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ON FIXED POINT THEOREMS SATISFYING SUZUKI $C(\alpha,\beta)$-CONTRACTIONS IN S-METRIC SPACE

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Abstract. The main aim of this paper is to introduce a generalized Suzuki-type contraction by employing two auxiliary functions α and β in S-metric spaces. Within this framework, we establish two fixed point theorems in the setting of complete S-metric spaces. In order to substantiate the originality and applicability of the main result, some consequences and relevant examples are provided.

Keywords: metric space; fixed point; Cauchy sequence; contraction.

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1. Introduction

Over the past few decades, scientists and engineers have been working to solve complex problems. The concepts such as fixed points and fixed point theorems play a crucial role in non-linear analysis to deal with such kind of problems. These theorems provide fundamental results that guarantee the existence and stability of solutions in various mathematical models, with wide-ranging applications in various fields. The Banach contraction principle [1] is widely recognized as a fundamental theorem in non-linear analysis. In last century, most of the works

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have been focused on generalizing Banach's work, either by extending the underlying metric spaces [2, 3] or by formulating more general contraction conditions [4–10].

In 2008, Suzuki [11] introduced weaker C-contractive condition and define a contraction mapping, commonly known as Suzuki contraction, and proved some fixed point theorems. The existence as well as uniqueness of fixed point of such types of mappings in metric spaces have also been extensively studied in [12, 13].

Alongside the classical theory of contraction mappings, particularly in last 50 years, the domain of metric spaces has been broadened and diversified, incorporating a wide range of novel approaches and generalizations. Some of the most significant examples of this trend are the 2-metric introduced by Gähler [14], the D-metric proposed by Dhage [15], and the G-metric formulated by Mustafa and Sims [16], S-metric space introduced by Sedghi et al. [17], S_b-metric space due to Souayah and Mlaiki [18], and Rohen et al. [19] and most recently extended parametric S_b-metric space formulated by Mani et al. [20]. These generalizations were subsequently employed to broaden the scope of fixed point theory.

Sedghi et al. [17] proposed the idea of S-metric space, by relaxing symmetry property, as a generalization of D-metric and G-metric space.

Definition 1.1. *[17] A function $S : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is said to be an S-metric on $\mathbb{X}$, where $\mathbb{X}$ is a non-empty set, if the following conditions hold:*

S1- $S(\zeta, \gamma, \nu) \geq 0$; for each $\zeta, \gamma, \nu \in \mathbb{X}$;

S2- $S(\zeta, \gamma, \nu) = 0$ if and only if $\zeta = \gamma = \nu$;

S3- $S(\zeta, \gamma, \nu) \leq S(\zeta, \zeta, \lambda) + S(\gamma, \gamma, \lambda) + S(\nu, \nu, \lambda)$ for each $\zeta, \gamma, \nu, \lambda \in \mathbb{X}$.

Moreover, the pair $(\mathbb{X}, S)$ is called an S-metric space.

In addition, the authors presented specific examples of S-metric space to highlight the structural differences between S-metric space and other generalized metric spaces.

Example 1.2. *[17] Let $\mathbb{X}$ be a set of real number and $S : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be a function defined as*

$$S(\zeta, \gamma, \nu) = |\zeta - \nu| + |\gamma - \nu|,$$

for all $\zeta, \gamma, \nu \in \mathbb{R}$. Then S is an S-metric on $\mathbb{R}$.

Example 1.3. *[17] Let $\mathbb{X}$ be set of real number and $S:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to[0,\infty)$ be a function defined as*

$$S(\zeta,\gamma,\nu)=|\gamma+\nu-2\zeta|+|\gamma-\nu|,$$

for all $\zeta,\gamma,\nu\in\mathbb{R}$. Then S is an S-metric on $\mathbb{R}$.

Definition 1.4. *[17] Let $(\mathbb{X},S)$ be an S-metric space. Then a sequence $\{\zeta_j\}\in\mathbb{X}$ is said to*

*i) **converge** to a point $\zeta\in\mathbb{X}$, if there exists $j_0\in\mathbb{N}$ such that for every $\varepsilon>0$,*

$$S(\zeta_j,\zeta_j,\zeta)<\varepsilon,\quad \forall\, j\geq j_0.$$

ii) be a Cauchy sequence, if there exists $j_0\in\mathbb{N}$ such that for every $\varepsilon>0$,

$$S(\zeta_j,\zeta_m,\zeta)<\varepsilon,\qquad \forall\, j,m\geq j_0.$$

Definition 1.5. *[17] An S-metric space $\mathbb{X}$ is said to be complete if every Cauchy sequence in $\mathbb{X}$ converges to a point in $\mathbb{X}$.*

Theorem 1.6. *[21] Let $(\mathbb{X},S)$ be an S-metric space such that $\mathbb{X}$ is complete. Define a map $\mathscr{B}:\mathbb{X}\to\mathbb{X}$ such that for all $\zeta,\gamma\in\mathbb{X}$, it satisfies*

$$S(\mathscr{B}\zeta,\mathscr{B}\zeta,\mathscr{B}\gamma)\leq LS(\zeta,\zeta,\gamma),$$

where $0\leq L<1$. Then $\mathscr{B}$ has a unique fixed point $u\in\mathbb{X}$.
Furthermore, for any $\zeta\in\mathbb{X}$, we have $\lim_{j\to\infty}\mathscr{B}^j(\zeta)=u$ with

$$S\left(\mathscr{B}^j(\zeta),\mathscr{B}^j(\zeta),u\right)\leq\frac{2L^j}{1-L}S(\zeta,\zeta,\mathscr{B}(\zeta)).$$

Lemma 1.7. *[21] Within the context of S-metric space, the function S satisfies,*

$$S(\zeta,\zeta,\gamma)=S(\gamma,\gamma,\zeta).$$

In last 10 years, fixed point problems have generated significant attention from researchers across various disciplines by using the various types of contractive conditions for mappings in S-metric spaces that involve simulation functions, control functions, C-class functions, Suzuki type functions and many more. For instances: Mlaiki et al. [22] proved some fixed point theorems using the set of simulation functions on an S-metric space. Ansari et al. [23] proved some coupled fixed point results in partially ordered S-metric spaces. Shahraki et al. [24], in 2020, proved some fixed point theorems by considering more general form of Suzuki type function in S-metric spaces. Babu and Kumssa [25] proved the existence and uniqueness of fixed points for contractive mappings involving control functions. In 2022, Saluja [26] proved some results via C-class fucntions in S-metric spaces. In 2023, Qaralleh et al. [27] proposed an extension by placing two functions instead of constants as another generalization of S-metric spaces, named it as extended metric spaces of type (α,β) and proved several fixed point theorems. Further, in 2024, Saluja et al. [28] proved fixed point theorems for integral type contractions involving rational terms in the context of complete S-metric spaces and discuss their implications. Most recently Mani et al. [29] proved some fixed point theorems for two mappings (not necessary continuous), satisfying generalized contractions involving rational expressions in the setting of extended parametric Sb-metric spaces.

Following definition is due to Khan et al. [30], will be quite useful in proving our main results. Let us denote the classes of functions Φ (due to Khan [29]) as follows:

i) $\Phi = \{\alpha | \alpha : [0,\infty) \to [0,\infty)$ is a monotone increasing and continuous function with $\alpha(\lambda) = 0$ if and only if $\lambda = 0\}$.

In this work, we first define a generalized Suzuki type contraction using auxiliary functions, called it as Suzuki $C(\alpha,\beta)$- contraction and then utilized it to prove two fixed point results on complete S-metric spaces. At last, some consequences and examples are presented to demonstrate the validity of the proved result. The results presented herein generalize, extend, and enhance various existing findings in the literature.

2. Main Results

In this section, we begin by presenting a lemma that is essential for establishing our main results.

Lemma 2.1. *Let $\mathbb{X}$ be a non-empty set and $S : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \to [0,\infty)$ be a function such that the pair $(\mathbb{X},S)$ is an S-metric space. Further assume that sequence $\{\zeta_j\} \in \mathbb{X}$ satisfies*

$$\lim_{j\to\infty} S(\zeta_j,\zeta_j,\zeta_{j+1}) = 0.$$

If $\{\zeta_j\}$ is not a Cauchy sequence in $\mathbb{X}$ then there exist an ε >0 and sequences of positive integers (m_k) and (j_k) with $(m_k) > (j_k) > k$ such that

$$S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k}) > \varepsilon$$

and

$$S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}}) \leq \varepsilon.$$

Then following holds:

$$\lim_{k\to\infty} S(\zeta_{j_k}, \zeta_{j_k}, \zeta_{m_k}) = \varepsilon$$
$$\lim_{k\to\infty} S(\zeta_{m_{k+1}}, \zeta_{m_{k+1}}, \zeta_{j_k}) = \varepsilon$$
$$\lim_{k\to\infty} S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \zeta_{m_{k+1}}) = \varepsilon.$$

Proof. Its given that the sequence $\{\zeta_j\} \in \mathbb{X}$ satisfies

$$\lim_{j\to\infty} S(\zeta_j, \zeta_j, \zeta_{j+1}) = 0. \tag{1}$$

Also, $\{\zeta_j\} \subset \mathbb{X}$ is not Cauchy, so we can choose an $\varepsilon > 0$ and two sequences $\{m_k\}$ (least positive integer) and $\{j_k\}$ of positive integers with $m_k > j_k > k$ such that

$$S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k}) > \varepsilon \tag{2}$$

and

$$S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{j_k}) \leq \varepsilon. \tag{3}$$

On taking the lower limit $k \to \infty$ in Eq.(2), we get

$$\varepsilon < \lim_{k\to\infty} \inf S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k}). \tag{4}$$

Using triangle inequality in $S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k})$ we get,

$$\begin{aligned} S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k}) &\leq 2S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{m_{k-1}}) + S(\zeta_{j_k}, \zeta_{j_k}, \zeta_{m_{k-1}}) \\ &= 2S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{m_{k-1}}) + S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{j_k}). \end{aligned} \tag{5}$$

From Eq.(1) , Eq. (3) and taking the upper limit as $k \to \infty$ in Eq. (5), we have

$$\limsup_{k\to\infty} S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k}) < \varepsilon. \tag{6}$$

Combining Eq. (4) and Eq. (6), we obtain

$$\lim_{k\to\infty} S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k}) = \varepsilon. \tag{7}$$

Again using triangle inequality, we have

$$\begin{aligned} S(\zeta_{m_{k+1}}, \zeta_{m_{k+1}}, \zeta_{j_k}) &\le 2S(\zeta_{m_{k+1}}, \zeta_{m_{k+1}}, \zeta_{m_k}) + S(\zeta_{j_k}, \zeta_{j_k}, \zeta_{m_k}) \\ &= 2S(\zeta_{m_{k+1}}, \zeta_{m_{k+1}}, \zeta_{m_k}) + S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k}). \end{aligned}$$

From Eq.(1) and Eq. (7) and taking $\lim_{k\to\infty}$ on both sides, we get

$$\lim_{k\to\infty} S(\zeta_{m_{k+1}}, \zeta_{m_{k+1}}, \zeta_{j_k}) = \varepsilon. \tag{8}$$

Again using triangle inequality, we have

$$\begin{aligned} S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \zeta_{m_{k+1}}) &\le 2S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \zeta_{j_k}) + S(\zeta_{m_{k+1}}, \zeta_{m_{k+1}}, \zeta_{j_k}) \\ &= 2S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \zeta_{j_k}) + S(\zeta_{m_{k+1}}, \zeta_{m_{k+1}}, \zeta_{j_k}). \end{aligned}$$

By using Eq.(1) and Eq. (8) and taking $\lim_{k\to\infty}$ on both sides, we get

$$\lim_{k\to\infty} S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \zeta_{m_{k+1}}) = \varepsilon.$$

This completes the proof of the lemma. □

Next, under the context of S-metric space, we introduce the definition of Suzuki $C(\alpha,\beta)$-contraction.

Definition 2.2. *Let $(\mathbb{X}, S)$ be an S-metric space. A map $\mathscr{B} : \mathbb{X} \to \mathbb{X}$ is said to be a Suzuki $C(\alpha,\beta)$-contraction, if for all $\zeta, \gamma, \nu \in \mathbb{X}$,*

$$\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \le S(\zeta,\gamma,\nu) \quad \textit{implies} \quad \alpha(S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu)) \le \beta(S(\zeta,\gamma,\nu)), \tag{9}$$

where $\beta : [0,\infty) \to [0,\infty)$ is a continuous function and $\alpha \in \Phi$ are such that for any $\lambda > 0$

$$\beta(\lambda) < \alpha(\lambda). \tag{10}$$

Remark 2.3. *In the above Definition 2.2, if we let $\alpha(\lambda) = \beta(\lambda) = \lambda$, it reduces to the Suzuki type contraction in S-metric spaces.*

Remark 2.4. *If we further let $\alpha(\lambda) = \beta(\lambda) = \lambda$, and suppose $d : \mathbb{X} \times \mathbb{X} \to [0,\infty)$ be a function defined as $d(\zeta,\gamma) = S(\zeta,\zeta,\gamma)$, Then Definition 2.2 reduces to the Suzuki type contraction in metric spaces.*

Consider the following example of a Suzuki $C(\alpha,\beta)$-contraction in S-metric spaces.

Example 2.5. *Let $\mathbb{X} = [0,1)$. Define the metric $S : \mathbb{X}^3 \to [0,\infty)$ by*

$$S(\zeta,\gamma,\nu) = \max\{\zeta,\gamma,\nu\},$$

for all $\zeta,\gamma,\nu \in \mathbb{X}$ such that the pair $(\mathbb{X},S)$ is an S-metric spaces.
Define a map $\mathscr{B} : \mathbb{X} \to \mathbb{X}$ by

$$\mathscr{B}(\zeta) = \ln\left(1 + \frac{2\zeta}{3}\right),$$

for all $\zeta \in \mathbb{X}$.
Further, define two auxiliary functions $\alpha,\beta : [0,\infty) \to [0,\infty)$ by

$$\alpha(\lambda) = \lambda \text{ and } \beta(\lambda) = \frac{\lambda}{\lambda+1}.$$

Clearly, $\alpha \in \Phi$ and β is a continuous function such that it satisfies Eq.(10).
Moreover, for all $\zeta,\gamma,\nu \in \mathbb{X}$, Map $\mathscr{B}$ satisfies the inequality (9).
Thus the map $\mathscr{B}$ is a Suzuki $C(\alpha,\beta)$-contraction in S-metric spaces.

Let's begin with our first result.

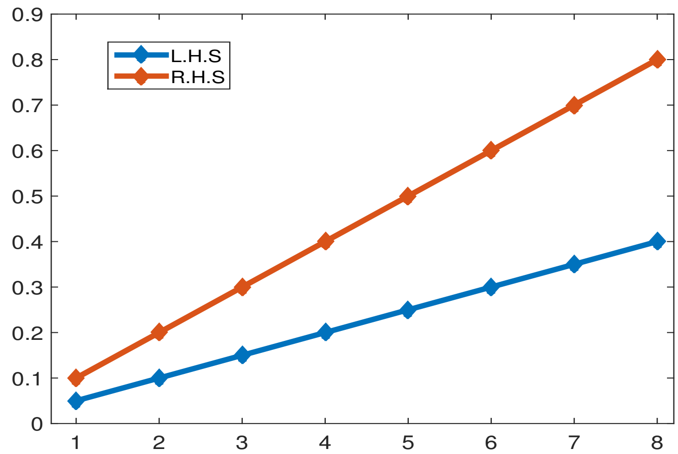
Theorem 2.6. *Let $(\mathbb{X},S)$ be a complete S-metric space and suppose that $\mathscr{B} : \mathbb{X} \to \mathbb{X}$ is a Suzuki $C(\alpha,\beta)$-contraction. Then $\mathscr{B}$ has a unique fixed point.*

Proof. Choose an $\zeta_0 \in \mathbb{X}$ as an arbitrary point and let $\mathscr{B}\zeta_0 = \zeta_1$.
Since,

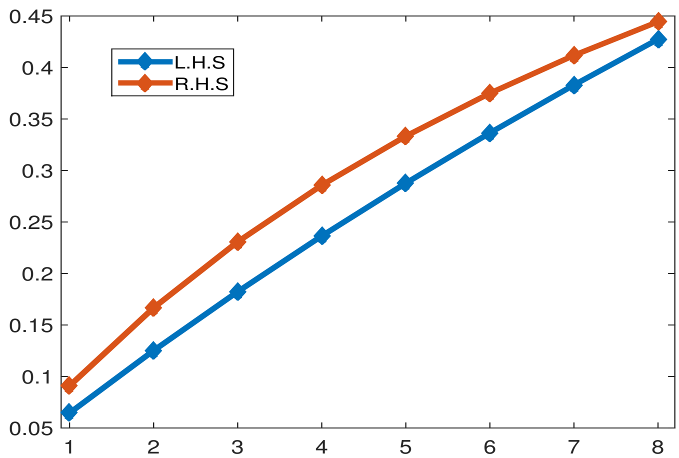
$$\frac{1}{2}S(\zeta_0,\zeta_0,\mathscr{B}\zeta_0) \le S(\zeta_0,\zeta_0,\zeta_1)$$

then, from Eq. (9) on substituting $\zeta = \gamma = \zeta_0$ and $\nu = \zeta_1$

$$\alpha(S(\zeta_1,\zeta_1,\zeta_2)) = \alpha(S(\mathscr{B}\zeta_0,\mathscr{B}\zeta_0,\mathscr{B}\zeta_1)) \le \beta(S(\zeta_0,\zeta_0,\zeta_1)).$$



(A) Behaviour of the inequality $\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \leq S(\zeta,\gamma,\nu)$



(B) Behaviour of the inequality $\alpha(S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu)) \leq \beta(S(\zeta,\gamma,\nu))$

FIGURE 1. Graphical Behaviour of the inequality (9) in context of Example 2.5

Let $\mathscr{B}\zeta_1 = \zeta_2$. Since

$$\frac{1}{2}S(\zeta_1,\zeta_1,\mathscr{B}\zeta_1) \leq S(\zeta_1,\zeta_1,\zeta_2)$$

again, from Eq. (9) on substituting $\zeta = \gamma = \zeta_1$ and $\nu = \zeta_2$, we have

$$\alpha(S(\zeta_2,\zeta_2,\zeta_3)) = \alpha(S(\mathscr{B}\zeta_1,\mathscr{B}\zeta_1,\mathscr{B}\zeta_2)) \leq \beta(S(\zeta_1,\zeta_1,\zeta_2)).$$

By continuing this process, we can construct a sequence $\{\zeta_j\}$ such that

$$\zeta_j = \mathscr{B}\zeta_{j-1}.$$

Since

$$\frac{1}{2}S(\zeta_j,\zeta_j,\mathscr{B}\zeta_j) \leq S(\zeta_j,\zeta_j,\zeta_{j+1}),$$

then, again on substituting $\zeta=\gamma=\zeta_j$ and $\nu=\zeta_{j+1}$ in Eq.(9), we obtain

$$\alpha(S(\zeta_{j+1},\zeta_{j+1},\zeta_{j+2}))=\alpha(S(\mathscr{B}\zeta_j,\mathscr{B}\zeta_j,\mathscr{B}\zeta_{j+1}))\leq\beta(S(\zeta_j,\zeta_j,\zeta_{j+1})). \tag{11}$$

Since α and β are continuous functions so

$$S(\zeta_{j+1},\zeta_{j+1},\zeta_{j+2})\leq S(\zeta_j,\zeta_j,\zeta_{j+1}).$$

Similarly, by proceeding as above, we get

$$S(\zeta_j,\zeta_j,\zeta_{j+1})\leq S(\zeta_{j-1},\zeta_{j-1},\zeta_j).$$

Thus, we get a sequence of non-increasing functions such that for any $\lambda\geq 0$, we have

$$\lim_{j\to\infty}S(\zeta_j,\zeta_j,\zeta_{j+1})=\lambda. \tag{12}$$

Suppose $\lambda>0$, then on taking $\lim_{j\to\infty}$ in Eq.(11), we get

$$\alpha(\lambda)\leq\beta(\lambda),$$

which is a contradiction to given condition (10). Hence, Eq.(12) gives that

$$\lim_{j\to\infty}S(\zeta_j,\zeta_j,\zeta_{j+1})=0.$$

Claim: $\{\zeta_j\}$ is a Cauchy sequence. Assume on contrary that the sequence $\{\zeta_j\}$ is not Cauchy then for an $\varepsilon>0$ there exists sub-sequences of positive integers $m_k>j_k>k$ such that

$$S(\zeta_{m_k},\zeta_{m_k},\zeta_{j_k})>\varepsilon \quad\text{and}\quad S(\zeta_{m_k},\zeta_{m_k},\zeta_{j_{k-1}})\leq\varepsilon. \tag{13}$$

Also for this $\varepsilon>0$, the convergence of sequence $\{S(\zeta_j,\zeta_j,\zeta_{j+1})\}$ implies that there exists some $N_0\in\mathbb{N}$ such that

$$S(\zeta_j,\zeta_j,\zeta_{j+1})<\varepsilon.$$

For all $j\geq N_0$, let $N_1=\max\{m_i,N_0\}$. Thus, for all $m_k>j_k\geq N_1$, we have

$$\begin{aligned}
&S(\zeta_{m_k},\zeta_{m_k},\zeta_{m_{k+1}})\leq\varepsilon<S(\zeta_{m_k},\zeta_{m_k},\zeta_{j_k})\\
&S(\zeta_{m_k},\zeta_{m_k},\zeta_{m_{k+1}})\leq 2S(\zeta_{m_k},\zeta_{m_k},\zeta_{j_{k-1}})+S(\zeta_{j_k},\zeta_{j_k},\zeta_{j_{k-1}})\\
&\qquad\leq 2S(\zeta_{m_k},\zeta_{m_k},\zeta_{j_{k-1}}),
\end{aligned}$$

and so

$$\frac{1}{2}S(\zeta_{m_k},\zeta_{m_k},\zeta_{m_{k+1}}) \leq S(\zeta_{m_k},\zeta_{m_k},\zeta_{j_{k-1}}).$$

Therefore, from Eq. (9), on substituting $\zeta = \gamma = \zeta_{m_k}$ and $\nu = \zeta_{j_{k-1}}$ we get

$$\begin{aligned}\alpha(S(\zeta_{m_{k+1}},\zeta_{m_{k+1}},\zeta_{j_k})) &= \alpha(S(\mathscr{B}\zeta_{m_k},\mathscr{B}\zeta_{m_k},\mathscr{B}\zeta_{j_{k-1}}))\\ &\leq \beta(S(\zeta_{m_k},\zeta_{m_k},\zeta_{j_k-1})).\end{aligned}$$

Taking $\lim_{k\to\infty}$ in above inequality, and make use of Lemma 2.1 and Eq. (13), we obtain

$$\alpha(\varepsilon) \leq \beta(\varepsilon),$$

which contradicts to Eq.(10). Thus our assumption is wrong. Hence, $\{\zeta_j\}$ is a Cauchy sequence.

Since $(\mathbb{X},S)$ is a complete S-metric space so the sequence $\{\zeta_j\}$ converges along with all its sub-sequences to some point say ω i.e.

$$\lim_{j\to\infty}\zeta_j = \lim_{j\to\infty}\mathscr{B}\zeta_j = \omega.$$

Alternatively, we have

$$\lim_{j\to\infty}S(\zeta_j,\zeta_j,\omega) = 0,$$

for all $j \in \mathbb{N}$.

Next we will prove that ω is a fixed point of $\mathscr{B}$, that is , we have to show that $S(\omega,\omega,\mathscr{B}\omega) = 0$.

Now, we claim that for each $j \in \mathbb{N}$, one of the following relations hold good:

$$\text{(14)} \qquad \begin{cases}\frac{1}{2}S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right) \leq S\left(\zeta_j,\zeta_j,\omega\right) \quad \text{or}\\ \\ \frac{1}{2}S\left(\zeta_{j+1},\zeta_{j+1},\mathscr{B}\zeta_{j+1}\right) \leq S\left(\zeta_{j+1},\zeta_{j+1},\omega\right).\end{cases}$$

Suppose inequality (14), is not true for every $j \in \mathbb{N}$. Therefore for some $j \geq 1$, we have

$$\frac{1}{2}S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right) \geq S\left(\zeta_j,\zeta_j,\omega\right) \text{ and } \frac{1}{2}S\left(\zeta_{j+1},\zeta_{j+1},\mathscr{B}\zeta_{j+1}\right) \geq S\left(\zeta_{j+1},\zeta_{j+1},\omega\right).$$

On using property (S3) of Definition 1.1, we have

$$\begin{aligned} S\left(\zeta_j,\zeta_j,\zeta_{j+1}\right) &\leq S\left(\zeta_j,\zeta_j,\omega\right)+S\left(\zeta_j,\zeta_j,\omega\right)+S\left(\zeta_{j+1},\zeta_{j+1},\omega\right) \\ &\leq \frac{1}{2}\left[S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)+S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)+S\left(\zeta_{j+1},\zeta_{j+1},\mathscr{B}\zeta_{j+1}\right)\right] \\ &\leq \frac{1}{2}\left[S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)+S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)+S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)\right] \\ &\leq \frac{3}{2}S\left(\zeta_j,\zeta_j,\zeta_{j+1}\right). \end{aligned}$$

This is a contradiction, and hence we have our claim, and so inequality (14) is true for all $j \in \mathbb{N}$.
Inequality (14) implies that (from Eq. (9))

$$\alpha(S(\mathscr{B}\zeta_j,\mathscr{B}\zeta_j,\mathscr{B}\omega)) \leq \beta(S(\zeta_j,\zeta_j,\omega)).$$

Taking the lim as $j \to \infty$ in above inequality and using the fact that α and β are continuous, we have

$$\alpha(S(\omega,\omega,\mathscr{B}\omega)) \leq \beta(0) = 0,$$

which is only possible if $S(\omega,\omega,\mathscr{B}\omega) = 0$, and so, ω is the fixed point of $\mathscr{B}$.
In order to guarantee the uniqueness of fixed point, assume on contrary that there exists another common fixed point κ, different from ω of map $\mathscr{B}$ in $\mathbb{X}$ such that $\mathscr{B}\kappa = \kappa$ and $\mathscr{B}\omega = \omega$. Clearly, $S(\omega,\omega,\kappa) \neq 0$.
Substitute $\zeta = \gamma = \omega$ and $\nu = \kappa$ in Eq.(9), we have

$$\frac{1}{2}S(\omega,\omega,\mathscr{B}\omega) = 0 \leq S(\omega,\omega,\kappa)$$

implies

$$\alpha(S(\mathscr{B}\omega,\mathscr{B}\omega,\mathscr{B}\kappa)) \leq \beta(S(\omega,\omega,\kappa)), \tag{15}$$

further implies that,

$$\alpha(S(\omega,\omega,\kappa)) \leq \beta(S(\omega,\omega,\kappa)),$$

which contradicts Eq.(10). Thus our assumption is wrong, and so $S(\omega,\omega,\kappa) = 0$. Hence, $\omega = \kappa$. This proves that map $\mathscr{B}$ has a unique fixed point . □

In the subsequent result, we establish a fixed-point theorem for a self-mapping that satisfies a more general form of Suzuki $C(\alpha,\beta)$-contraction involving rational expressions.

Theorem 2.7. *Let $(\mathbb{X}, S)$ be a complete S-metric space and let $\mathscr{B} : \mathbb{X} \to \mathbb{X}$ be a self mapping. Suppose also that for all $\zeta, \gamma, \nu \in \mathbb{X}$,*

$$\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \le S(\zeta,\gamma,\nu) \quad \textit{implies} \quad \alpha(S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu)) \le \beta(Q(\zeta,\gamma,\nu)), \tag{16}$$

where

$$Q(\zeta,\gamma,\nu) = \max\left\{ \begin{array}{c} S(\zeta,\gamma,\nu), S(\zeta,\zeta,\mathscr{B}\zeta), S(\nu,\nu,\mathscr{B}\nu), \\ \dfrac{S(\nu,\nu,\mathscr{B}\zeta)S(\zeta,\zeta,\mathscr{B}\nu)}{1+S(\zeta,\gamma,\nu)}, \dfrac{S(\nu,\nu,\mathscr{B}\nu)}{1+S(\zeta,\gamma,\nu)} \end{array} \right\},$$

$\alpha \in \Phi$ and β is a continuous function such that for any $\lambda > 0$

$$\beta(\lambda) < \alpha(\lambda). \tag{17}$$

Then $\mathscr{B}$ has a unique fixed point.

Proof. Choose an $\zeta_0 \in \mathbb{X}$ as an arbitrary point and let $\mathscr{B}\zeta_0 = \zeta_1$. If $\zeta_0 = \zeta_1$, then ζ_0 is the fixed point of the $\mathscr{B}$. Hence we have the proof.

Without loss of generality, we assume that $\zeta_0 \neq \zeta_1$. Construct a sequence $\{\zeta_j\} \in \mathbb{X}$ such that

$$\zeta_j = \mathscr{B}\zeta_{j-1} = \mathscr{B}^j\zeta_0.$$

Since

$$\frac{1}{2}S(\zeta_j,\zeta_j,\mathscr{B}\zeta_j) \le S(\zeta_j,\zeta_j,\zeta_{j-1}),$$

then on substituting $\zeta = \gamma = \zeta_j$ and $\nu = \zeta_{j-1}$ in Eq. (16), gives that

$$\alpha(S(\zeta_{j+1},\zeta_{j+1},\zeta_j)) = \alpha(S(\mathscr{B}\zeta_j,\mathscr{B}\zeta_j,\mathscr{B}\zeta_{j-1})) \le \beta(Q(\zeta_j,\zeta_j,\zeta_{j-1})),$$

where

$$\begin{aligned} Q(\zeta_j,\zeta_j,\zeta_{j-1}) &= \max\left\{ \begin{array}{c} S(\zeta_j,\zeta_j,\zeta_{j-1}), S(\zeta_j,\zeta_j,\mathscr{B}\zeta_j), S(\zeta_{j-1},\zeta_{j-1},\mathscr{B}\zeta_{j-1}), \\ \dfrac{S(\zeta_{j-1},\zeta_{j-1},\mathscr{B}\zeta_j)S(\zeta_j,\zeta_j,\mathscr{B}\zeta_{j-1})}{1+S(\zeta_j,\zeta_j,\zeta_{j-1})}, \dfrac{S(\zeta_{j-1},\zeta_{j-1},\mathscr{B}\zeta_{j-1})}{1+S(\zeta_j,\zeta_j,\zeta_{j-1})} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} S(\zeta_j,\zeta_j,\zeta_{j-1}), S(\zeta_j,\zeta_j,\zeta_{j+1}), S(\zeta_{j-1},\zeta_{j-1},\zeta_j), \\ \dfrac{S(\zeta_{j-1},\zeta_{j-1},\zeta_{j+1})S(\zeta_j,\zeta_j,\zeta_j)}{1+S(\zeta_j,\zeta_j,\zeta_{j-1})}, \dfrac{S(\zeta_{j-1},\zeta_{j-1},\zeta_j)}{1+S(\zeta_j,\zeta_j,\zeta_{j-1})} \end{array} \right\} \\ &= \max\left\{ S(\zeta_j,\zeta_j,\zeta_{j-1}), S(\zeta_j,\zeta_j,\zeta_{j+1}) \right\} \\ &= \max\left\{ S(\zeta_j,\zeta_j,\zeta_{j+1}), S(\zeta_j,\zeta_j,\zeta_{j-1}) \right\}. \end{aligned}$$

Two alternatives emerge here:

Alter. 1- Either $Q(\zeta_j,\zeta_j,\zeta_{j-1}) = S(\zeta_j,\zeta_j,\zeta_{j+1})$

Alter. 2- or $Q(\zeta_j,\zeta_j,\zeta_{j-1}) = S(\zeta_j,\zeta_j,\zeta_{j-1})$.

Suppose Alter.1- is true, then we obtain that

$$\alpha(S(\zeta_{j+1},\zeta_{j+1},\zeta_j)) \leq \beta(S(\zeta_{j+1},\zeta_{j+1},\zeta_j)),$$

which contradicts to Eq.(17).

Thus from Alter.2, we have

$$\alpha(S(\zeta_{j+1},\zeta_{j+1},\zeta_j)) \leq \beta(S(\zeta_j,\zeta_j,\zeta_{j-1})). \tag{18}$$

Since α and β are continuous functions, so

$$S(\zeta_j,\zeta_j,\zeta_{j+1}) \leq S(\zeta_{j-1},\zeta_{j-1},\zeta_j).$$

Similarly, by proceeding as above, we get

$$S(\zeta_{j+1},\zeta_{j+1},\zeta_{j+2}) \leq S(\zeta_j,\zeta_j,\zeta_{j+1}).$$

Thus, we get a sequence of non-increasing functions such that for any $\lambda \geq 0$, we have

$$\lim_{j\to\infty} S(\zeta_j,\zeta_j,\zeta_{j+1}) = \lambda. \tag{19}$$

Suppose $\lambda > 0$, then on taking $\lim_{j\to\infty}$ in Eq.(18), we get

$$\alpha(\lambda) \leq \beta(\lambda),$$

which is a contradiction to given condition (17). Hence, Eq. (19) gives that

$$\lim_{j\to\infty} S(\zeta_j,\zeta_j,\zeta_{j+1}) = 0.$$

Claim: $\{\zeta_j\}$ is a Cauchy sequence. Assume on contrary that the sequence $\{\zeta_j\}$ is not Cauchy then for an $\varepsilon > 0$ there exists sub-sequences of positive integers $m_k > j_k > k$ such that

$$S(\zeta_{m_k},\zeta_{m_k},\zeta_{j_k}) > \varepsilon \quad \text{and} \quad S(\zeta_{m_k},\zeta_{m_k},\zeta_{j_{k-1}}) \leq \varepsilon.$$

Also for this $\varepsilon > 0$, the convergence of sequence $\left\{S\left(\zeta_j, \zeta_j, \zeta_{j+1}\right)\right\}$ implies that there exists some $N_0 \in \mathbb{N}$ such that

$$S\left(\zeta_j, \zeta_j, \zeta_{j+1}\right) < \varepsilon.$$

For all $j \geq N_0$, let $N_1 = \max\{m_i, N_0\}$. Thus, for all $m_k > j_k \geq N_1$, we have

$$\begin{aligned} S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{m_{k+1}}) &\leq \varepsilon < S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k}) \\ S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{m_{k+1}}) &\leq 2S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}}) + S(\zeta_{j_k}, \zeta_{j_k}, \zeta_{j_{k-1}}) \\ &\leq 2S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}}) \end{aligned}$$

and so

$$\frac{1}{2}S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{m_{k+1}}) \leq S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}}).$$

Therefore, from Eq. (16), on substituting $\zeta = \gamma = \zeta_{m_k}$ and $\nu = \zeta_{j_{k-1}}$ we get

$$\begin{aligned} \alpha(S(\zeta_{m_{k+1}}, \zeta_{m_{k+1}}, \zeta_{j_k})) &= \alpha(S(\mathscr{B}\zeta_{m_k}, \mathscr{B}\zeta_{m_k}, \mathscr{B}\zeta_{j_{k-1}})) \\ &\leq \beta(Q(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k} - 1)), \end{aligned} \tag{20}$$

where

$$\begin{aligned} Q(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}}) &= \max \left\{ \begin{array}{c} S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}}), S(\zeta_{m_k}, \zeta_{m_k}, \mathscr{B}\zeta_{m_k}), S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \mathscr{B}\zeta_{j_{k-1}}), \\ \dfrac{S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \mathscr{B}\zeta_{m_k}) S(\zeta_{m_k}, \zeta_{m_k}, \mathscr{B}\zeta_{j_{k-1}})}{1 + S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}})}, \\ \dfrac{S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \mathscr{B}\zeta_{j_{k-1}})}{1 + S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}})} \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}}), S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{m_{k+1}}), S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \zeta_{j_k}), \\ \dfrac{S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \zeta_{m_{k+1}}) S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_k})}{1 + S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}})}, \dfrac{S(\zeta_{j_{k-1}}, \zeta_{j_{k-1}}, \zeta_{j_k})}{1 + S(\zeta_{m_k}, \zeta_{m_k}, \zeta_{j_{k-1}})} \end{array} \right\}. \end{aligned}$$

Taking $\lim_{k \to \infty}$, and on using Lemma 2.1 in above equality and in Eq. (20), we obtain

$$\alpha(\varepsilon) \leq \beta(\varepsilon),$$

which is again a contradiction to Eq.(17). Thus our assumption is wrong. Hence, $\{\zeta_j\}$ is a Cauchy sequence.

Since $(\mathbb{X}, S)$ is a complete S-metric space so the sequence $\left\{\zeta_j\right\}$ converges along with all its sub-sequences to some point say ω i.e.

$$\lim_{j\to\infty}\zeta_j=\lim_{j\to\infty}\mathscr{B}\zeta_j=\omega.$$

Alternatively, we have

$$\lim_{j\to\infty}S(\zeta_j,\zeta_j,\omega)=0,$$

for all $j\in\mathbb{N}$.

Next we will prove that ω is a fixed point of $\mathscr{B}$, that is , we have to show that $S(\omega,\omega,\mathscr{B}\omega)=0$.

Now, we claim that for each $j\in\mathbb{N}$, one of the following relations hold good:

$$\text{(21)}\qquad \begin{cases} \frac{1}{2}S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)\le S\left(\zeta_j,\zeta_j,\omega\right) \quad \text{or} \\ \\ \frac{1}{2}S\left(\zeta_{j+1},\zeta_{j+1},\mathscr{B}\zeta_{j+1}\right)\le S\left(\zeta_{j+1},\zeta_{j+1},\omega\right). \end{cases}$$

Suppose inequality (21), is not true for every $j\in\mathbb{N}$. Therefore for some $j\ge 1$, we have

$$\frac{1}{2}S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)\ge S\left(\zeta_j,\zeta_j,\omega\right) \text{ and } \frac{1}{2}S\left(\zeta_{j+1},\zeta_{j+1},\mathscr{B}\zeta_{j+1}\right)\ge S\left(\zeta_{j+1},\zeta_{j+1},\omega\right).$$

On using property (S3) of Definition 1.1, we have

$$\begin{aligned} S\left(\zeta_j,\zeta_j,\zeta_{j+1}\right) &\le S\left(\zeta_j,\zeta_j,\omega\right)+S\left(\zeta_j,\zeta_j,\omega\right)+S\left(\zeta_{j+1},\zeta_{j+1},\omega\right) \\ &\le \frac{1}{2}\left[S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)+S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)+S\left(\zeta_{j+1},\zeta_{j+1},\mathscr{B}\zeta_{j+1}\right)\right] \\ &\le \frac{1}{2}\left[S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)+S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)+S\left(\zeta_j,\zeta_j,\mathscr{B}\zeta_j\right)\right] \\ &\le \frac{3}{2}S\left(\zeta_j,\zeta_j,\zeta_{j+1}\right). \end{aligned}$$

This is a contradiction, and hence we have our claim, and so inequality (21) is true for all $j\in\mathbb{N}$.

Inequality (21) implies that (from Eq. (16))

$$\alpha(S(\mathscr{B}\zeta_j,\mathscr{B}\zeta_j,\mathscr{B}\omega))\le\beta(Q(\zeta_j,\zeta_j,\omega)).$$

Taking lim as $j\to\infty$ in above inequality, and using the fact that the maps α and β are continuous, we have

$$\text{(22)}\qquad \alpha(S(\omega,\omega,\mathscr{B}\omega))=\alpha(\lim_{j\to\infty}S(\mathscr{B}\zeta_j,\mathscr{B}\zeta_j,\mathscr{B}\omega))\le\beta(\lim_{j\to\infty}Q(\zeta_j,\zeta_j,\omega)).$$

where

$$\lim_{j\to\infty} Q(\zeta_j,\zeta_j,\omega) = \lim_{j\to\infty} \max\left\{\begin{array}{c} S(\zeta_j,\zeta_j,\omega), S(\zeta_j,\zeta_j,\mathscr{B}\zeta_j), \\ S(\omega,\omega,\mathscr{B}\omega), \\ \dfrac{S(\omega,\omega,\mathscr{B}\zeta_j)S(\zeta_j,\zeta_j,\mathscr{B}\omega)}{1+S(\zeta_j,\zeta_j,\omega)}, \\ \dfrac{S(\omega,\omega,\mathscr{B}\omega)}{1+S(\zeta_j,\zeta_j,\omega)} \end{array}\right\}$$

$$= S(\omega,\omega,\mathscr{B}\omega).$$

Thus from Eq. (22), we have

$$\alpha(S(\omega,\omega,\mathscr{B}\omega)) \leq \beta(S(\omega,\omega,\mathscr{B}\omega)),$$

which is a contradiction to Eq.(17) and only possible $S(\omega,\omega,\mathscr{B}\omega)=0$. Thus, we have our claim that ω is the fixed point of $\mathscr{B}$.

In order to guarantee the uniqueness of fixed point, assume on contrary that there exists another common fixed point κ, different from ω of map $\mathscr{B}$ in $\mathbb{X}$ such that $\mathscr{B}\kappa=\kappa$ and $\mathscr{B}\omega=\omega$. Clearly, $S(\omega,\omega,\kappa)\neq 0$.

Substitute $\zeta=\gamma=\omega$ and $\nu=\kappa$ in Eq.(16), we have

$$\frac{1}{2}S(\omega,\omega,\mathscr{B}\omega) = 0 \leq S(\omega,\omega,\kappa)$$

implies

$$\alpha(S(\omega,\omega,\kappa)) = \alpha(S(\mathscr{B}\omega,\mathscr{B}\omega,\mathscr{B}\kappa)) \leq \beta(Q(\omega,\omega,\kappa)), \tag{23}$$

where

$$Q(\omega,\omega,\kappa) = \max\left\{\begin{array}{c} S(\omega,\omega,\kappa), S(\omega,\omega,\mathscr{B}\omega), S(\kappa,\kappa,\mathscr{B}\kappa), \\ \dfrac{S(\kappa,\kappa,\mathscr{B}\omega)S(\omega,\omega,\mathscr{B}\kappa)}{1+S(\kappa,\kappa,\omega)}, \dfrac{S(\omega,\omega,\mathscr{B}\kappa)}{1+S(\omega,\omega,\kappa)} \end{array}\right\}$$

$$= \max\{S(\omega,\omega,\kappa),0,0,S(\omega,\omega,\kappa)\}$$

$$= S(\omega,\omega,\kappa).$$

Thus Eq. (23) gives that

$$\alpha(S(\omega,\omega,\kappa)) \leq \beta(S(\omega,\omega,\kappa)),$$

which contradicts Eq.(17). Thus our assumption is wrong. Hence, $\omega = \kappa$.
This proves that $\mathscr{B}$ has a unique fixed point. □

If we take $\alpha(\lambda) = \beta(\lambda) = \lambda$ in Theorem 2.6, and in Theorem 2.7 respectively, we obtain the following results.

Corollary 2.8. *Let $(\mathbb{X}, S)$ be a complete S-metric space and $\mathscr{B} : \mathbb{X} \to \mathbb{X}$ be a self mapping such that for each $\zeta, \gamma, \nu \in \mathbb{X}$,*

$$\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \leq S(\zeta,\gamma,\nu)$$

implies

$$S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu) \leq S(\zeta,\gamma,\nu).$$

Then $\mathscr{B}$ has a unique fixed point.

Corollary 2.9. *Let $(\mathbb{X}, S)$ be a complete S-metric space and $\mathscr{B} : \mathbb{X} \to \mathbb{X}$ be a self mapping such that for each $\zeta, \gamma, \nu \in \mathbb{X}$,*

$$\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \leq S(\zeta,\gamma,\nu)$$

implies

$$S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu) \leq Q(\zeta,\gamma,\nu),$$

where

$$Q(\zeta,\gamma,\nu) = \max\left\{\begin{array}{c} S(\zeta,\gamma,\nu), S(\zeta,\zeta,\mathscr{B}\zeta), S(\nu,\nu,\mathscr{B}\nu), \\ \dfrac{S(\nu,\nu,\mathscr{B}\zeta)S(\zeta,\zeta,\mathscr{B}\nu)}{1+S(\zeta,\gamma,\nu)}, \dfrac{S(\nu,\nu,\mathscr{B}\nu)}{1+S(\zeta,\gamma,\nu)} \end{array}\right\}.$$

Then $\mathscr{B}$ has a unique fixed point.

Remark 2.10. *i) Corollary 2.8 is the extension of Suzuki result from complete metric space to complete S-metric space.*

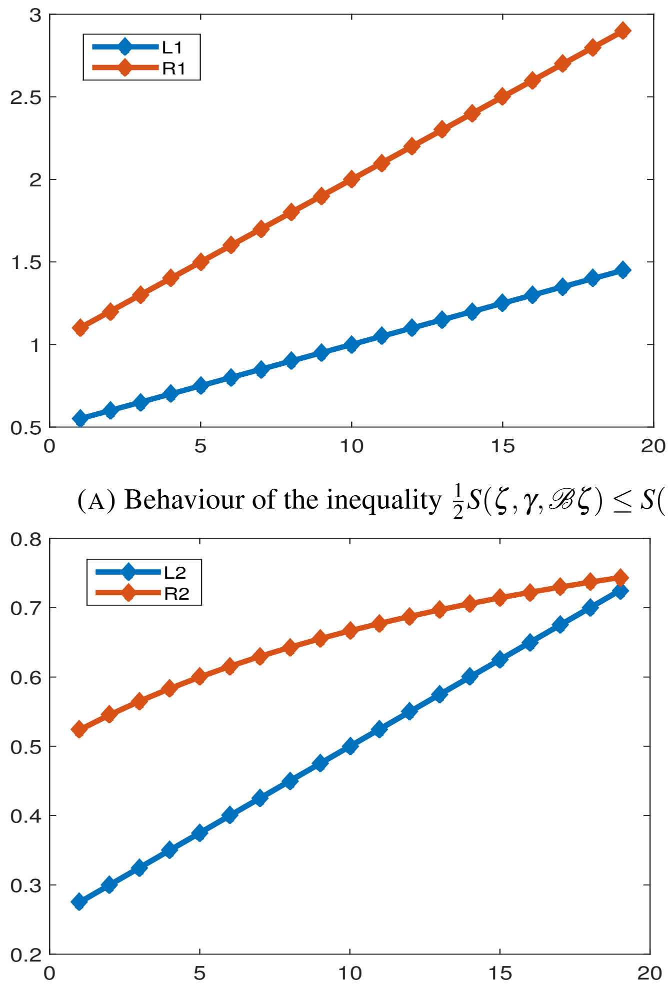
ii) Corollary 2.9 is the more general form of Suzuki result in complete S-metric space involving rational expressions.

3. Numerical Illustrations

Example 3.1. *Let $\mathbb{X} = [0,3]$. Define the metric $S : \mathbb{X}^3 \to [0,\infty)$ by*

$$S(\zeta,\gamma,\nu)=\max\{\zeta,\gamma,\nu\},$$

for all $\zeta,\gamma,\nu\in\mathbb{X}$ *such that the pair* $(\mathbb{X},S)$ *is a complete S-metric space.*



(A) Behaviour of the inequality $\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta)\leq S(\zeta,\gamma,\nu)$

(B) Behaviour of the inequality $\alpha(S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu))\leq\beta(S(\zeta,\gamma,\nu))$

FIGURE 2. Graphical Behaviour of the inequality (9) in context of Example 3.1

Define a map $\mathscr{B}:\mathbb{X}\to\mathbb{X}$ *by*

$$\mathscr{B}(\zeta)=\frac{\zeta}{4},$$

for all $\zeta\in\mathbb{X}$.

Further, define two auxiliary functions $\alpha,\beta:[0,\infty)\to[0,\infty)$ *by*

$$\alpha(\lambda)=\lambda \text{ and } \beta(\lambda)=\frac{\lambda}{\lambda+1},$$

for all $\lambda > 0$. Clearly, $\alpha \in \Phi$ and β is a continuous function such that it satisfies Eq.(10). Then from Eq. (9), we have

$$\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \leq S(\zeta,\gamma,\nu)$$
$$\frac{1}{2}\max\left\{\zeta,\gamma,\frac{\zeta}{4}\right\} \leq \max\{\zeta,\gamma,\nu\},$$

which satisfies and implies

$$\alpha(S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu)) \leq \beta(S(\zeta,\gamma,\nu))$$
$$\max\left\{\frac{\zeta}{4},\frac{\gamma}{4},\frac{\nu}{4}\right\} \leq \frac{\max\{\zeta,\nu,\gamma\}}{\max\{\zeta,\nu,\gamma\}+1}.$$

All the conditions of Th.2.6 are satisfied so 0 is the unique fixed point for $\mathscr{B}$.

Example 3.2. *Let $\mathbb{X} = [0,2]$. Define the metric $S : \mathbb{X}^3 \to [0,\infty)$ by*

$$S(\zeta,\gamma,\nu) = \max\{\zeta,\gamma,\nu\},$$

for all $\zeta,\gamma,\nu \in \mathbb{X}$ such that the pair $(\mathbb{X},S)$ is a complete S-metric space.

Define a map $\mathscr{B} : \mathbb{X} \to \mathbb{X}$ by

$$\mathscr{B}(\zeta) = \ln\left\{1+\frac{\zeta}{3}\right\},$$

for all $\zeta \in \mathbb{X}$.

Further, define two auxiliary functions $\alpha,\beta : [0,\infty) \to [0,\infty)$ by

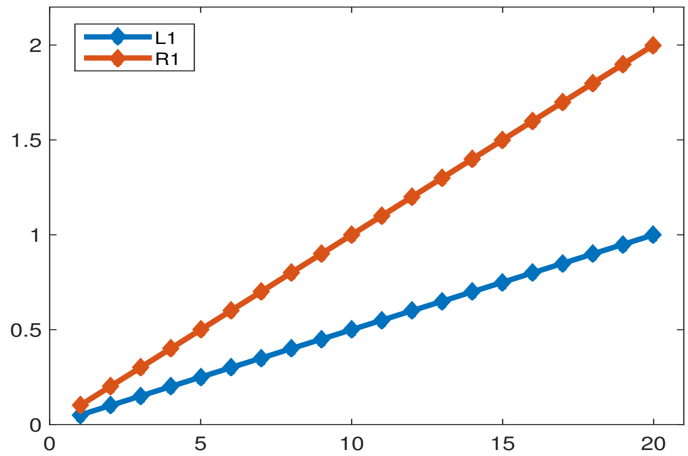
$$\alpha(\lambda) = \lambda \text{ and } \beta(\lambda) = \frac{\lambda}{\lambda+1},$$

for all $\lambda > 0$.Clearly, $\alpha \in \Phi$ and β is a continuous function such that it satisfies Eq.(10). Then from Eq. (9), we have

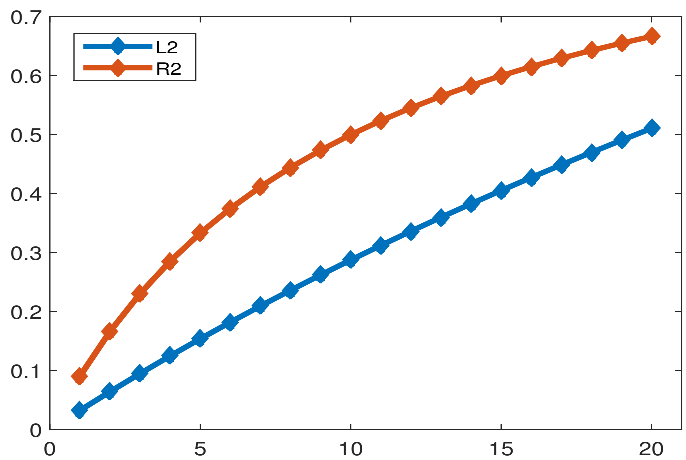
$$\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \leq S(\zeta,\gamma,\nu)$$
$$\frac{1}{2}\max\left\{\zeta,\gamma,\ln\left\{1+\frac{\zeta}{3}\right\}\right\} \leq \max\{\zeta,\gamma,\nu\},$$

which satisfies and implies

$$\alpha(S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu)) \leq \beta(S(\zeta,\gamma,\nu))$$
$$\max\left\{\ln\left\{1+\frac{\zeta}{3}\right\},\ln\left\{1+\frac{\gamma}{3}\right\},\ln\left\{1+\frac{\nu}{3}\right\}\right\} \leq \frac{\max\{\zeta,\nu,\gamma\}}{\max\{\zeta,\nu,\gamma\}+1}.$$



(A) Behaviour of the inequality $\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \leq S(\zeta,\gamma,\nu)$



(B) Behaviour of the inequality $\alpha(S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu)) \leq \beta(S(\zeta,\gamma,\nu))$

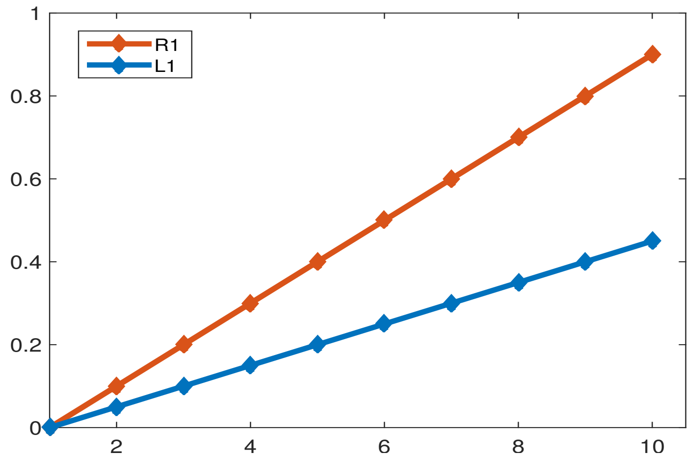
FIGURE 3. Graphical Behaviour of the inequality (9) in context of Example 3.2

All the conditions of Th.2.6 are satisfied so 0 *is the unique fixed point for* $\mathscr{B}$.

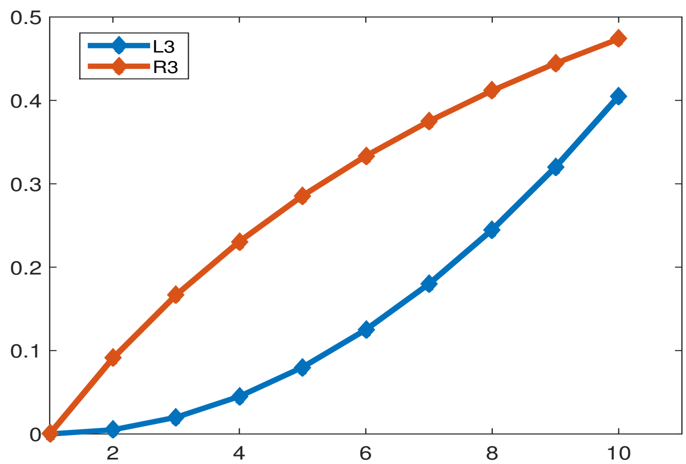
Example 3.3. *Let* $\mathbb{X} = [0,1)$. *Define the metric* $S : \mathbb{X}^3 \to [0,\infty)$ *by*

$$S(\zeta,\gamma,\nu) = \max\{\zeta,\gamma,\nu\},$$

for all $\zeta,\gamma,\nu \in \mathbb{X}$ *such that the pair* $(\mathbb{X},S)$ *is a complete S-metric space.*



(A) Behaviour of the inequality $\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \leq S(\zeta,\gamma,\nu)$



(B) Behaviour of the inequality $\alpha(S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu)) \leq \beta(Q(\zeta,\gamma,\nu))$

FIGURE 4. Graphical Behaviour of the inequality (16) in context of Example 3.3

Define a map $\mathscr{B}:\mathbb{X}\to\mathbb{X}$ *by*

$$\mathscr{B}(\zeta)=\frac{\zeta^2}{2},$$

for all $\zeta\in\mathbb{X}$.

Further, define two auxiliary functions $\alpha,\beta:[0,\infty)\to[0,\infty)$ *by*

$$\alpha(\lambda)=\lambda \text{ and } \beta(\lambda)=\frac{\lambda}{\lambda+1},$$

for all $\lambda > 0$.Clearly, $\alpha \in \Phi$ and β is a continuous function such that it satisfies Eq.(17).

From Eq. (16)

$$\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \leq S(\zeta,\gamma,\nu)$$
$$\frac{1}{2}\max\left\{\zeta,\gamma,\frac{\zeta^2}{2}\right\} \leq \max\{\zeta,\gamma,\nu\},$$

which satisfies and implies

$$\begin{aligned}\alpha(S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu)) &\leq \beta(Q(\zeta,\gamma,\nu))\\ &\leq \beta\left(\max\left\{\begin{array}{c} S(\zeta,\gamma,\nu), S(\zeta,\zeta,\mathscr{B}\zeta), S(\nu,\nu,\mathscr{B}\nu),\\ \frac{S(\nu,\nu,\mathscr{B}\zeta)S(\zeta,\zeta,\mathscr{B}\nu)}{1+S(\zeta,\gamma,\nu)}, \frac{S(\nu,\nu,\mathscr{B}\nu)}{1+S(\zeta,\gamma,\nu)}\end{array}\right\}\right)\\ &\leq \beta(S(\zeta,\gamma,\nu))\\ \max\left\{\frac{\zeta^2}{2},\frac{\gamma^2}{2},\frac{\nu^2}{2}\right\} &\leq \frac{\max\{\zeta,\nu,\gamma\}}{\max\{\zeta,\nu,\gamma\}+1}.\end{aligned}$$

Hence, all the conditions of Th 2.7 are satisfied so 0 is the unique fixed point for $\mathscr{B}$.

Example 3.4. *Let $\mathbb{X} = [0,\infty]$. Define the metric $S : \mathbb{X}^3 \to [0,\infty)$ by*

$$S(\zeta,\gamma,\nu) = \max\{\zeta,\gamma,\nu\},$$

for all $\zeta,\gamma,\nu \in \mathbb{X}$ such that the pair $(\mathbb{X},S)$ is a complete S-metric space.

Define a map $\mathscr{B} : \mathbb{X} \to \mathbb{X}$ by

$$\mathscr{B}(\zeta) = \frac{1}{2}\sin\zeta,$$

for all $\zeta \in \mathbb{X}$.

Further, define two auxiliary functions $\alpha,\beta : [0,\infty) \to [0,\infty)$ by

$$\alpha(\lambda) = \lambda \text{ and } \beta(\lambda) = \frac{\lambda}{\lambda+1},$$

for all $\lambda > 0$.Clearly, $\alpha \in \Phi$ and β is a continuous function such that it satisfies Eq.(17).

From Eq. (16) we have

$$\frac{1}{2}S(\zeta,\gamma,\mathscr{B}\zeta) \leq S(\zeta,\gamma,\nu)$$
$$\frac{1}{2}\max\left\{\zeta,\gamma,\frac{1}{2}\sin\zeta\right\} \leq \max\{\zeta,\gamma,\nu\},$$

which satisfies and implies

$$\begin{aligned}\alpha(S(\mathscr{B}\zeta,\mathscr{B}\gamma,\mathscr{B}\nu)) &\leq \beta(Q(\zeta,\gamma,\nu)) \\ &\leq \beta\left(\max\left\{\begin{array}{c} S(\zeta,\gamma,\nu), S(\zeta,\zeta,\mathscr{B}\zeta), S(\nu,\nu,\mathscr{B}\nu), \\ \frac{S(\nu,\nu,\mathscr{B}\zeta)S(\zeta,\zeta,\mathscr{B}\nu)}{1+S(\zeta,\gamma,\nu)}, \frac{S(\nu,\nu,\mathscr{B}\nu)}{1+S(\zeta,\gamma,\nu)} \end{array}\right\}\right) \\ &\leq \beta(S(\zeta,\gamma,\nu))\end{aligned}$$

$$\max\left\{\frac{1}{2}\sin\zeta, \frac{1}{2}\sin\gamma, \frac{1}{2}\sin\nu\right\} \leq \frac{\max\{\zeta,\nu,\gamma\}}{\max\{\zeta,\nu,\gamma\}+1}.$$

Thus all the conditions of Th 2.7 are satisfied so 0 *is the unique fixed point for* $\mathscr{B}$.

4. Conclusion

In this study, we initially define and introduce the concept of the Suzuki $C(\alpha,\beta)$ condition within the framework of S-metric spaces. A supporting lemma (Lemma 2.1), which is instrumental in establishing the Cauchy behavior of sequences, is presented. Furthermore, two main theorems (Theorem 2.6 and Theorem 2.7) are proved to ensure the existence and uniqueness of fixed points under the proposed condition. To substantiate the theoretical results, several illustrative examples, including graphical representations, are provided. The findings of this work not only extend but also generalize various existing results in the current literature.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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