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FIXED POINT OF HARDY-ROGERS CONTRACTION MAPPINGS IN NON-SOLID CONE χ_b-METRIC SPACE WITH APPLICATIONS

ZAHIA DJEDID[1], IQBAL M. BATIHA[2,3,*], SHARIFA ALSHARIF[4], SANA ABU-GHURRA[5], MAZIN ALJAZZAZI[6], JAMILA JAWDAT[7]

[1]Department of Mathematics, The University of Science and Technology Houari Boumedience, Bab Ezzouar, Algeria
[2]Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan
[3]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, UAE
[4]Department of Mathematics, Yarmouk University, Irbid, Jordan
[5]Department of Mathematics, Faculty of Science, Applied Science Private University, Amman, Jordan
[6]Department of Mathematics, The University of Jordan, Amman, Jordan
[7]Department of Mathematics, Zarqa University, Zarqa, Jordan

Abstract. The concept of cone χ_b-metric spaces was recently introduced in 2023 as a generalization of both cone metric spaces and traditional χ-metric spaces, allowing for the possibility that the underlying cones are non-solid. The main objective of this article is to establish certain fixed point results for Hardy–Rogers-type mappings and to investigate the T-stability of Picard's iteration in the framework of non-solid cone χ_b-metric spaces. As an application, we employ the main results to demonstrate the existence and uniqueness of solutions to a nonlinear integral equation.

Keywords: non-solid cone; cone χ_b-metric spaces; T-stability; fixed point.

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*Corresponding author
E-mail address: i.batiha@zuj.edu.jo
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1. Introduction

In recent decades, nonlinear functional analysis—particularly fixed point theory—has been extended to a variety of abstract spaces [1, 2, 3, 4]. It has been widely applied to numerous scientific problems, bridging both applied and pure mathematical methods, especially in relation to computational issues [5, 6, 7, 8]. Fixed point theory has played a key role in modeling and analyzing a broad range of real-world phenomena and applications, including the study of differential and integral equations, as well as problems in physics, engineering, biology, and the social sciences [9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

In 1992, one of the most influential and foundational results in this area, known as the Banach contraction mapping theorem (see [19]), was established by the Polish mathematician Stefan Banach. Since then, the field of fixed point theory has witnessed significant developments and wide-ranging applications.

In some developments in non-convex analysis, an ordering on the space was introduced via a cone. In [20], Huang and Zhang introduced the notion of a cone metric space as a generalization of conventional metric spaces. Within that framework, several fixed point theorems for contractive mappings were established, thereby extending classical results from standard metric spaces to cone metric spaces.

Notably, Hussain and Shah [21] introduced the concept of cone b-metric spaces, which provide a broader structure generalizing b-metric spaces. Several corresponding fixed point results were developed within this setting, as reported in [22]–[23]. Subsequently, Khamsi [24] and Khamsi and Hussain [25] revisited these ideas and introduced the concept of metric-type spaces, further enriching the theory and expanding its applicability. On the other hand, Filipović and Kukić investigated certain classical contraction principles of Hardy–Rogers [26] in the context of b-metric spaces, without assuming continuity of the underlying metric.

In recent years, the stability of fixed point iteration methods has attracted significant research interest across various branches of mathematics. Numerous studies have been conducted on the stability of iterative procedures in different types of spaces. Among these, Picard's iteration stands out as one of the most fundamental methods, and its stability plays a crucial role in many fields.

The concept of Υ-stability for iterative procedures has also been investigated within cone metric spaces, particularly by Asadi [27] and Yousefi [28] (see also [29] for more on the Υ-stability of Picard's iteration). More recently, in 2023, we introduced the concept of cone χ_b-metric spaces [30], establishing several properties related to cones with empty interior that contain semi-interior points. Within this framework, we demonstrated the validity of several classical fixed point results—such as those by Banach, Kannan, and Chatterjea—even in the presence of cones that are neither solid nor normal.

This article investigates classical contraction principles of Hardy–Rogers in the framework of cone χ_b-metric spaces, with particular emphasis on settings involving non-solid and non-normal cones. In addition, we explore the Υ-stability of various iterative schemes used to approximate fixed points of mappings that satisfy the Hardy–Rogers contraction condition, specifically within e-complete cone χ_b-metric spaces. As an application, we establish a result concerning the existence and uniqueness of solutions to a nonlinear integral equation. Illustrative examples are also provided to support and validate the theoretical findings.

Definition 1. [19] *Let χ be a vector space over $\mathbb{R}$, and let 0_χ denote the zero element in χ. A nonempty closed convex subset $\chi^+ \subset \chi$ is said to be a* cone *if it satisfies the following conditions:*

(1) For every $\kappa \in \chi^+$ and $\sigma \geq 0$, it holds that $\sigma\kappa \in \chi^+$;
(2) If $\kappa \in \chi^+$ and $-\kappa \in \chi^+$, then $\kappa = 0_\chi$.

Definition 2. [20] *Let χ be a real normed space and χ^+ a positive cone in χ. A relation $\preceq$ is called a* partial ordering *on χ if*

$$\kappa, \rho \in \chi \text{ and } \kappa \preceq \rho \text{ if and only if } \rho - \kappa \in \chi^+.$$

It follows directly that

$$\kappa \in \chi^+ \text{ if and only if } 0_\chi \preceq \kappa.$$

Consistent with the work of Huang and Zhang [20], the following definitions and results are necessary for the subsequent development.

Definition 3. [20] *Let χ be a vector space over $\mathbb{R}$ equipped with a partial order relation "$\preceq$". We say that χ is an* ordered space *if the following conditions hold:*

- *For all $\kappa, w, \theta \in \chi$, if $\kappa \preceq w$, then $\kappa + \theta \preceq w + \theta$;*
- *For all $\delta \in \mathbb{R}^+$ and $\kappa \in \chi$ with $\kappa \succeq 0_\chi$, it holds that $\delta\kappa \succeq 0_\chi$.*

Furthermore, if χ is equipped with a norm $\|\cdot\|$, then it is called a *normed ordered space*.

Definition 4. [20] *Let χ be a normed ordered space. Then*

(1) The cone χ^+ is said to be solid *if* $\operatorname{int}(\chi^+) \neq \varnothing$*;*

(2) The cone χ^+ is said to be normal *if there exists a constant $\iota > 0$ such that*

$$0_\chi \preceq \kappa \preceq \rho \;\Rightarrow\; \|\kappa\| \leq \iota \|\rho\|,$$

for all $\kappa, \rho \in \chi$. The smallest such constant ι is called the normality constant *of χ^+.*

(3) The cone χ^+ is said to be regular *if every increasing sequence bounded above is convergent. That is, if a sequence $\{\rho_n\}_{n\geq 1}$ satisfies*

$$\rho_1 \preceq \rho_2 \preceq \cdots \preceq \rho_n \preceq \cdots \preceq a, \quad \text{for some } a \in \chi^+,$$

then there exists $\rho \in \chi^+$ such that

$$\|\rho_n - \rho\| \to 0 \quad \text{as } n \to \infty.$$

It is evident that every regular cone is also normal. We now proceed to define cone metric and b-metric spaces.

Definition 5. [20] *Let $\mathscr{X}$ be a non-empty set, and let χ be a partially ordered vector space with respect to the positive cone χ^+. A mapping $d : \mathscr{X} \times \mathscr{X} \to \chi$ is called a* cone metric *if it satisfies the following conditions for all $\kappa, \rho, w \in \mathscr{X}$:*

(1) $0_\chi \preceq d(\kappa, \rho)$ and $d(\kappa, \rho) = 0_\chi$ if and only if $\kappa = \rho$;

(2) $d(\kappa, \rho) = d(\rho, \kappa)$;

(3) $d(\kappa, \rho) \preceq d(\kappa, w) + d(w, \rho)$.

Then, d is called a cone metric on $\mathscr{X}$, and the pair $(\mathscr{X}, d)$ is referred to as a *cone metric space*.

Example 1. [31] *Let* $\chi = \mathbb{R}^3$ *and define the cone* $\chi^+ = \left\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1, z_2, z_3 \geq 0\right\}$. *Let* $\mathscr{X} = \mathbb{R}$. *Define the mapping* $d^\chi : \mathscr{X} \times \mathscr{X} \to \chi$ *by*

$$d^\chi(\kappa, \rho) = (z_1 |\kappa - \rho|, z_2 |\kappa - \rho|, z_3 |\kappa - \rho|),$$

where $z_1, z_2, z_3 \geq 0$. *Then,* $(\mathscr{X}, d^\chi)$ *is a cone metric space. Consequently, the cone* χ^+ *is normal with a normality constant* $z = 1$.

Definition 6. [32] *Let* $\mathscr{X}$ *be a non-empty set and let* $d : \mathscr{X} \times \mathscr{X} \to \mathbb{R}^+$. *Then* d *is called a* b-metric *on* $\mathscr{X}$ *if there exists a constant* $b \geq 1$ *such that for all* $\kappa, \rho, w \in \mathscr{X}$*, the following conditions are satisfied:*

(1) $d(\kappa, \rho) = 0$ *if and only if* $\kappa = \rho$*;*

(2) $d(\kappa, \rho) = d(\rho, \kappa)$*;*

(3) $d(\kappa, \rho) \leq b[d(\kappa, w) + d(w, \rho)]$.

Under these conditions, the pair $(\mathscr{X}, d)$ is called a *b-metric space*.

2. THE CONE χ_b-METRIC SPACE

Let χ be an ordered normed space. We denote its zero element by 0_χ and its positive cone by χ^+. Let $\mathscr{U} = \{\kappa \in \chi : \|\kappa\| \leq 1\}$ denote the closed unit ball in χ, and let $\mathscr{U}_+ = \mathscr{U} \cap \chi^+$ denote the positive part of the unit ball.

Definition 7. *A point* $\kappa_0 \in \chi^+$ *is said to be a* semi-interior point *of* χ^+ *if there exists a real number* $z > 0$ *such that*

$$\kappa_0 - z\mathscr{U}_+ \subseteq \chi^+.$$

Throughout this work, we denote by $(\chi^+)^\oslash$ the set of all semi-interior points of χ^+. It is important to note that every interior point of χ^+ is also a semi-interior point. However, the converse does not hold, as demonstrated by the following example (Example 2.5 in [33]).

Example 2. *Let* $\chi_i = \mathbb{R}^2$ *be ordered pointwise and endowed with the norm* $\|\cdot\|_k$*, where*

$$\|(\kappa, \rho)\|_k = \begin{cases} |\kappa| + |\rho|, & \text{if } \kappa\rho \geq 0, \\ \max\{|\kappa|, |\rho|\} - \dfrac{k-1}{k} \min\{|\kappa|, |\rho|\}, & \text{if } \kappa\rho < 0 \end{cases}$$

We designate D_i as the unit ball of χ_i, which corresponds to the polygon with vertices:

$$(-i,i),\quad (-1,0),\quad (0,-1),\quad (i,-i),\quad (1,0),\quad (0,1).$$

Let

$$\chi = \left\{ \begin{array}{l} \kappa = (\kappa_i)_{i\in\mathbb{N}},\ \kappa_i = (\kappa_i^1, \kappa_i^2) \in \chi_i \text{ such that} \\ \|\kappa_i\|_i \leq m_k,\ \text{where } m_k > 0 \text{ depends on } k \end{array} \right\}.$$

Assume that χ is ordered using $\chi^+ = \left\{\rho = (\rho_i) \in \chi : \rho_i \in \mathbb{R}_+^2 \text{ for all } i\right\}$, and is equipped with the norm

$$\|\rho\|_\infty = \sup_{i\in\mathbb{N}} \|\rho_i\|_i .$$

Let $\mathscr{Z} = \chi^+ - \chi^+$ be the subspace of χ generated by χ^+, ordered by $\mathscr{Z}^+ = \chi^+$. Now, define $\mathbf{1} = (\rho_i) \in \chi$ by $\rho_i = (1,1)$ for every i. Then, $\mathbf{1}$ is not an interior point of $\mathscr{Z}^+$. Indeed, for any positive integer m, define $\kappa = (\kappa_i)$ in $\mathscr{Z}$ such that $\kappa_m = (-2,2)$ and $\kappa_i = (0,0)$ for all $i \neq m$. It is easy to verify that

$$\|\kappa\|_\infty = \frac{2}{m}, \quad \text{but} \quad \mathbf{1} + \kappa \notin \chi^+.$$

Therefore, $\mathbf{1} + \lambda\mathscr{U}_+ \nsubseteq \chi^+$ for any $\lambda > 0$, and hence $\mathbf{1}$ is not an interior point of χ^+. However, in the same way, it can be shown that $\operatorname{int}(\chi^+) = \varnothing$*, while the point $\mathbf{1} = (\rho_i)$ is a* semi-interior *point of χ^+.*

Now, let χ be a normed space equipped with the order induced by its positive cone χ^+. For $\kappa, \rho \in \chi^+$, we write $\kappa \lll \rho$ if and only if $\rho - \kappa \in (\chi^+)^\ominus$. It is easy to verify that

$$\kappa \in \left(\chi^+\right)^\ominus \text{ if and only if } 0_\chi \lll \kappa.$$

The following outlines the topological properties associated with semi-interior points in χ-metric spaces, as defined in [34].

Definition 8. *A sequence $\{\rho_n\}$ in χ^+ is said to be an e-sequence if for each $0_\chi \lll e$, there exists a natural number k such that $\rho_n \lll e$ for all $n > k$.*

It is straightforward to observe that (ρ_n) is an e-sequence if $\rho_n \to 0_\chi$ as $n \to \infty$.

Proposition 1. *If $\kappa, \rho \in \chi$ and $\rho \lll \kappa$, then $\rho \preceq \kappa$.*

Proposition 2. *If the condition* $0_\chi \preceq u \lll e$ *holds for every* $e \in (\chi^+)^\oslash$, *then it can be concluded that* $u = 0_\chi$.

Proposition 3. *Let* $0 \leq \lambda < 1$ *be a constant, and let* $u \in \chi^+$ *such that* $u \preceq \lambda u$. *Then* $u = 0_\chi$.

Lemma 1. *If* $\rho \lll \kappa + e$ *holds for every* $e \in (\chi^+)^\oslash$, *then* $\rho \lll \kappa$.

We now introduce the concept of the e-sequence in E-metric spaces as defined in [34].

Proposition 4. *Let* $\{\kappa_n\}$ *and* $\{\rho_n\}$ *be two e-sequences in* χ. *For constants* $\gamma, \nu \geq 0$, *the sequence* $\{\gamma\kappa_n + \nu\rho_n\}$ *is also an e-sequence.*

Proposition 5. *Let* $\{\kappa_n\}$ *be a sequence in* χ *such that* $\kappa_n \to 0_\chi$ *as* $n \to \infty$. *Then* $\{\kappa_n\}$ *is an e-sequence.*

Proposition 6. *Let* $\{\kappa_n\}$ *and* $\{\rho_n\}$ *be two sequences in* χ *satisfying* $\kappa_n \preceq \rho_n$ *and* $\rho_n \to 0_\chi$ *as* $n \to \infty$. *Then* $\{\kappa_n\}$ *is an e-sequence.*

Remark 1. *Let* $\{\kappa_n\}$ *and* $\{\rho_n\}$ *be two sequences in* χ *such that* $\kappa_n \preceq \rho_n$. *Then* $\{\kappa_n\}$ *is an e-sequence provided that* $\{\rho_n\}$ *is an e-sequence.*

Proposition 7. *Let* $0 \leq \lambda < 1$ *be a constant, and let* $\{\kappa_n\}$ *and* $\{\rho_n\}$ *be sequences in* χ^+ *satisfying*

$$\kappa_{n+1} \preceq \lambda\kappa_n + \rho_n. \tag{1}$$

Then $\{\kappa_n\}$ *is an e-sequence if* $\{\rho_n\}$ *is an e-sequence.*

We now state the following definition of cone χ_b-metric space.

Definition 9 ([30])**.** *Let* $\mathscr{X}$ *be a non-empty set and let* $b \geq 1$ *be a real number. Let* χ^+ *denote the positive cone of the ordered space* χ, *with* $(\chi^+)^\ominus \neq \emptyset$. *A function* $d^\chi : \mathscr{X} \times \mathscr{X} \to \chi^+$ *is called a cone* χ_b*-metric if it satisfies the following conditions for all* $\kappa, \rho, \lambda \in \mathscr{X}$:

- **(a):** $d^\chi(\kappa,\rho) = 0$ *if and only if* $\kappa = \rho$;
- **(b):** $d^\chi(\kappa,\rho) = d^\chi(\rho,\kappa)$;
- **(c):** $d^\chi(\kappa,\rho) \preceq b[d^\chi(\kappa,\lambda) + d^\chi(\lambda,\rho)]$.

Additionally, $(\mathscr{X}, d^\chi)$ is called a cone χ_b-metric space.

Example 3 ([30])**.** *Let $L^p([0,1])$, with $0 < p < 1$, be the space of all real-valued functions h such that*

$$\int_0^1 |h(\tau)|^p \, d\tau < \infty.$$

We consider $\mathscr{X} = L^p([0,1])$, $\chi = \mathbb{R}^2$, and define the positive cone $\chi^+ = \{(s,\tau) \in \chi \mid s,\tau \geq 0\} \subset \mathbb{R}^2$. Define the function $d^\chi : \mathscr{X} \times \mathscr{X} \to \chi$ by

$$d^\chi(\Lambda,\Delta) = \left(z_1 \left(\int_0^1 |\Lambda(\tau) - \Delta(\tau)|^p \, d\tau \right)^{\frac{1}{p}}, \, z_2 \left(\int_0^1 |\Lambda(\tau) - \Delta(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \right),$$

where $z_1, z_2 \geq 0$ are constants. Therefore, $(\mathscr{X}, d^\chi)$ forms a cone χ_b-metric space with a normal and solid cone, characterized by the coefficient $b = 2^{\frac{1}{p}-1}$.

In this context, we aim to explore the concepts of e-convergence and e-completeness within the framework of a cone χ_b-metric space.

Definition 10 ([30])**.** *Let χ be an ordered normed space such that $(\chi^+)^\oslash$ is non-empty. Assume that $(\mathscr{X}, d^\chi)$ is a cone χ_b-metric space. Additionally, let (r_n) be a sequence in $\mathscr{X}$ and let $r \in \mathscr{X}$. Then:*

- (i) *The sequence (r_n) is said to be e-convergent to r if for every $0_\chi \lll e$, there exists an index $n_0 \in \mathbb{N}$ such that*

$$d^\chi(r_n, r) \lll e, \quad \textit{for all } n \geq n_0.$$

In this case, we write $\lim\limits_{n\to\infty} r_n = r$ or $r_n \xrightarrow{e} r$.

- (ii) *The sequence (r_n) is called an e-Cauchy sequence if for every $0_\chi \lll e$, there exists an integer $n_0 \in \mathbb{N}$ such that*

$$d^\chi(r_n, r_m) \lll e, \quad \textit{for all } n, m \geq n_0.$$

- (iii) *The space $(\mathscr{X}, d^\chi)$ is said to be e-complete if every e-Cauchy sequence in $\mathscr{X}$ is e-convergent.*

3. Fixed Point Theorems in Cone χ_b-Metric Space

In this section, we recall a well-known definition and establish several theorems. We then present a number of applications of fixed point theory concerning Hardy-Rogers type mappings within the framework of cone χ_b-metric spaces.

Definition 11. *Let $(\mathscr{X}, d^{\chi})$ be a cone χ_b-metric space with closed positive cone χ^+ such that $(\chi^+)^{\ominus} \neq \emptyset$. A mapping $\Upsilon : \mathscr{X} \to \mathscr{X}$ is said to be of Hardy-Rogers type if it satisfies*

$$(2) \qquad \begin{aligned} d^{\chi}(\Upsilon\kappa, \Upsilon\rho) &\preceq \delta_1 d^{\chi}(\kappa, \rho) + \delta_2 d^{\chi}(\kappa, \Upsilon\kappa) + \delta_3 d^{\chi}(\rho, \Upsilon\rho) \\ &\quad + \delta_4 d^{\chi}(\kappa, \Upsilon\rho) + \delta_5 d^{\chi}(\rho, \Upsilon\kappa), \end{aligned}$$

for all $\kappa, \rho \in \mathscr{X}$, where $\delta_i \geq 0$ for $i = 1,2,3,4,5$, and the condition

$$\sum_{i=1}^{5} \delta_i < 1$$

is satisfied.

We now proceed to present the following lemma, which serves as an essential component in the continuation of our analysis.

Lemma 2. *Let $(\mathscr{X}, d^{\chi})$ be an e-complete cone χ_b-metric space with closed positive cone χ^+ such that $(\chi^+)^{\oslash} \neq \emptyset$. Let $\{\kappa_n\}$ be a sequence in $\mathscr{X}$ satisfying*

$$(3) \qquad d^{\chi}(\kappa_n, \kappa_{n+1}) \preceq \delta\, d^{\chi}(\kappa_{n-1}, \kappa_n) \quad \text{for all } n = 1, 2, \ldots,$$

where $0 < \delta < \frac{1}{b}$ is a constant. Then, the sequence $\{\kappa_n\}$ is an e-Cauchy sequence in $\mathscr{X}$.

Proof. Assume that $\{\kappa_n\}$ is a contractive sequence in $\mathscr{X}$. This implies that there exists a real number $\delta \in \left(0, \frac{1}{b}\right)$ such that

$$d^{\chi}(\kappa_n, \kappa_{n+1}) \preceq \delta\, d^{\chi}(\kappa_{n-1}, \kappa_n) \preceq \delta^2\, d^{\chi}(\kappa_{n-2}, \kappa_{n-1}) \preceq \cdots \preceq \delta^n\, d^{\chi}(\kappa_0, \kappa_1).$$

Now, for $n > m$, we can express:

$$\begin{aligned} d^{\chi}(\kappa_m, \kappa_n) &\preceq b\,(d^{\chi}(\kappa_m, \kappa_{m+1}) + d^{\chi}(\kappa_{m+1}, \kappa_n)) \\ &\preceq b\, d^{\chi}(\kappa_m, \kappa_{m+1}) + b^2\,(d^{\chi}(\kappa_{m+1}, \kappa_{m+2}) + d^{\chi}(\kappa_{m+2}, \kappa_n)) \end{aligned}$$

$$\begin{aligned}
&\preceq \cdots \\
&\preceq b\,d^{\chi}(\kappa_m,\kappa_{m+1}) + b^2\,d^{\chi}(\kappa_{m+1},\kappa_{m+2}) + \cdots + b^{n-m-1}\,d^{\chi}(\kappa_{n-2},\kappa_{n-1}) \\
&\quad + b^{n-m}\,d^{\chi}(\kappa_{n-1},\kappa_n).
\end{aligned}$$

Now, from (3) and the condition $b\delta < 1$, it follows that

$$\begin{aligned}
d^{\chi}(\kappa_m,\kappa_n) &\preceq \left(b\delta^m + b^2\delta^{m+1} + \cdots + b^{n-m}\delta^{n-1}\right) d^{\chi}(\kappa_1,\kappa_0) \\
&= b\delta\,\delta^m\left(1 + b\delta + (b\delta)^2 + \cdots + (b\delta)^{n-m-1}\right) d^{\chi}(\kappa_1,\kappa_0) \\
&\preceq \frac{b\delta^m\,(1-(b\delta)^{n-m})}{1-b\delta}\,d^{\chi}(\kappa_1,\kappa_0).
\end{aligned}$$

Let $0_{\chi} \lll e$ be given. Choose $\rho > 0$ such that $e - \rho\,B_+ \subseteq \chi^+$, and select $k_1 \in \mathbb{N}$ such that

$$\frac{b\delta^m\,(1-(b\delta)^{n-m})}{1-b\delta}\,d^{\chi}(\kappa_1,\kappa_0) \in \frac{\rho}{2}B_+ \quad \text{for all } m,n \geq k_1.$$

Therefore, we have

$$e - \frac{b\delta^m\,(1-(b\delta)^{n-m})}{1-b\delta}\,d^{\chi}(\kappa_1,\kappa_0) - \frac{\rho}{2}B_+ \subseteq e - \rho B_+ \subseteq \chi^+.$$

Hence, we have

$$d^{\chi}(\kappa_m,\kappa_n) \preceq \frac{b\delta^m\,(1-(b\delta)^{n-m})}{1-b\delta}\,d^{\chi}(\kappa_1,\kappa_0) \lll e, \quad \text{for all } n,m \geq k_1.$$

As a result, the sequence (κ_n) is demonstrated to be an e-Cauchy sequence under the given conditions. □

Subsequently, we examine several applications in fixed point theory, focusing specifically on Hardy–Rogers type mappings within the framework of e-complete cone χ_b-metric spaces.

Theorem 1. *Let $(\mathscr{X},d^{\chi})$ be an e-complete cone χ_b-metric space with closed positive cone χ^+ such that $(\chi^+)^{\ominus} \neq \emptyset$. Let $\Upsilon : \mathscr{X} \to \mathscr{X}$ be a mapping defined on $\mathscr{X}$. Assume there exist nonnegative coefficients δ_i for $i = 1,\ldots,5$ such that*

$$\delta_1 + \delta_2 + \delta_3 + b(\delta_4 + \delta_5) < 1 \quad \textit{and} \quad \delta_2 + \delta_5 < \frac{1}{b^2}, \tag{4}$$

and for all $\kappa,\rho \in \mathscr{X}$,

$$\begin{aligned} d^\chi(\Upsilon\kappa,\Upsilon\rho) \preceq\ & \delta_1 d^\chi(\kappa,\rho)+\delta_2 d^\chi(\kappa,\Upsilon\kappa)+\delta_3 d^\chi(\rho,\Upsilon\rho) \\ & + \delta_4 d^\chi(\kappa,\Upsilon\rho)+\delta_5 d^\chi(\rho,\Upsilon\kappa). \end{aligned} \tag{5}$$

Then Υ *has a unique fixed point in* $\mathscr{X}$, *and for each* $\kappa \in \mathscr{X}$, *the iterative sequence* $\{\Upsilon^n\kappa\}_{n\geq 0}$ *e-converges to this fixed point.*

Note that condition (4) is satisfied, for example, when $\sum_{i=1}^{5}\delta_i < 1$. Notably, when $b=1$, inequality (4) reduces to the classical Hardy–Rogers condition defined in the context of cone χ-metric spaces.

Proof. Let $\kappa_0 \in \mathscr{X}$ be chosen, and consider the iterative sequence

$$\kappa_{n+1} = \Upsilon\kappa_n = \Upsilon^{n+1}\kappa_0, \quad \text{for } n \geq 1.$$

By applying condition (5), we obtain

$$\begin{aligned} d^\chi(\kappa_n,\kappa_{n+1}) &= d^\chi(\Upsilon\kappa_{n-1},\Upsilon\kappa_n) \\ &\preceq \delta_1 d^\chi(\kappa_{n-1},\kappa_n)+\delta_2 d^\chi(\kappa_{n-1},\Upsilon\kappa_{n-1})+\delta_3 d^\chi(\kappa_n,\Upsilon\kappa_n) \\ &\quad + \delta_4 d^\chi(\kappa_{n-1},\Upsilon\kappa_n)+\delta_5 d^\chi(\kappa_n,\Upsilon\kappa_{n-1}) \\ &\preceq (\delta_1+\delta_2) d^\chi(\kappa_{n-1},\kappa_n)+\delta_3 d^\chi(\kappa_n,\kappa_{n+1})+\delta_4 d^\chi(\kappa_{n-1},\kappa_{n+1}) \\ &\quad + \delta_5 d^\chi(\kappa_n,\kappa_n) \\ &= (\delta_1+\delta_2) d^\chi(\kappa_{n-1},\kappa_n)+\delta_3 d^\chi(\kappa_n,\kappa_{n+1})+\delta_4 d^\chi(\kappa_{n-1},\kappa_{n+1}) \\ &\preceq (\delta_1+\delta_2) d^\chi(\kappa_{n-1},\kappa_n)+\delta_3 d^\chi(\kappa_n,\kappa_{n+1}) \\ &\quad + b\delta_4\left[d^\chi(\kappa_{n-1},\kappa_n)+d^\chi(\kappa_n,\kappa_{n+1})\right] \\ &= (\delta_1+\delta_2+b\delta_4) d^\chi(\kappa_{n-1},\kappa_n)+(\delta_3+b\delta_4) d^\chi(\kappa_n,\kappa_{n+1}). \end{aligned} \tag{6}$$

On the other hand, it follows that

$$\begin{aligned} d^\chi(\kappa_n,\kappa_{n+1}) &= d^\chi(\Upsilon\kappa_{n-1},\Upsilon\kappa_n) \\ &\preceq \delta_1 d^\chi(\kappa_n,\kappa_{n-1})+\delta_2 d^\chi(\kappa_n,\Upsilon\kappa_n)+\delta_3 d^\chi(\kappa_{n-1},\Upsilon\kappa_{n-1}) \end{aligned}$$

$$\begin{aligned} &+\delta_4 d^{\chi}(\kappa_n,\Upsilon\kappa_{n-1})+\delta_5 d^{\chi}(\kappa_{n-1},\Upsilon\kappa_n)\\ &\preceq(\delta_1+\delta_3)d^{\chi}(\kappa_{n-1},\kappa_n)+\delta_2 d^{\chi}(\kappa_n,\kappa_{n+1})+\delta_4 d^{\chi}(\kappa_n,\kappa_n)\\ &+b\delta_5\left[d^{\chi}(\kappa_{n-1},\kappa_n)+d^{\chi}(\kappa_n,\kappa_{n+1})\right]\\ &=(\delta_1+\delta_3+b\delta_5)d^{\chi}(\kappa_{n-1},\kappa_n)+(\delta_2+b\delta_5)d^{\chi}(\kappa_n,\kappa_{n+1}). \qquad (7)\end{aligned}$$

Adding up (6) and (7) yields

$$\begin{aligned} 2d^{\chi}(\kappa_n,\kappa_{n+1}) &\preceq(2\delta_1+\delta_2+\delta_3+b(\delta_4+\delta_5))\,d^{\chi}(\kappa_{n-1},\kappa_n)\\ &\quad+(\delta_2+\delta_3+b(\delta_4+\delta_5))\,d^{\chi}(\kappa_n,\kappa_{n+1}),\end{aligned}$$

Hence,

$$d^{\chi}(\kappa_n,\kappa_{n+1})\preceq\frac{2\delta_1+\delta_2+\delta_3+b(\delta_4+\delta_5)}{2-(\delta_2+\delta_3+b(\delta_4+\delta_5))}d^{\chi}(\kappa_{n-1},\kappa_n). \qquad (8)$$

Since $0\leq\sum_{i=1}^{3}\delta_i+b\,(\delta_4+\delta_5)<1$, we have

$$\delta=\frac{2\delta_1+\delta_2+\delta_3+b\,(\delta_4+\delta_5)}{2-(\delta_2+\delta_3+b\,(\delta_4+\delta_5))}<1.$$

From equation (8) and Lemma 2, we deduce that $\{\kappa_n\}$ is an e-Cauchy sequence in $\mathscr{X}$. Since $(\mathscr{X},d^{\chi})$ is assumed to be e-complete, there exists an element $\kappa\in\mathscr{X}$ such that $\{\kappa_n\}$ e-converges to κ. In the subsequent analysis, we prove that κ is a fixed point of Υ. Indeed, using (5), we have

$$\begin{aligned} d^{\chi}(\Upsilon\kappa,\kappa) &\preceq b\,(d^{\chi}(\Upsilon\kappa,\kappa_n)+d^{\chi}(\kappa_n,\kappa))\\ &\preceq b\Big(\delta_1 d^{\chi}(\kappa,\kappa_{n-1})+\delta_2 d^{\chi}(\kappa,\Upsilon\kappa)+\delta_3 d^{\chi}(\kappa_{n-1},\Upsilon\kappa_{n-1})\\ &\quad+\delta_4 d^{\chi}(\kappa,\Upsilon\kappa_{n-1})+\delta_5 d^{\chi}(\kappa_{n-1},\Upsilon\kappa)+d^{\chi}(\kappa_n,\kappa)\Big)\end{aligned}$$

or

$$\begin{aligned} d^{\chi}(\Upsilon\kappa,\kappa) &\preceq b\Big(\delta_1 d^{\chi}(\kappa,\kappa_{n-1})+\delta_2 d^{\chi}(\kappa,\Upsilon\kappa)+b\delta_3\left[d^{\chi}(\kappa_{n-1},\kappa)+d^{\chi}(\kappa,\Upsilon\kappa_{n-1})\right]\\ &\quad+\delta_4 d^{\chi}(\kappa,\kappa_n)+b\delta_5\left[d^{\chi}(\kappa_{n-1},\kappa)+d^{\chi}(\kappa,\Upsilon\kappa)\right]+d^{\chi}(\kappa_n,\kappa)\Big)\\ &\preceq\left(b\delta_1+b^2\delta_3+b^2\delta_5\right)d^{\chi}(\kappa_{n-1},\kappa)+\left(b\delta_2+b^2\delta_5\right)d^{\chi}(\Upsilon\kappa,\kappa)\end{aligned}$$

$$+\left(b+b^2\delta_3+b\delta_4\right)d^{\chi}(\kappa_n,\kappa).$$

This means that

$$d^{\chi}(\Upsilon\kappa,\kappa)\preceq\frac{b\delta_1+b^2\delta_3+b^2\delta_5}{1-b^2(\delta_2+\delta_5)}d^{\chi}(\kappa_{n-1},\kappa)+\frac{b+b^2\delta_3+b\delta_4}{1-b^2(\delta_2+\delta_5)}d^{\chi}(\kappa_n,\kappa)\triangleq\beta_n, \tag{9}$$

where $k_1=\dfrac{b\delta_1+b^2\delta_3+b^2\delta_5}{1-b^2(\delta_2+\delta_5)}$ and $k_2=\dfrac{b+b^2\delta_3+b\delta_4}{1-b^2(\delta_2+\delta_5)}$ are positive numbers. Since $\{\kappa_n\}$ e-converges to κ, it follows that $\{d^{\chi}(\kappa_n,\kappa)\}$ e-converges to 0_{χ}. Thus, $\{\beta_n\}$ also e-converges to 0_{χ}. Hence, by applying result (9), for any $e\ggg 0_{\chi}$, there exists an integer n_1 such that for all $n\geqslant n_1$, we have

$$d^{\chi}(\Upsilon\kappa,\kappa)\lll e. \tag{10}$$

According to Proposition 2, $d^{\chi}(\Upsilon\kappa,\kappa)=0_{\chi}$, i.e., κ is identified as a fixed point of Υ. To establish the uniqueness of the fixed point, assume ρ is another fixed point of Υ. By utilizing (5), it follows that

$$\begin{aligned} d^{\chi}(\kappa,\rho) &= d^{\chi}(\Upsilon\kappa,\Upsilon\rho)\\ &\preceq \delta_1 d^{\chi}(\kappa,\rho)+\delta_2 d^{\chi}(\rho,\Upsilon\rho)+\delta_3 d^{\chi}(\kappa,\Upsilon\kappa)\\ &\quad+\delta_4 d^{\chi}(\rho,\Upsilon\kappa)+\delta_5 d^{\chi}(\kappa,\Upsilon\rho)\\ &= (\delta_1+\delta_4+\delta_5)\,d^{\chi}(\kappa,\rho), \end{aligned}$$

As $0\leq\delta_1+\delta_4+\delta_5\leq\sum\limits_{i=1}^{3}\delta_i+b(\delta_4+\delta_5)<1$, it follows from Lemma 2 that

$$d^{\chi}(\kappa,\rho)=0_{\chi}.$$

Therefore, we conclude that $\kappa=\rho$. □

Example 4. *Analogous to Example 2, we define the mapping $d^{\chi}:\mathscr{X}\times\mathscr{X}\to\chi$ by*

$$d^{\chi}(\kappa_1,\kappa_2)=\left(\|\kappa_1-\kappa_2\|_{\infty},\|\kappa_1-\kappa_2\|_{\infty}\right).$$

This means that $d^{\chi}(\kappa_1,\kappa_2)=\{d_n\}$, where $d_n=\left(\|\kappa_1-\kappa_2\|_{\infty},\|\kappa_1-\kappa_2\|_{\infty}\right)$ for all $n\in\mathbb{N}$. Now, consider a mapping $\Upsilon:\mathscr{X}\to\mathscr{X}$ such that $\Upsilon\kappa=\frac{\kappa}{5}$.

$$d^{\chi}(\Upsilon\kappa_1,\Upsilon\kappa_2)=\left(\|\Upsilon\kappa_1-\Upsilon\kappa_2\|_{\infty},\|\Upsilon\kappa_1-\Upsilon\kappa_2\|_{\infty}\right),\ \textit{for each}\ \kappa_1,\kappa_2\in\mathscr{X}.$$

But, we have

$$\begin{aligned}\|\Upsilon\kappa_1-\Upsilon\kappa_2\|_\infty &= \left\|\frac{1}{5}\kappa_1-\frac{1}{5}\kappa_2\right\|_\infty\\ &= \frac{1}{5}\|\kappa_1-\kappa_2\|_\infty\\ &\le \frac{1}{5}\|\kappa_1-\Upsilon\kappa_1\|_\infty+\frac{1}{5}\|\kappa_2-\Upsilon\kappa_2\|_\infty+\frac{1}{5}\|\Upsilon\kappa_1-\Upsilon\kappa_2\|_\infty\\ &\le \frac{1}{4}\left(\|\kappa_1-\Upsilon\kappa_1\|_\infty+\|\kappa_2-\Upsilon\kappa_2\|_\infty\right).\end{aligned}$$

So, we get

$$\begin{aligned}d^\chi(\Upsilon\kappa_1,\Upsilon\kappa_2) &\preceq \frac{1}{4}\left(\|\kappa_1-\Upsilon\kappa_1\|_\infty+\|\kappa_2-\Upsilon\kappa_2\|_\infty,\|\kappa_1-\Upsilon\kappa_1\|_\infty+\|\kappa_2-\Upsilon\kappa_2\|_\infty\right)\\ &= \frac{1}{4}\left[\left(\|\kappa_1-\Upsilon\kappa_1\|_\infty,\|\kappa_1-\Upsilon\kappa_1\|_\infty\right)+\left(\|\kappa_2-\Upsilon\kappa_2\|_\infty,\|\kappa_2-\Upsilon\kappa_2\|_\infty\right)\right]\\ &= \frac{1}{4}d^\chi(\kappa_1,\Upsilon\kappa_1)+\frac{1}{4}d^\chi(\kappa_2,\Upsilon\kappa_2)\\ &= \delta_2 d^\chi(\kappa_1,\Upsilon\kappa_1)+\delta_3 d^\chi(\kappa_2,\Upsilon\kappa_2),\end{aligned}$$

where $\delta_2=\delta_3=\frac{1}{4}$ and $\delta_1=\delta_4=\delta_5=0$. Therefore, all the conditions of Theorem 1 are satisfied, and thus Υ has a unique fixed point.

Theorem 2. *Let $(\mathscr{X},d^\chi)$ be an e-complete cone χ_b-metric space with a closed positive cone χ^+ such that $\chi^+\neq\emptyset$. Assume that Υ is a self-mapping on $\mathscr{X}$. If there exist nonnegative coefficients δ_i, $i=1,\ldots,5$, such that*

$$\delta_1+b(\delta_2+\delta_3)+\delta_4+\delta_5<1,$$

and for all $\kappa,\rho\in\mathscr{X}$,

$$\begin{aligned}d^\chi(\Upsilon\kappa,\Upsilon\rho) &\preceq \delta_1 d^\chi(\kappa,\rho)+\delta_2 d^\chi(\kappa,\Upsilon\kappa)+\delta_3 d^\chi(\rho,\Upsilon\rho)\\ &\quad+\delta_4 d^\chi(\kappa,\Upsilon\rho)+\delta_5 d^\chi(\rho,\Upsilon\kappa),\end{aligned}\tag{11}$$

then the Picard iteration is Υ-stable.

Proof. Let ρ be an arbitrary fixed point of Υ. Consider a sequence $\{\kappa_n\}$ in $\mathscr{X}$ such that the sequence $\{d^\chi(\kappa_{n+1},\Upsilon\kappa_n)\}$ is e-convergent. Referring to condition (11), we obtain

$$d^\chi(\Upsilon\kappa_n,\rho) = d^\chi(\Upsilon\kappa_n,\Upsilon\rho)$$

$$
\begin{aligned}
&\preceq \delta_1 d^\chi(\kappa_n,\rho)+\delta_2 d^\chi(\kappa_n,\Upsilon\kappa_n)+\delta_3 d^\chi(\rho,\Upsilon\rho)\\
&\quad +\delta_4 d^\chi(\kappa_n,\Upsilon\rho)+\delta_5 d^\chi(\rho,\Upsilon\kappa_n)\\
&= \delta_1 d^\chi(\kappa_n,\rho)+\delta_2 d^\chi(\kappa_n,\kappa_{n+1})+\delta_3 d^\chi(\rho,\rho)\\
&\quad +\delta_4 d^\chi(\kappa_n,\rho)+\delta_5 d^\chi(\rho,\kappa_{n+1})\\
&\preceq (\delta_1+\delta_4)\, d^\chi(\kappa_n,\rho)+b\delta_2\left[d^\chi(\kappa_n,\rho)+d^\chi(\rho,\kappa_{n+1})\right]\\
&\quad +\delta_5 d^\chi(\rho,\kappa_{n+1})\\
&\preceq (\delta_1+b\delta_2+\delta_4)\, d^\chi(\kappa_n,\rho)+(b\delta_2+\delta_5)\, d^\chi(\rho,\kappa_{n+1}). \qquad (12)
\end{aligned}
$$

From another perspective, we observe that

$$
\begin{aligned}
d^\chi(\Upsilon\kappa_n,\rho) &= d^\chi(\rho,\Upsilon\kappa_n)=d^\chi(\Upsilon\rho,\Upsilon\kappa_n)\\
&\preceq \delta_1 d^\chi(\rho,\kappa_n)+\delta_2 d^\chi(\rho,\Upsilon\rho)+\delta_3 d^\chi(\kappa_n,\Upsilon\kappa_n)\\
&\quad +\delta_4 d^\chi(\rho,\Upsilon\kappa_n)+\delta_5 d^\chi(\kappa_n,\Upsilon\rho)\\
&= \delta_1 d^\chi(\kappa_n,\rho)+\delta_2 d^\chi(\rho,\rho)+\delta_3 d^\chi(\kappa_n,\kappa_{n+1})\\
&\quad +\delta_4 d^\chi(\rho,\kappa_{n+1})+\delta_5 d^\chi(\kappa_n,\rho)\\
&\preceq \delta_1 d^\chi(\kappa_n,\rho)+\delta_2 d^\chi(\rho,\rho)+b\delta_3\left[d^\chi(\kappa_n,\rho)+d^\chi(\rho,\kappa_{n+1})\right]\\
&\quad +\delta_4 d^\chi(\rho,\kappa_{n+1})+\delta_5 d^\chi(\kappa_n,\rho)\\
&\preceq (\delta_1+b\delta_3+\delta_5)\, d^\chi(\kappa_n,\rho)+(b\delta_3+\delta_4)\, d^\chi(\rho,\kappa_{n+1}). \qquad (13)
\end{aligned}
$$

We add up inequalities (12) and (13), which yields

$$
\begin{aligned}
2d^\chi(\Upsilon\kappa_n,\rho) &\preceq (2\delta_1+b\delta_2+b\delta_3+\delta_4+\delta_5)\, d^\chi(\kappa_n,\rho)\\
&\quad +(b\delta_2+b\delta_3+\delta_4+\delta_5)\, d^\chi(\rho,\kappa_{n+1}).
\end{aligned}
$$

This implies that

$$
d^\chi(\Upsilon\kappa_n,\rho)\preceq \frac{2\delta_1+b(\delta_2+\delta_3)+\delta_4+\delta_5}{2-(b(\delta_2+\delta_3)+\delta_4+\delta_5)} d^\chi(\kappa_n,\rho).
$$

Now, since

$$
0\le \delta_1+b(\delta_2+\delta_3)+\delta_4+\delta_5<1,
$$

we have

$$\delta = \frac{2\delta_1 + b(\delta_2+\delta_3)+\delta_4+\delta_5}{2-(b(\delta_2+\delta_3)+\delta_4+\delta_5)} < 1,$$

and

$$d^\chi(\Upsilon\kappa_n,\rho) \preceq \delta\, d^\chi(\kappa_n,\rho).$$

By defining $\eta_n = d^\chi(\kappa_n,\rho)$ and $\gamma_n = d^\chi(\kappa_{n+1},\Upsilon\kappa_n)$, we obtain

$$\begin{aligned}
\eta_{n+1} &= d^\chi(\kappa_{n+1},\rho) \\
&\preceq b\left(d^\chi(\kappa_{n+1},\Upsilon\kappa_n) + d^\chi(\Upsilon\kappa_n,\rho)\right) \\
&\preceq b\left(d^\chi(\kappa_{n+1},\Upsilon\kappa_n) + d^\chi(\Upsilon\kappa_n,\Upsilon\rho)\right) \\
&\preceq b\gamma_n + \delta b\,\eta_n \\
&\preceq b(\delta\eta_n + \gamma_n).
\end{aligned}$$

Since $\{\gamma_n\}$ is an e-sequence, by applying Proposition 7, we deduce that $\{\eta_n\}$ is also an e-sequence. Therefore, $\{\kappa_n\}$ converges to ρ as $n\to\infty$. This implies that the Picard iteration is Υ-stable. □

Theorem 3. *Let $(\mathscr{X},d^\chi)$ be an e-complete cone χ_b-metric space with a closed positive cone χ^+ such that $(\chi^+)^\ominus \neq \emptyset$. Let Υ be a mapping on $\mathscr{X}$. Assume that there exist nonnegative coefficients δ_i, $i=1,\ldots,5$, such that*

$$2b^2\delta_1 + 2b(\delta_2+\delta_3) + 2b^3(\delta_4+\delta_5) \leq 2 \quad \textit{and} \quad \delta_1+\delta_4+\delta_5 < \frac{1}{b^3},$$

and for all $\kappa,\rho\in\mathscr{X}$,

$$\begin{aligned}
d^\chi(\Upsilon\kappa,\Upsilon\rho) \preceq\ & \delta_1 d^\chi(\kappa,\rho) + \delta_2 d^\chi(\kappa,\Upsilon\kappa) + \delta_3 d^\chi(\rho,\Upsilon\rho) \\
& + \delta_4 d^\chi(\kappa,\Upsilon\rho) + \delta_5 d^\chi(\rho,\Upsilon\kappa).
\end{aligned}$$

Then, the sequence $\{d^\chi(\kappa_n,\Upsilon\kappa_n)\}$ is an e-sequence if and only if the sequence $\{d^\chi(\kappa_{n+1},\Upsilon\kappa_n)\}$ is also an e-sequence.

Proof. Suppose that $\{\kappa_n\}$ is a sequence in $\mathscr{X}$. Define

$$\varpi_n = d^\chi(\kappa_n,\Upsilon\kappa_n) \quad \text{and} \quad \eta_n = d^\chi(\kappa_{n+1},\Upsilon\kappa_n).$$

If $\{\varpi_n\}$ is an e-sequence, then

$$\begin{aligned}
\eta_n &= d^{\chi}(\kappa_{n+1},\Upsilon\kappa_n)\\
&\preceq b\left(d^{\chi}(\kappa_{n+1},\Upsilon\kappa_{n+1})+d^{\chi}(\Upsilon\kappa_{n+1},\Upsilon\kappa_n)\right)\\
&\preceq b\Big(d^{\chi}(\kappa_{n+1},\Upsilon\kappa_{n+1})+\delta_1 d^{\chi}(\kappa_{n+1},\kappa_n)+\delta_2 d^{\chi}(\kappa_{n+1},\Upsilon\kappa_{n+1})\\
&\quad +\delta_3 d^{\chi}(\kappa_n,\Upsilon\kappa_n)+\delta_4 d^{\chi}(\kappa_{n+1},\Upsilon\kappa_n)+\delta_5 d^{\chi}(\kappa_n,\Upsilon\kappa_{n+1})\Big)\\
&\preceq b\Big(d^{\chi}(\kappa_{n+1},\Upsilon\kappa_{n+1})+b\delta_1\left[d^{\chi}(\kappa_{n+1},\Upsilon\kappa_n)+d^{\chi}(\kappa_n,\Upsilon\kappa_n)\right]\\
&\quad +\delta_2 d^{\chi}(\kappa_{n+1},\Upsilon\kappa_{n+1})+\delta_3 d^{\chi}(\kappa_n,\Upsilon\kappa_n)+\delta_4 d^{\chi}(\kappa_{n+1},\Upsilon\kappa_n)\\
&\quad +b\delta_5\left[d^{\chi}(\kappa_n,\Upsilon\kappa_n)+b\left(d^{\chi}(\Upsilon\kappa_n,\kappa_{n+1})+d^{\chi}(\kappa_{n+1},\Upsilon\kappa_{n+1})\right)\right]\Big)\\
&= \left(b+b\delta_2+b^3\delta_5\right)\varpi_{n+1}+\left(b^2\delta_1+b\delta_3+b^2\delta_5\right)\varpi_n+\left(b^2\delta_1+b\delta_4+b^3\delta_5\right)\eta_n.
\end{aligned}$$

It is clear that

$$\begin{aligned}
\eta_n &\preceq \frac{b+b\delta_2+b^3\delta_5}{1-b^2\delta_1-b\delta_4-b^3\delta_5}\varpi_{n+1}+\frac{b^2\delta_1+b\delta_4+b^3\delta_5}{1-b^2\delta_1-b\delta_4-b^3\delta_5}\varpi_n\\
&\preceq \frac{b+b\delta_2+b^3\delta_5}{1-b^3(\delta_1+\delta_4+\delta_5)}\varpi_{n+1}+\frac{b^2\delta_1+b\delta_4+b^3\delta_5}{1-b^3(\delta_1+\delta_4+\delta_5)}\varpi_n\triangleq\gamma_n,
\end{aligned}$$

where $k_3=\dfrac{b+b\delta_2+b^3\delta_5}{1-b^3(\delta_1+\delta_4+\delta_5)}$ and $k_4=\dfrac{b^2\delta_1+b\delta_4+b^3\delta_5}{1-b^3(\delta_1+\delta_4+\delta_5)}$ are positive constants.Since $\{\varpi_n\}$ is an e-sequence, it follows that (ϖ_n) e-converges to 0_χ, and hence (γ_n) also e-converges to 0_χ. Consequently, for any given $e\ggg 0_\chi$, there exists $\kappa>0$ such that $\eta_n\preceq\gamma_n\lll e$. Therefore, the sequence (δ_n) e-converges to 0_χ, and $\{\eta_n\}$ is an e-sequence.

In contrast, assume that $\{\eta_n\}$ is an e-sequence. Then, among other implications, we deduce that:

$$\begin{aligned}
\varpi_n &= d^{\chi}(\kappa_n,\Upsilon\kappa_n)\preceq b\left(d^{\chi}(\kappa_n,\Upsilon\kappa_{n-1})+d^{\chi}(\Upsilon\kappa_{n-1},\Upsilon\kappa_n)\right)\\
&\preceq b\Big(d^{\chi}(\kappa_n,\Upsilon\kappa_{n-1})+\delta_1 d^{\chi}(\kappa_n,\kappa_{n-1})+\delta_2 d^{\chi}(\kappa_n,\Upsilon\kappa_n)+\delta_3 d^{\chi}(\kappa_{n-1},\Upsilon\kappa_{n-1})\\
&\qquad +\delta_4 d^{\chi}(\kappa_n,\Upsilon\kappa_{n-1})+\delta_5 d^{\chi}(\kappa_{n-1},\Upsilon\kappa_n)\Big)\\
&\preceq b\Bigg(d^{\chi}(\kappa_n,\kappa_n)+b\delta_1\left[d^{\chi}(\kappa_n,\Upsilon\kappa_{n-1})+d^{\chi}(\Upsilon\kappa_{n-1},\kappa_{n-1})\right]+\delta_2 d^{\chi}(\kappa_n,\kappa_{n+1})
\end{aligned}$$

$$
\begin{aligned}
& +\delta_3 d^\chi(\kappa_{n-1},\kappa_n)+\delta_4 d^\chi(\kappa_n,\kappa_n)+b\delta_5\Big[d^\chi(\kappa_{n-1},\Upsilon\kappa_{n-1}) \\
& +b\left(d^\chi(\Upsilon\kappa_{n-1},\kappa_n)+d^\chi(\kappa_n,\Upsilon\kappa_n)\right)\Big]\Big) \\
= \; & \left(b\delta_2+b^3\delta_5\right)d^\chi(\kappa_n,\kappa_{n+1})+\left(b^2\delta_1+b\delta_3+b^2\delta_5\right)d^\chi(\kappa_{n-1},\kappa_n) \\
& +\left(b+b^2\delta_1+b\delta_4+b^3\delta_5\right)d^\chi(\kappa_n,\kappa_n) \\
= \; & \left(b\delta_2+b^3\delta_5\right)\varpi_n+\left(b^2\delta_1+b\delta_3+b^2\delta_5\right)\varpi_{n-1}+\left(b+b^2\delta_1+b\delta_4+b^3\delta_5\right)\delta_{n-1},
\end{aligned}
$$

which implies that

$$
\left(1-b\delta_2-b^3\delta_5\right)\varpi_n \preceq \left(b^2\delta_1+b\delta_3+b^3\delta_5\right)\varpi_{n-1}+\left(b+b^2\delta_1+b\delta_4+b^3\delta_5\right)\delta_{n-1}. \tag{14}
$$

In addition, we observe that

$$
\begin{aligned}
\varpi_n = \; & d^\chi(x_n,\Upsilon x_n) \preceq b\left(d^\chi(x_n,\Upsilon x_{n-1})+d^\chi(\Upsilon x_{n-1},\Upsilon x_n)\right) \\
\preceq \; & b\Big(d^\chi(x_n,\Upsilon x_{n-1})+\delta_1 d^\chi(x_{n-1},x_n)+\delta_2 d^\chi(x_{n-1},\Upsilon x_{n-1}) \\
& +\delta_3 d^\chi(x_n,\Upsilon x_n)+\delta_4 d^\chi(x_{n-1},\Upsilon x_n)+\delta_5 d^\chi(x_n,\Upsilon x_{n-1})\Big) \\
\preceq \; & b\Big(d^\chi(x_n,x_n)+b\delta_1\left[d^\chi(x_{n-1},\Upsilon x_{n-1})+d^\chi(\Upsilon x_{n-1},x_n)\right]+\delta_2 d^\chi(x_{n-1},x_n) \\
& +\delta_3 d^\chi(x_n,x_{n+1})+b\delta_4\left[d^\chi(x_{n-1},\Upsilon x_{n-1})+b\left(d^\chi(\Upsilon x_{n-1},x_n)+d^\chi(x_n,\Upsilon x_n)\right)\right] \\
& +\delta_5 d^\chi(x_n,x_n)\Big)
\end{aligned}
$$

or

$$
\begin{aligned}
\varpi_n \preceq \; & \left(b\delta_3+b^3\delta_4\right)d^\chi(x_n,x_{n+1})+\left(b^2\delta_1+b\delta_2+b^2\delta_4\right)d^\chi(x_{n-1},x_n) \\
& +\left(b+b^2\delta_1+b^3\delta_4+b\delta_5\right)d^\chi(x_n,x_n) \\
= \; & \left(b\delta_3+b^3\delta_4\right)\varpi_n+\left(b^2\delta_1+b\delta_2+b^2\delta_4\right)\varpi_{n-1}+\left(b+b^2\delta_1+b^3\delta_4+b\delta_5\right)\eta_{n-1},
\end{aligned}
$$

which implies

$$
\left(1-b\delta_3-b^3\delta_4\right)\varpi_n \preceq \left(b^2\delta_1+b\delta_2+b^2\delta_4\right)\varpi_{n-1}+\left(b+b^2\delta_1+b^3\delta_4+b\delta_5\right)\eta_{n-1}. \tag{15}
$$

Adding (14) and (15), we obtain

$$\varpi_n \preceq \frac{2b^2\delta_1 + b(\delta_2+\delta_3) + b^3(\delta_4+\delta_5)}{2 - b(\delta_2+\delta_3) - b^3(\delta_4+\delta_5)}\varpi_{n-1} + \frac{2b + 2b^2\delta_1 + b(\delta_4+\delta_5) + b^3(\delta_4+\delta_5)}{2 - b(\delta_2+\delta_3) - b^3(\delta_4+\delta_5)}\eta_{n-1}.$$

Given that

$$0 \leq \delta = \frac{2b^2\delta_1 + b(\delta_2+\delta_3) + b^3(\delta_4+\delta_5)}{2 - b(\delta_2+\delta_3) - b^3(\delta_4+\delta_5)} < 1,$$

by applying Proposition 7, it follows that the sequence $\{\varpi_n\}$ is an e-sequence. $\square$

Theorem 4. *Let $(\mathscr{X}, p^\chi)$ be an e-complete χ_b-metric space equipped with a closed positive cone χ^+ such that $(\chi^+)^\ominus \neq \emptyset$. If the mapping $\Upsilon : \mathscr{X} \to \mathscr{X}$ satisfies*

$$p^\chi(\Upsilon\kappa, \Upsilon\rho) \preceq \delta\, p^\chi(\kappa,\rho), \quad \textit{for all } \kappa,\rho \in \mathscr{X} \textit{ and some } \delta \in [0,1),$$

then Υ has a unique fixed point in $\mathscr{X}$, and the iterative sequence $(\Upsilon^n\kappa)_{n\geq 0}$ e-converges to this unique fixed point for any initial point $\kappa \in \mathscr{X}$.

Proof. The result follows directly from the previous theorem by setting $\delta_2 = \delta_3 = \delta_4 = \delta_5 = 0$. $\square$

Remark 2. *If we consider the case where $\delta_1 = \delta_4 = \delta_5 = 0$ and $\delta_2 = \delta_3 \in \left[0, \frac{1}{2}\right)$ (resp., $\delta_1 = \delta_2 = \delta_3 = 0$ and $\delta_4 = \delta_5 \in \left[0, \frac{1}{2}\right)$) in the previous definition, then the inequality (5) corresponds to the Kannan fixed point theorem (resp., Chatterjea fixed point theorem). Also, if $\delta_4 = \delta_5 = 0$ and $\delta_1 = \delta_2 = \delta_3 \neq 0$ in (5), the condition is said to represent the Riech fixed point theorem.*

4. Applications

In this section, without assuming the solidness of the underlying cone, we present several examples to demonstrate that the main results serve as effective tools for verifying the uniqueness of solutions to fixed point equations.

Problem 1. *Let $\mathscr{X} = L[0,1]$ denote the set of all real-valued Lebesgue integrable functions defined on the interval $[0,1]$. Consider the following nonlinear integral equation:*

$$\int_0^1 \Lambda(\tau, h(s))\, ds = h(\tau), \tag{16}$$

where $\Lambda : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a real-valued Lebesgue measurable function. Suppose there exists a function $\Delta : [0,1] \to \mathbb{R}$ such that

$$0 < \int_0^1 \Delta(s)\, ds < 1,$$

and for almost every $v \in [0,1]$ and all $y_1, y_2 \in \mathbb{R}$, the following Lipschitz-type condition holds:

$$|\Lambda(v,y_1) - \Lambda(v,y_2)| \leq \Delta(v)\, |y_1 - y_2|\, .$$

Then equation (16) *admits a unique non-negative solution in $L[0,1]$.*

Proof. We now verify that all the conditions of Theorem 1 are satisfied. Define the operator $\Upsilon : L[0,1] \to L[0,1]$ by

$$\Upsilon\varpi(\tau) = \int_0^1 \Lambda(r, h(s))\, ds.$$

Moreover, let the cone χ^+ be defined as

$$\chi^+ = \{k \in L[0,1] : k(t) \geq 0 \text{ for a.e. } t \in [0,1]\}\, .$$

Define the function $d^\chi : \mathscr{Z} \times \mathscr{Z} \to \chi^+$ by

$$d^\chi(\varpi, \varphi) = e^t \left(\int_0^1 |\varpi(t) - \varphi(t)|\, dt \right).$$

It is clear that $(\mathscr{Z}, d^\chi)$ is an e-complete cone χ_b-metric space. Now we verify that all the conditions of Theorem 1 are satisfied. We observe

$$\begin{aligned}
d\left(\Upsilon\varpi, \Upsilon\varphi\right) &= e^x \left(\int_0^1 |\Upsilon\varpi(r) - \Upsilon\varphi(r)|\, dr \right) \\
&= e^x \left(\int_0^1 \left| \int_0^1 \left(\Lambda(r, \varpi(s)) - \Lambda(r, \varphi(s))\right) ds \right| dr \right) \\
&\leq e^x \left(\int_0^1 \int_0^1 |\Lambda(r, \varpi(s)) - \Lambda(r, \varphi(s))|\, ds\, dr \right) \\
&\leq e^x \left(\int_0^1 \int_0^1 \Delta(r)\, |\varpi(s) - \varphi(s)|\, ds\, dr \right) \\
&= e^x \left(\int_0^1 |\varpi(s) - \varphi(s)|\, ds \right) \left(\int_0^1 \Delta(r)\, dr \right) \\
&< \delta \cdot e^x \left(\int_0^1 |\varpi(s) - \varphi(s)|\, ds \right) \\
&= \delta \cdot d\left(\varpi, \varphi\right),
\end{aligned}$$

where $\delta = \int_0^1 \Delta(r)\,dr < 1$. Hence, the inequality takes the form:

$$d(\Upsilon\varpi, \Upsilon\varphi) \preceq \delta_1 d(\varpi, \varphi) + \delta_2 d(\varpi, \Upsilon\varpi) + \delta_3 d(\varphi, \Upsilon\varphi) + \delta_4 d(\varpi, \Upsilon\varphi) + \delta_5 d(\Upsilon\varpi, \varphi),$$

with parameters $\delta_1 = \delta$, and $\delta_2 = \delta_3 = \delta_4 = \delta_5 = 0$. Given that $b = 1$, it follows that

$$\delta_1 + \delta_2 + \delta_3 + b(\delta_4 + \delta_5) < 1 \quad \text{and} \quad \delta_2 + \delta_5 < \frac{1}{b^2}.$$

As a result, all the conditions of Theorem 11 are satisfied. Hence, the mapping Υ has a unique fixed point. In other words, equation (16) admits a unique solution. □

Problem 2. *Let $L[0,1]$ denote the space of real-valued functions $\varpi(\tau)$ such that*

$$\int_0^1 |\varpi(\tau)|\,d\tau < +\infty.$$

Define $\mathscr{X} = L[0,1]$, and let

$$\chi^+ = \{\varpi \in L[0,1] : \varpi(\tau) \geq 0 \text{ for a.e. } \tau \in [0,1]\}.$$

Define the function $d^\chi : \mathscr{X} \times \mathscr{X} \to \chi^+$ by

$$d^\chi(\varpi, \omega) = \left(\kappa \int_0^1 |\varpi - \omega|\,d\tau, \rho \int_0^1 |\varpi - \omega|\,d\tau\right),$$

where $\kappa, \rho \geq 0$ are constants. Then $(\mathscr{X}, d^\chi)$ is a non-solid cone χ_b-metric space. Define the mapping $\Upsilon : \mathscr{X} \to \mathscr{X}$ by

$$\Upsilon\varpi(\tau) = \frac{1}{4}\ln(1 + |\varpi(\tau)|).$$

Proof. Consider the well-known inequality $0 < \ln(1+\tau) < \tau$ for all $\tau > 0$. Then we compute

$$\begin{aligned}
d^\chi(\Upsilon\varpi, \Upsilon\omega) &= \left(\kappa \int_0^1 \left|\frac{1}{4}\ln(1+|\varpi(\tau)|) - \frac{1}{4}\ln(1+|\omega(\tau)|)\right| d\tau,\right.\\
&\qquad \left.\rho \int_0^1 \left|\frac{1}{4}\ln(1+|\varpi(\tau)|) - \frac{1}{4}\ln(1+|\omega(\tau)|)\right| d\tau\right)\\
&= \frac{1}{4}\left(\kappa \int_0^1 \left|\ln\left(\frac{1+|\varpi(\tau)|}{1+|\omega(\tau)|}\right)\right| d\tau, \rho \int_0^1 \left|\ln\left(\frac{1+|\varpi(\tau)|}{1+|\omega(\tau)|}\right)\right| d\tau\right)\\
&= \frac{1}{4}\left(\kappa \int_0^1 \left|\ln\left(1 + \frac{|\varpi(\tau)| - |\omega(\tau)|}{1+|\omega(\tau)|}\right)\right| d\tau,\right.\\
&\qquad \left.\rho \int_0^1 \left|\ln\left(1 + \frac{|\varpi(\tau)| - |\omega(\tau)|}{1+|\omega(\tau)|}\right)\right| d\tau\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{4}\left(\kappa \int_0^1 \left|\ln\left(1+\frac{|\varpi(\tau)-\omega(\tau)|}{1+|\omega(\tau)|}\right)\right| d\tau,\right.\\
&\qquad\qquad \left.\rho \int_0^1 \left|\ln\left(1+\frac{|\varpi(\tau)-\omega(\tau)|}{1+|\omega(\tau)|}\right)\right| d\tau\right)\\
&\leq \frac{1}{4}\left(\kappa \int_0^1 |\varpi(\tau)-\omega(\tau)|\, d\tau, \rho \int_0^1 |\varpi(\tau)-\omega(\tau)|\, d\tau\right)\\
&= \frac{1}{4} d^{\chi}(\varpi,\omega)\\
&= \delta_1 d^{\chi}(\varpi,\omega)+\delta_2 d^{\chi}(\varpi,\Upsilon\varpi)+\delta_3 d^{\chi}(\omega,\Upsilon\omega)+\delta_4 d^{\chi}(\varpi,\Upsilon\omega)+\delta_5 d^{\chi}(\Upsilon\varpi,\omega),
\end{aligned}
$$

where $\delta_1 = \frac{1}{4}$ and $\delta_2 = \delta_3 = \delta_4 = \delta_5 = 0$. As a result, since all the conditions outlined in Theorem 1 are satisfied, the mapping Υ has a unique fixed point in $\mathscr{X}$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests.

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