k})\\
&\quad *E(z_{n-2},z_{n-1},z_n,\frac{t}{k})*E(x_{n-2},x_{n-1},x_n,\frac{t}{k})*E(y_{n-2},y_{n-1},y_n,\frac{t}{k})\\
&\geq [E(x_{n-2},x_{n-1},x_n,\frac{t}{k})]^3*[E(y_{n-2},y_{n-1},y_n,\frac{t}{k})]^3*[E(z_{n-2},z_{n-1},z_n,\frac{t}{k})]^3\\
&\geq [E(x_{n-3},x_{n-2},x_{n-1},\frac{t}{k^2})]^{3^2}*[E(y_{n-3},y_{n-2},y_{n-1},\frac{t}{k^2})]^{3^2}\\
&\quad *[E(z_{n-3},z_{n-2},z_{n-1},\frac{t}{k^2})]^{3^2}\\
&\geq [E(x_{n-4},x_{n-3},x_{n-2},\frac{t}{k^3})]^{3^3}*[E(y_{n-4},y_{n-3},y_{n-2},\frac{t}{k^3})]^{3^3}\\
&\quad *[E(z_{n-4},z_{n-3},z_{n-2},\frac{t}{k^3})]^{3^3}\\
&\vdots\\
&\geq [E(x_0,x_1,x_2,\frac{t}{k^{n-1}})]^{3^{(n-1)}}*[E(y_0,y_1,y_2,\frac{t}{k^{n-1}})]^{3^{(n-1)}}\\
&\quad *[E(z_0,z_1,z_2,\frac{t}{k^{n-1}})]^{3^{(n-1)}}
\end{aligned}$$

$$\longrightarrow 1 \text{ as } n\to\infty \tag{1}$$

In the same way

$$\begin{aligned}
E(x_{n+2},x_{n+1},x_n,kt) &\geq [E(x_2,x_1,x_0,\frac{t}{k^{n-1}})]^{3^{(n-1)}}*[E(y_2,y_1,y_0,\frac{t}{k^{n-1}})]^{3^{(n-1)}}\\
&\quad *[E(z_2,z_1,z_0,\frac{t}{k^{n-1}})]^{3^{(n-1)}}
\end{aligned}$$

$$\longrightarrow 1 \text{ as } n\to\infty \tag{2}$$

From 1 and 2, for $m,n \in \mathcal{N}$ with $m > n$,

$$
\begin{aligned}
&E(x_m, x_m, x_n, kt)\\
&\geq E(x_{n+1}, x_m, x_n, \frac{kt}{2}) * E(x_m, x_{n+1}, x_n, \frac{kt}{2})\\
&\geq E(x_n, x_{n+1}, x_{n+2}, \frac{kt}{4}) * E(x_{n+1}, x_{n+2}, x_{n+3}, \frac{kt}{8})\\
&\quad * E(x_{n+2}, x_{n+3}, x_{n+4}, \frac{kt}{16}) * \ldots * E(x_{m-2}, x_{m-1}, x_m, \frac{kt}{2^{m-n}})\\
&\quad * E(x_{n+2}, x_{n+1}, x_n, \frac{kt}{4}) * E(x_{n+3}, x_{n+2}, x_{n+1}, \frac{kt}{8})\\
&\quad * E(x_{n+4}, x_{n+3}, x_{n+2}, \frac{kt}{16}) * \ldots * E(x_m, x_{m-1}, x_{m-2}, \frac{kt}{2^{m-n}})\\
&\geq [E(x_0, x_1, x_2, \frac{t}{2^2 k^{n-1}})]^{3^{(n-1)}} * [E(y_0, y_1, y_2, \frac{t}{2^2 k^{n-1}})]^{3^{(n-1)}}\\
&\quad * [E(z_0, z_1, z_2, \frac{t}{2^2 k^{n-1}})]^{3^{(n-1)}} * \ldots * [E(x_0, x_1, x_2, \frac{t}{2^{m-n} k^{m-1}})]^{3^{(m-1)}}\\
&\quad * [E(y_0, y_1, y_2, \frac{t}{2^{m-n} k^{m-1}})]^{3^{(m-1)}} * [E(z_0, z_1, z_2, \frac{t}{2^{m-n} k^{m-1}})]^{3^{(m-1)}}\\
&\quad * [E(x_2, x_1, x_0, \frac{t}{2^2 k^{n-1}})]^{3^{(n-1)}} * [E(y_2, y_1, y_0, \frac{t}{2^2 k^{n-1}})]^{3^{(n-1)}}\\
&\quad * [E(z_2, z_1, z_0, \frac{t}{2^2 k^{n-1}})]^{3^{(n-1)}} * \ldots * [E(x_2, x_1, x_0, \frac{t}{2^{m-n} k^{m-1}})]^{3^{(m-1)}}\\
&\quad * [E(y_2, y_1, y_0, \frac{t}{2^{m-n} k^{m-1}})]^{3^{(m-1)}} * [E(z_2, z_1, z_0, \frac{t}{2^{m-n} k^{m-1}})]^{3^{(m-1)}}\\
&\longrightarrow 1 \text{ as } n \to \infty
\end{aligned}
$$

Thus (x_n) is a cauchy sequence in $(X, E, *)$. Similarly (y_n) and (z_n) are cauchy sequences in $(X, E, *)$.

Step 2: To prove (x, y, z) is a tripled fixed point of F.

Since X is a complete $E-$fuzzy metric space there exists some x, y and z in X such that $x_n \to x$, $y_n \to y$ and $z_n \to z$.

$$
\begin{aligned}
E(x_{n+1}, F(x,y,z), F(x,y,z), t) &\geq E(x_{n+1}, F(x,y,z), F(x,y,z), kt)\\
&\geq E(F(x_n, y_n, z_n), F(x,y,z), F(x,y,z), kt)\\
&\geq E(x_n, x, x, t) * E(y_n, y, y, t) * E(z_n, z, z, t).
\end{aligned}
$$

Since E is continuous on its variables and as $n \to \infty$,

$E(x, F(x,y,z), F(x,y,z), t) = 1$, which implies $F(x,y,z) = x$.

Similarly $F(y,z,x) = y$ and $F(z,x,y) = z$.

Hence (x,y,z) is a tripled fixed point of F.

Step 3: To show $x = y = z$.

$$\begin{aligned}
E(x,y,z,kt) &= E(F(x,y,z),F(y,z,x),F(z,x,y),kt) \\
&\geq E(x,y,z,t) * E(y,z,x,t) * E(z,x,y,t) \\
&\geq [E(x,y,z,t)]^3 \\
&\geq [E(x,y,z,\frac{t}{k})]^{3^3} \\
&\geq [E(x,y,z,\frac{t}{k^2})]^{3^{3^2}} \\
&\vdots \\
&\geq [E(x,y,z,\frac{t}{k^n})]^{3^{3^{3^n}}} \\
&\longrightarrow 1 \text{ as } n \to \infty
\end{aligned}$$

Hence $x = y = z$.

Step 4: To show uniqueness of x.

Suppose x' is another point such that $x' = F(x',x',x')$. Then

$$\begin{aligned}
E(x,x,x',kt) &= E(F(x,x,x)F(x,x,x),F(x',x',x'),kt) \\
&\geq E(x,x,x',t) * E(x,x,x',t) * E(x,x',x',t) \\
&\geq [E(x,x,x',t)]^3 \\
&\geq [E(x,x,x',\frac{t}{k})]^{3^3} \\
&\geq [E(x,x,x',\frac{t}{k^2})]^{3^{3^2}} \\
&\vdots \\
&\geq [E(x,x,x',\frac{t}{k^n})]^{3^{3^n}} \\
&\longrightarrow 1 \text{ as } n \to \infty.
\end{aligned}$$

Thus $x = x'$. Hence F has unique tripled fixed point (x,x,x).

$\square$

Theorem 3.2. Let $(X,E,*)$ be a complete $E-$fuzzy metric space and $F : X\times X\times X \to X$ be a mapping such that $\forall\, x,y,z,a,b,c,u,v,w \in X$, $t>0$ and $0<k<1$,

$$E(F(x,y,z),F(a,b,c),F(u,v,w),kt) \geq \min\{E(x,a,u,t),E(y,b,v,t),E(z,c,w,t)\}.$$

Then there exists unique $x\in X$ such that $x = F(x,x,x)$.

Proof. Let $x_0,y_0,z_0 \in X$, we can construct three sequences $(x_n),(y_n)$ and (z_n) in X such that for $n \geq 0$

$$x_{n+1} = F(x_n,y_n,z_n)$$

$$y_{n+1} = F(y_n,z_n,x_n)$$

$$z_{n+1} = F(z_n,x_n,y_n).$$

Step 1: To prove (x_n), (y_n) and (z_n) are cauchy sequences in X.

$$\begin{aligned} E(x_n,x_{n+1},x_{n+2},kt) &= E(F(x_{n-1},y_{n-1},z_{n-1}),F(x_n,y_n,z_n),F(x_{n+1},y_{n+1},z_{n+1}),kt) \\ &\geq \min\{E(x_{n-1},x_n,x_{n+1},t),E(y_{n-1},y_n,y_{n+1},t),E(z_{n-1},z_n,z_{n+1},t)\} \end{aligned}$$

Similarly,

$$E(y_n,y_{n+1},y_{n+2},kt) \geq \min\{E(y_{n-1},y_n,y_{n+1},t),E(z_{n-1},z_n,z_{n+1},t),E(x_{n-1},x_n,x_{n+1},t)\}$$

and

$$E(z_n,z_{n+1},z_{n+2},kt) \geq \min\{E(z_{n-1},z_n,z_{n+1},t),E(x_{n-1},x_n,x_{n+1},t),E(y_{n-1},y_n,y_{n+1},t)\}$$

Hence

$$\begin{aligned} E(x_n,x_{n+1},x_{n+2},kt) &\geq \min\{E(x_{n-1},x_n,x_{n+1},t),E(y_{n-1},y_n,y_{n+1},t), \\ &\qquad E(z_{n-1},z_n,z_{n+1},t)\} \\ &\geq \min\{E(x_{n-2},x_{n-1},x_n,\frac{t}{k}),E(y_{n-2},y_{n-1},y_n,\frac{t}{k}), \\ &\qquad E(z_{n-2},z_{n-1},z_n,\frac{t}{k})\} \\ &\geq \min\{E(x_{n-3},x_{n-2},x_{n-1},\frac{t}{k^2}),E(y_{n-3},y_{n-2},y_{n-1},\frac{t}{k^2}), \\ &\qquad E(z_{n-3},z_{n-2},z_{n-1},\frac{t}{k^2})\} \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&\geq \min\{E(x_0,x_1,x_2,\frac{t}{k^{n-1}}),E(y_0,y_1,y_2,\frac{t}{k^{n-1}}), \\
&\qquad E(z_0,z_1,z_2,\frac{t}{k^{n-1}})\} \\
&\longrightarrow 1 \text{ as } n\to\infty, \text{ since } \lim_{t\to\infty} E(x,y,z,t)=1 \qquad (3)
\end{aligned}
$$

Similar way, we get for $n \geq 0$

$$E(x_{n+2},x_{n+1},x_n,kt) \longrightarrow 1 \text{ as } n\to\infty \qquad (4)$$

From 3 and 4, for $m,n \in \mathcal{N}$ with $m > n$

$$
\begin{aligned}
E(x_m,x_m,x_n,kt) &\geq E(x_{n+1},x_m,x_n,\frac{kt}{2}) * E(x_m,x_{n+1},x_n,\frac{kt}{2}) \\
&\geq E(x_n,x_{n+1},x_{n+2},\frac{kt}{4}) * E(x_{n+1},x_{n+2},x_{n+3},\frac{kt}{8}) \\
&\quad * E(x_{n+2},x_{n+3},x_{n+4},\frac{kt}{16}) * \ldots * E(x_{m-2},x_{m-1},x_m,\frac{kt}{2^{m-n}}) \\
&\quad * E(x_{n+2},x_{n+1},x_n,\frac{kt}{4}) * E(x_{n+3},x_{n+2},x_{n+1},\frac{kt}{8}) \\
&\quad * E(x_{n+4},x_{n+3},x_{n+2},\frac{kt}{16}) * \ldots * E(x_m,x_{m-1},x_{m-2},\frac{kt}{2^{m-n}}) \\
&\longrightarrow 1 \text{ as } n\to\infty
\end{aligned}
$$

Hence (x_n) is a cauchy sequence in $(X,E,*)$. Similarly (y_n) and (z_n) are cauchy sequences in $(X,E,*)$.

Step 2: To show (x,y,z) is a tripled fixed point of F.

Since X is a complete $E-$fuzzy metric space, the sequences (x_n), (y_n) and (z_n) are converges to some x,y and z in X.

$$
\begin{aligned}
E(x_{n+1},F(x,y,z),F(x,y,z),t) &\geq E(x_{n+1},F(x,y,z),F(x,y,z),kt) \\
&\geq E(F(x_n,y_n,z_n),F(x,y,z),F(x,y,z),kt) \\
&\geq \min\{E(x_n,x,x,t),E(y_n,y,y,t),E(z_n,z,z,t)\}.
\end{aligned}
$$

Since E is continuous on its variables and as $n\to\infty$,

$E(x,F(x,y,z),F(x,y,z),t) = 1$, which implies $F(x,y,z) = x$.

Similarly $F(y,z,x) = y$ and $F(z,x,y) = z$.

Hence (x,y,z) is a tripled fixed point of F.

Step 3: To show $x = y = z$.

Consider

$$\begin{aligned} E(x,y,z,kt) &= E(F(x,y,z),F(y,z,x),F(z,x,y),kt) \\ &\geq \min\{E(x,y,z,t),E(y,z,x,t),E(z,x,y,t)\}. \end{aligned}$$

Thus $E(x,y,z,kt) \geq E(x,y,z,t)$.

Also, $E(x,y,z,t) \geq E(x,y,z,kt)$, since E is non-decreasing with respect to t. Thus $E(x,y,z,t) = E(x,y,z,kt)$, $0 < k < 1$ and $t > 0$. This happens only for $x = y = z$. Hence (x,x,x) is the tripled fixed point of F.

Step 4: To show uniqueness of x.

Suppose x' is another point such that $x' = F(x',x',x')$.

Then

$$\begin{aligned} E(x',x,x,kt) &= E(F(x',x',x'),F(x,x,x),F(x,x,x),kt) \\ &\geq \min\{E(x',x,x,,t),E(x',x,x,t),E(x',x,x,t)\} \\ &\geq E(x',x,x,t) \end{aligned}$$

Thus, $E(x',x,x,t) \geq E(x',x,x,kt) \geq E(x',x,x,t)$, implies $x' = x$.

Hence F has unique tripled fixed point x, that is $x = F(x,x,x)$. □

Following results are the generalization of above results using the ϕ function.

Theorem 3.3. Let $(X,E,*)$ be a complete $E-$fuzzy metric space and $F : X \times X \times X \to X$ be a continuous mapping such that $\forall\, x,y,z,a,b,c,u,v,w \in X$ and $t > 0$

$$E(F(x,y,z),F(a,b,c),F(u,v,w),\phi(t)) \geq E(x,a,u,t) * E(y,b,v,t) * E(z,c,w,t).$$

Then there exists unique $x \in X$ such that $x = F(x,x,x)$.

Theorem 3.4. Let $(X,E,*)$ be a complete $E-$fuzzy metric space and $F : X \times X \times X \to X$ be a continuous mapping such that $\forall\, x,y,z,a,b,c,u,v,w \in X$ and $t > 0$,

$$E(F(x,y,z),F(a,b,c),F(u,v,w),\phi(t)) \geq \min\{E(x,a,u,t),E(y,b,v,t),E(z,c,w,t)\}.$$

Then there exists unique $x \in X$ such that $x = F(x,x,x)$.

Now the tripled coincidence point theorems of the mappings F and g are established.

Theorem 3.5. Let $(X,E,*)$ be a complete $E-$fuzzy metric space. Let $F : X \times X \times X \to X$ and $g : X \to X$ be mappings such that $\forall x,y,z,a,b,c,u,v,w \in X$,

$$E(F(x,y,z),F(a,b,c),F(u,v,w),\phi(t)) \geq E(g(x),g(a),g(u),t) * E(g(y),g(b),g(v),t) * E(g(z),g(c),g(w),t).$$

Assume that F and g satisfy the following conditions

- $F(X \times X \times X) \subseteq g(X)$
- g is continuous and commutes with F

Then there exists a unique x in X such that $g(x) = F(x,x,x) = x$.

Corollary 3.1. Let $(X,E,*)$ be a complete $E-$fuzzy metric space. Let $F : X \times X \times X \to X$ and $g : X \to X$ be mappings such that $\forall x,y,z,a,b,c,u,v,w \in X$ and $k, 0 < k < 1$,

$$E(F(x,y,z),F(a,b,c),F(u,v,w),kt) \geq E(g(x),g(a),g(u),t) * E(g(y),g(b),g(v),t) * E(g(z),g(c),g(w),t).$$

Suppose that F and g satisfy the following conditions

- $F(X \times X \times X) \subseteq g(X)$
- g is continuous and commutes with F

Then there exists a unique x in X such that $g(x) = F(x,x,x) = x$.

Proof. Taking $\phi(t) = kt$ in Theorem 3.5 with $0 < k < 1$, we get the result. □

Theorem 3.6. Let $(X,E,*)$ be a complete $E-$fuzzy metric space. Let $F : X \times X \times X \to X$ and $g : X \to X$ be mappings such that $\forall x,y,z,a,b,c,u,v,w \in X$,

$$E(F(x,y,z),F(a,b,c),F(u,v,w),\phi(t)) \geq \min\{E(g(x),g(a),g(u),t),E(g(y),g(b),g(v),t), E(g(z),g(c),g(w),t)\}.$$

If F and g satisfy the following conditions

- $F(X \times X \times X) \subseteq g(X)$
- g is continuous and commutes with F

Then there exists a unique x in X such that $g(x) = F(x,x,x) = x$.

Corollary 3.2. Let $(X,E,*)$ be a complete $E-$fuzzy metric space. Let $F : X \times X \times X \to X$ and $g : X \to X$ be mappings such that $\forall x,y,z,a,b,c,u,v,w \in X$ and $k, 0 < k < 1$,

$$E(F(x,y,z),F(a,b,c),F(u,v,w),kt) \geq \min\{E(g(x),g(a),g(u),t),E(g(y),g(b),g(v),t),$$
$$E(g(z),g(c),g(w),t)\}.$$

If F and g satisfy the following conditions

- $F(X \times X \times X) \subseteq g(X)$
- g is continuous and commutes with F

Then there exists a unique x in X such that $g(x) = F(x,x,x) = x$.

Proof. Taking $\phi(t) = kt$ in Theorem 3.6, we get the result. □

4. APPLICATION OF TRIPLED FIXED POINT THEOREMS

As an application of the results in section 3, solutions of system of three nonlinear integral equations are studied by considering a map and shows that there exists a unique solution of system if and only if there exists tripled fixed point of the map.

Let $X = C[a,b]$ be the set of all continuous real-valued functions on $[a,b]$ and define $E-$fuzzy metric $E : X^3 \times (0,\infty) \to [0,1]$ as $E(x,y,z,t') = [exp(\frac{G(x,y,z)}{t'})]^{-1}$ where $G : X^3 \to R$ is defined as $G(x,y,z) = sup_{t\in[a,b]}|x(t)-y(t)| + sup_{t\in[a,b]}|y(t)-z(t)| + sup_{t\in[a,b]}|z(t)-x(t)|$. Let the $t-$norm $*$ is defined as $x*y = x.y$ and the ϕ function, $\phi(t') = \lambda t'$, $0 < \lambda < 1$. Then $(X,E,*)$ is a complete $E-$ fuzzy metric space. Consider the system of nonlinear non homogeneous fredholm integral equations of the second kind,

$$\begin{aligned} x_1(t) &= g(t) + \lambda \int_a^b f(t,s,x_1(s),x_2(s),x_3(s))ds \\ x_2(t) &= g(t) + \lambda \int_a^b f(t,s,x_2(s),x_3(s),x_1(s))ds \end{aligned}$$

$$x_3(t) = g(t)+\lambda\int_a^b f(t,s,x_3(s),x_1(s),x_2(s))ds$$

Where $t,s\in[a,b]$, $0<a<b<\infty$ and λ is a parameter, $0<\lambda<1$. Consider the following conditions,

(1) $g:[a,b]\to R^+$ and $f:[a,b]\times[a,b]\times R^+\times R^+\times R^+\to R^+$ are continuous.

(2) there exists an integrable function $k:[a,b]\times[a,b]\to R^+$ such that

$$|f(t,s,x_1(s),x_2(s),x_3(s))-f(t,s,y_1(s),y_2(s),y_3(s))|<k(t,s)[|x_1(s)-y_1(s)| + |x_2(s)-y_2(s)|+|x_3(s)-y_3(s)|]$$

where $t,s\in[a,b]$.

(3) $sup_{t\in[a,b]}|\int_a^b k(t,s)ds|\leq\frac{1}{b-a}$

Under these conditions a solution exists for the above system of integral equations. We can show the existence and uniqueness of solution of the above system of equations by using Theorem 3.5.

For, define the mapping $F:X\times X\times X\to X$ as

$$F(x_1,x_2,x_3)(t)=g(t)+\lambda\int_a^b f(t,s,x_1(s),x_2(s),x_3(s))ds$$

where $t,s\in[a,b]$ and $x_1,x_2,x_3\in X$.

Now,

$G(F(x_1,x_2,x_3)(t),F(y_1,y_2,y_3)(t),F(z_1,z_2,z_3)(t))$

$$\begin{aligned}&= sup_{t\in[a,b]}|F(x_1,x_2,x_3)(t)-F(y_1,y_2,y_3)(t)|\\ &\quad +sup_{t\in[a,b]}|F(y_1,y_2,y_3)(t)-F(z_1,z_2,z_3)(t)|\\ &\quad +sup_{t\in[a,b]}|F(z_1,z_2,z_3)(t)-F(x_1,x_2,x_3)(t)|\end{aligned}$$

$$\begin{aligned}&\leq sup_{t\in[a,b]}|\lambda\int_a^b f(t,s,x_1(s),x_2(s),x_3(s))ds-\lambda\int_a^b f(t,s,y_1(s),y_2(s),y_3(s))ds|\\ &\quad +sup_{t\in[a,b]}|\lambda\int_a^b f(t,s,y_1(s),y_2(s),y_3(s))ds-\lambda\int_a^b f(t,s,z_1(s),z_2(s),z_3(s))ds|\\ &\quad +sup_{t\in[a,b]}|\lambda\int_a^b f(t,s,z_1(s),z_2(s),z_3(s))ds-\lambda\int_a^b f(t,s,x_1(s),x_2(s),x_3(s))ds|\end{aligned}$$

$$\leq \lambda (sup_{t\in[a,b]}|\int_a^b f(t,s,x_1(s),x_2(s),x_3(s))ds - \int_a^b f(t,s,y_1(s),y_2(s),y_3(s))ds|$$
$$+sup_{t\in[a,b]}|\int_a^b f(t,s,y_1(s),y_2(s),y_3(s))ds - \int_a^b f(t,s,z_1(s),z_2(s),z_3(s))ds|$$
$$+sup_{t\in[a,b]}|\int_a^b f(t,s,z_1(s),z_2(s),z_3(s))ds - \int_a^b f(t,s,x_1(s),x_2(s),x_3(s))ds|)$$

$$< \lambda sup_{t\in[a,b]}\int_a^b k(t,s)ds \int_a^b (sup_{t\in[a,b]}[|x_1(t)-y_1(t)|+|x_2(s)-y_2(t)|+|x_3(t)-y_3(t)|]$$
$$+sup_{t\in[a,b]}[|y_1(t)-z_1(t)|+|y_2(t)-z_2(t)|+|y_3(t)-z_3(t)|]+sup_{t\in[a,b]}[|z_1(t)-x_1(t)|$$
$$+|z_2(t)-x_2(t)|+|z_3(t)-x_3(t)|])ds$$

$$< \lambda (sup_{t\in[a,b]}[|x_1(t)-y_1(t)|+|x_2(s)-y_2(t)|+|x_3(t)-y_3(t)|]$$
$$+sup_{t\in[a,b]}[|y_1(t)-z_1(t)|+|y_2(t)-z_2(t)|+|y_3(t)-z_3(t)|]$$
$$+sup_{t\in[a,b]}[|z_1(t)-x_1(t)|+|z_2(t)-x_2(t)|+|z_3(t)-x_3(t)|])$$

$$< \lambda \left(G(x_1,y_1,z_1)+G(x_2,y_2,z_2)+G(x_3,y_3,z_3)\right).$$

Now,

$E(F(x_1,x_2,x_3)(t),F(y_1,y_2,y_3)(t),F(z_1,z_2,z_3)(t),\lambda t')$

$$\begin{aligned}
&= \left[exp\frac{G(F(x_1,x_2,x_3)(t),F(y_1,y_2,y_3)(t),F(z_1,z_2,z_3)(t))}{\lambda t'}\right]^{-1} \\
&\geq \left[exp\frac{\lambda(G(x_1,y_1,z_1)+G(x_2,y_2,z_2)+G(x_3,y_3,z_3))}{\lambda t'}\right]^{-1} \\
&\geq E(x_1,y_1,z_1,t')*E(x_2,y_2,z_2,t')*E(x_3,y_3,z_3,t')
\end{aligned}$$

Thus by Theorem 3.5 there exists unique tripled fixed point (x,x,x) of F. Since the triple (x,x,x) is a solution of system of equations if and only if (x,x,x) is a tripled fixed point of F. Thus (x,x,x) is a solution of the system of equations.

5. CONCLUSION

In this paper, tripled fixed point theorem of the mapping and tripled common fixed point theorems of two mappings are studied and applied these results to study existence and uniqueness for the solution of system of nonlinear non homogeneous Fredholm integral equation of second kind. Also these results can be established for fixed point of $N-$order in complete $E-$ fuzzy metric space and using different types of contractive conditions, we can develop these results in complete $E-$ fuzzy metric spaces.

Conflict of Interests

The authors declare that there is no conflict of interests.

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