Available online at http://scik.org
Adv. Fixed Point Theory, 2025, 15:47
https://doi.org/10.28919/afpt/9320
ISSN: 1927-6303

NONLINEAR CONTRACTION IN A (θ,ρ)-METRIC SPACE

KARIM CHAIRA[1], MUSTAPHA KABIL[1], SAMIH LAZAIZ[2], YASSINE NEBRASS[1,*]

[1]Laboratory of Mathematics, Computer Science and Applications (LMCSA), Hassan II University of Casablanca, Morocco

[2]ENSAM Casablanca,Hassan II University of Casablanca, Morocco

Abstract. Motivated by the concepts of extended b-metric and suprametric space, we define the concepts of extended θ-metric and extended (θ,ρ)-metric spaces. We also show some fixed point theorems for self-mappings defined on such spaces. Our results extend the results in [1] and [2]. Further some examples are given.

Keywords: extended θ-metric space; extended (θ,ρ)-space; fixed point theorem; Banach contraction.

2020 AMS Subject Classification: 54A05, 47H10, 47H09.

1. Introduction and Preliminaries

The generalisation of metric spaces to more abstract spaces comes from the relaxation of the triangular inequality to a distance functions such as partial metric space and b-metric spaces. The partial metric space was defined by Matthews in [3]. Since then, further research has been carried out to explore the usefulness of this distance function, see for example [4, 5, 6, 7].

The b-metric space was introduced by Czerwik [8, 9]. Furthermore, a fixed point theorem for this space was also established. Because of the importance of this distance function, it has been

*Corresponding author
E-mail address: nebyassine123@gmail.com
Received April 28, 2025

spread in different ways and has been the subject of a number of enhancements and adaptations for extensive applications, such as [10, 11, 12, 13].

The concept of rectangular distance was first introduced by Branciari [14]. Subsequently, several fixed point results have been established in spaces equipped with a rectangular distance function, as illustrated in [15, 16, 17, 18].

Several new distance functions have been developed by combining, relaxing, or extending certain axioms of existing distance functions. In addition, numerous fixed-point results have been obtained in sets endowed with these generalized distances [19, 20, 21, 22, 23].

Recently, a new weaken triangular inequalities has appeared, named by suprametric, and the extended b-metric. These were introduced by the authors in [1] and [24].

In this paper, we generalize these two concepts by introducing the extended θ-metric and extended (θ,ρ)-metric spaces. Finally, we establish some fixed point theorems for mappings satisfying nonlinear contractions, moreover we extend theorem 3.2 in [2] to the extended θ-metric.

Definition 1.1. Let X be a non empty set and $\theta : X\times X\to[1,\infty)$. A function d_θ is called an extended b-metric ([1]) if for all $x,y,z\in X$ it satisfies:

(d_1) $d_\theta(x,y)=0$ if and only if $x=y$,

(d_2) $d_\theta(x,y)=d_\theta(y,x)$

(d_3) $d_\theta(x,y)\leq\theta(x,y)(d_\theta(x,z)+d_\theta(z,y))$

The pair (X,d_θ) is called an extended b-metric space.

The suprametric space introduced in [24] as follows.

Definition 1.2. Let X be a non empty set and $\theta : X\times X\to[1,\infty)$, $\rho\in\mathbb{R}^+$. A function d_θ is called a suprametric if for all $x,y,z\in X$ it satisfies:

(d_1) $d_\theta(x,y)=0$ if and only if $x=y$

(d_2) $d_\theta(x,y)=d_\theta(y,x)$

(d_3) $d_\theta(x,y)\leq d_\theta(x,z)+d_\theta(z,y)+\rho d_\theta(x,z)d_\theta(z,y)$

The pair (X,d_θ) is called a suprametric space.

Let us recall a example of such spaces see e.g [2, 1]:

Example 1.3. Let $\gamma > 0$, and $d : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ a function given by:

$$d(x,y) = \gamma(e^{|x-y|}-1) \quad for \quad all \quad x,y\in\mathbb{R}$$

Then (X,d) is an suprametric space with $\rho = \frac{1}{\gamma}$.

Example 1.4. Let $X = C([a,b],\mathbb{R})$, and $d(x,y) = \sup_{t\in[a,b]} |x(t)-y(t)|^2$
Then (X,d) is an extended b-metric space with $\theta(x,y) = |x(t)|+|y(t)|+2$, where $\theta : X\times X \to [1,\infty)$.

Denote by $\mathbb{M}$ the set of matkowski functions [25], $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy:

- ψ is increasing on $\mathbb{R}^+$
- $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$.

Denote by $\mathbb{M}_b$ $(b \geq 1)$ the set of functions of $\mathbb{M}$ that satisfy:

$$\limsup_{n\to\infty} \frac{\psi^{n+1}(t)}{\psi^n(t)} < \frac{1}{b} \quad , \quad for \quad all \quad t > 0$$

Recall next the result of Czerwik [8] and the result in [2]:

Theorem 1.5. *Let (X,d_θ) be a complete b-metric space such that d_θ is a continuous functional, and $T : X \to X$ be a mapping, Assume there exists $\psi\in\mathbb{M}$ such that:*

$$d_\theta(Tx,Ty) \leq \psi(d_\theta(x,y)) \qquad for all \quad x,y\in X \tag{1}$$

Then, T has precisely one fixed point ξ. Moreover for each $y\in X$, $T^n y \to \xi$.

Theorem 1.6. *Let (X,d_θ) be a complete b-suprametric space such that d_θ is a continuous functional, and $T : X \to X$ be a mapping, Assume there exists $\psi\in\mathbb{M}_b$ such that:*

$$d_\theta(Tx,Ty) \leq \psi(d_\theta(x,y)) \qquad for all \quad x,y\in X \tag{2}$$

Then, T has precisely one fixed point ξ. Moreover for each $y\in X$, $T^n y \to \xi$.

Now we introduce the concept of extended θ-metric space.

Definition 1.7. Let X be a non empty set and $\theta : X \times X \to [1,\infty)$, $\rho \in \mathbb{R}^+$. A function d_θ is called a extended θ-metric if for all $x,y,z \in X$ it satisfies:

(d_1) $d_\theta(x,y) = 0$ if and only if $x = y$,

(d_2) $d_\theta(x,y) = d_\theta(y,x)$

(d_3) $d_\theta(x,y) \leq \theta(x,y)(d_\theta(x,z) + d_\theta(z,y)) + \rho d_\theta(x,z) d_\theta(z,y)$

The pair (X, d_θ) is called a extended θ-metric space.

Example 1.8. Let $X = [0,\infty[$, define $d : X \times X \to [0,\infty)$ by:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2}(x+y+1)^2 & \text{if } x \neq y, \end{cases}$$

and $\theta : X \times X \to [1,\infty)$ by $\theta(x,y) = \frac{x+y+1}{x+y}$, $\rho = 4$. Then (X,d,ρ) is an extend θ-metric space.

Proof. Clearly, conditions (d_1) and (d_2) holds.

Let $x,\, y and\, z \in X$,

Case 1: $x \neq y$, $y \neq z$ and $x \neq z$ .For $x \neq y$ we have:

$$\begin{aligned} d(x,y) = \frac{1}{2}(x+y+1)^2 &= \frac{1}{2}\left(x+\frac{1}{2}+y+\frac{1}{2}\right)^2 \\ &\leq \frac{1}{2}\left(x+\frac{1}{2}+z+\frac{1}{2}+y+\frac{1}{2}+z+\frac{1}{2}\right)^2 \\ &\leq \frac{1}{2}(x+z+1)^2 + \frac{1}{2}(z+y+1)^2 + (x+z+1)(z+y+1) \\ &\leq d(x,z) + d(z,y) + 4\left(\frac{1}{2}(z+y+1)^2\right)\left(\frac{1}{2}(z+x+1)^2\right) \\ &\leq d(x,z) + d(z,y) + 4d(x,z)d(z,y) \\ &\leq \frac{x+y+1}{x+y}(d(x,z)+d(z,y)) + 4d(x,z)d(z,y) \end{aligned}$$

Case 2: $x \neq y$, $x = z$, the inequality is verified because $\theta(x,y) \geq 1$.

Case 3: $x = y$ the inequality is verified because $\theta(x,y) \geq 1$.

Therefore

$$d(x,y) \leq \theta(x,y)(d(x,z)+d(z,y)) + \rho d(x,z)d(z,y),\ for all\, x,\, y and\, z \in X.$$

Hence (X,d,ρ) is an extended θ-metric space. but not extended b-metric,

for

$$x=4\ ,y=5\ and\ z=1$$

we have

$$d(x,y)=\frac{100}{2},\ d(x,z)=\frac{36}{2}\ ,d(y,z)=\frac{49}{2}\quad and\quad \theta(x,y)=\frac{10}{9}\ ,$$

so:

$$d(x,y)=\frac{100}{2}>\theta(x,y)(d(x,z)+d(y,z))=\frac{425}{9}$$

□

Example 1.9. Let $X=\mathbb{R}$, define $d:X\times X\to[0,\infty)$ by: $d(x,y)=\gamma(e^{|x-y|}-1)$ where $0<\gamma<\frac{\sqrt{e}-1}{2}$ and $\theta:X\times X\to[1,\infty)$ by $\theta(x,y)=\gamma(x^2+y^2)+1$, $\rho=\frac{1}{\gamma}$.
Then (X,d,ρ) is an extend θ- metric space.

Proof. Clearly, conditions (d_1) and (d_2) holds .

Let $x,\ y\,and\,z\in X$,

We have:

$$\begin{aligned}
d(x,y)&=\gamma(e^{|x-y|}-1)=\gamma(e^{|x-z+z-y|}-1)\\
&\leq\gamma(e^{|x-z|+|z-y|}-1)=\gamma(e^{|x-z|}-1)+\gamma(e^{|z-y|}-1)+\gamma(e^{|x-z|}-1)(e^{|z-y|}-1)\\
&\leq(\gamma(x^2+y^2)+1)\Big(\gamma(e^{|x-z|}-1)+\gamma(e^{|z-y|}-1)\Big)+\frac{1}{\gamma}\Big(\gamma(e^{|x-z|}-1)\gamma(e^{|z-y|}-1)\Big)
\end{aligned}$$

Therefore

$$d(x,y)\leq\theta(x,y)(d(x,z)+d(z,y))+\rho d(x,z)d(z,y)\ ,\ for\ all\ x,\ y\ and\ ,z\in X\ .$$

Hence (X,d,ρ) is an extended θ-metric space. □

Remark 1.10. *(X,d,ρ) in example 1.9 is not an extended b-metric.*

Proof. For

$$x=1,\quad y=0\quad and\quad z=\frac{1}{2}$$

We have:

$$d(x,y)=\gamma(e^{1}-1)\ ,\quad d(x,z)=\gamma(e^{\frac{1}{2}}-1)\ ,\quad d(y,z)=\gamma(e^{\frac{1}{2}}-1)\quad and\quad \theta(x,y)=\gamma+1$$

so:

$$d(x,y) = \gamma(e^1 - 1) > \theta(x,y)(d(x,z) + d(y,z)) = (\gamma+1)\gamma(2e^{\frac{1}{2}} - 2)$$

wich equivalent to:

$$(e^1 - 1) > (\gamma+1)(2e^{\frac{1}{2}} - 2)$$

$$e - 2(\gamma+1)e^{\frac{1}{2}} + 2\gamma + 1 > 0$$

The discriminant of this inequality: $\Delta = 4(\gamma+1)^2 - 4(2\gamma+1) = 4\gamma^2 > 0$. Then the polynomial $x^2 - 2(\gamma+1)x + 2\gamma + 1$ have two roote 1 and $2\gamma+1$, however $\gamma < \frac{\sqrt{e}-1}{2}$ therfore $2\gamma+1 < \sqrt{e}$ thus $e - 2(\gamma+1)e^{\frac{1}{2}} + 2\gamma + 1 > 0$ is verified, the proof is completed.

□

The concepts of convergence, Cauchy sequence and completeness can easily be extended to the case of an extended θ-metric space [26].

Definition 1.11. Let (X, d_θ) be a extended θ-metric space.

- A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$, for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.
- A sequence $\{x_n\}$ in X is said to be cauchy, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

Definition 1.12. An extended θ-metric space (X, d_θ) is complete if every Cauchy sequence in X is convergent.

Remark 1.13. *Assume that (X, d_θ) is an extended θ-metric space. If d_θ is continuous, then every convergent sequence has a unique limit.*

2. Main Results

The first main result is the following theorem.

Theorem 2.1. *Let (X, d_θ) be a complete extended θ-metric space such that d_θ is a continuous functional, and $T : X \to X$ be a mapping, Assume there exists $K \in [0,1)$ such that:*

$$d_\theta(Tx, Ty) \leq K d_\theta(x,y) \qquad for all \quad x, y \in X \tag{3}$$

where $K \in [0,1)$ such that for each $x_0 \in X$,

$\lim_{n,m\to\infty} K\theta(x_n,x_m)(\theta(x_{n+1},x_m)+\rho K^n d(x_0,x_1)) < 1$, *where $x_n = T^n x_0$, $n = 1,2,3,...,$,*

Then, T has precisely one fixed point ξ. Moreover for each $y \in X$, $T^n y \to \xi$.

Proof. We choose any $x_0 \in X$ be arbitrary, define the iterative sequence $\{x_n\}$ by:

$$x_0\,, \quad x_1 = Tx_0\,, \quad x_2 = Tx_1 \quad ,..., \quad x_n = T^n x_0$$

By triangular inequality and, for $m > n$ we have:

$$\begin{aligned} d_\theta(x_n,x_m) &\leq \theta(x_n,x_m)d_\theta(x_n,x_{n+1}) + \theta(x_n,x_m)d(x_{n+1},x_m) + \rho d_\theta(x_n,x_{n+1})d(x_{n+1},x_m) \\ &\leq \theta(x_n,x_m)d_\theta(x_n,x_{n+1}) + (\theta(x_n,x_m)+\rho d_\theta(x_n,x_{n+1}))d(x_{n+1},x_m) \end{aligned}$$

Similarly,

$$\begin{aligned} d_\theta(x_{n+1},x_m) &\leq \theta(x_{n+1},x_m)d_\theta(x_{n+1},x_{n+2}) + \theta(x_n,x_m)d(x_{n+2},x_m) + \rho d_\theta(x_{n+1},x_{n+2})d(x_{n+2},x_m) \\ &\leq \theta(x_{n+1},x_m)d_\theta(x_{n+1},x_{n+2}) + (\theta(x_{n+1},x_m)+\rho d_\theta(x_{n+1},x_{n+2}))d(x_{n+2},x_m) \end{aligned}$$

then:

$$\begin{aligned} d_\theta(x_n,x_m) &\leq \theta(x_n,x_m)K^n d_\theta(x_0,x_1) + (\theta(x_n,x_m)+\rho K^n d_\theta(x_0,x_1))\theta(x_{n+1},x_m)K^{n+1}d(x_0,x_1) \\ &\quad + (\theta(x_n,x_m)+\rho K^n d_\theta(x_0,x_1))(\theta(x_{n+1},x_m)+\rho K^{n+1}d_\theta(x_0,x_1))d_\theta(x_{n+2},x_m) \end{aligned}$$

And by induction we obtain:

$$\begin{aligned} d_\theta(x_n,x_m) &\leq \theta(x_n,x_m)K^n d(x_0,x_1) \\ &\quad + K^n d_\theta(x_0,x_1)\sum_{i=1}^{m-n-1}\theta(x_{n+i},x_m)K^i\prod_{\eta=0}^{i-1}(\theta(x_{n+\eta},x_m)+\rho K^{n+\eta}d_\theta(x_0,x_1)) \end{aligned}$$

since $K \in [0,1)$, it follows that:

$$\begin{aligned} d_\theta(x_n,x_m) &\leq \theta(x_n,x_m)K^n d(x_0,x_1) \\ &\quad + K^n d_\theta(x_0,x_1)\sum_{i=1}^{m-n-1}\theta(x_{n+i},x_m)K^i\prod_{\eta=0}^{i-1}(\theta(x_{n+\eta},x_m)+\rho K^{\eta}d_\theta(x_0,x_1)) \end{aligned}$$

Let:

$$U_i = \theta(x_{n+i},x_m)K^i\prod_{\eta=0}^{i-1}(\theta(x_{n+\eta},x_m)+\rho K^{\eta}d_\theta(x_0,x_1)) \quad \textit{and} \quad U_0 = \theta(x_n,x_m)K^n d(x_0,x_1)$$

By Ratio test $\sum_{i=0}^{\infty} U_i$ is converges, since

$$\lim_{i\to\infty}\left|\frac{U_{i+1}}{U_i}\right| \leq \lim_{i\to\infty} K\theta(x_{n+i+1},x_m)(\theta(x_{n+i},x_m)+\rho K^i d(x_0,x_1)) < 1$$

we can deduce that, $d_\theta(x_n,x_m)$ tends to zero as n,m tend to infinity, that suggests the sequence $\{x_n\}$ is Cauchy. Therefore by completeness of X, as a result of this $\{x_n\}$ converges to some $x \in X$.

We have:

$$d(Tx_n,Tx) \leq Kd(x_n,x)$$

Therefore as $n \to \infty$, thus , we conclude $x = Tx$.

To prove uniqueness, let us assume x_a and x_b are two fixed points of T,

$$\begin{aligned} d(x_a,x_b) &= d(Tx_a,Tx_b) \\ &\leq Kd(x_a,x_b) \\ &< d(x_a,x_b)\,,\ a\ contadiction \end{aligned}$$

Hence $x_a = x_b$, This completes the proof of the theorem.

□

Lemma 2.2. *For all $x \in [0,\infty)$, we have :* $(e^{|\frac{x}{3}-\frac{y}{3}|}-1) \leq \frac{1}{3}(e^{|x-y|}-1)$

Proof. Consider the function: $f : [0,\infty) \to \mathbb{R}$ defined by:

$$f(x) = \frac{1}{3}e^x - \frac{1}{3} - e^{\frac{x}{3}} + 1$$

We have:

$$\begin{aligned} f^{'}(x) &= \frac{1}{3}e^x - \frac{1}{3}e^{\frac{x}{3}} \\ &= \frac{1}{3}(e^x - e^{\frac{x}{3}}) \end{aligned}$$

since $e^x - e^{\frac{x}{3}} \geq 0 \quad for all\ x \geq 0$ then f is a increasing function. Therefore $f(x) \geq f(0)$ for all $x \geq 0$, but $f(0) = 0$. Hence $f(x) \geq f(0)$ for all $x \geq 0$

(i.e) $(e^{\frac{x}{3}}-1) \leq \frac{1}{3}(e^x - 1)$ for all $x \geq 0$ □

Example 2.3. Let $X=[0,1]$, define $d: X\times X\to[0,\infty)$ by: $d(x,y)=\frac{1}{4}(e^{|x-y|}-1)$ and $\theta: X\times X\to[1,\infty)$ by $\theta(x,y)=\frac{1}{4}(x^2+y^2)+1$, and $\rho=4$. Then (X,d) be a complete extended θ-metric space.

Define $T: X\to X$ by $Tx=\frac{x}{3}$. We have:

$$
\begin{aligned}
d(Tx,Ty) &= \frac{1}{4}(e^{|\frac{x}{3}-\frac{y}{3}|}-1)\\
&\le \frac{1}{12}(e^{|x-y|}-1)\\
&\le \frac{1}{3}.\frac{1}{4}(e^{|x-y|}-1)\\
&\le \frac{1}{3}d(x,y)
\end{aligned}
$$

Note that for each $x\in X$, $T^n x=\frac{x}{3^n}$. Then we obtain:

$$\lim_{n,m\to\infty}\theta(x_n,x_m)=\frac{1}{4}(\frac{x^2}{3^{2m}}+\frac{x^2}{3^{2n}})+1=1$$

therefore:

$$\lim_{n,m\to\infty}K\theta(x_n,x_m)[\theta(x_{n+1},x_m)+\rho K^n d(x_0,(x_1)]=\frac{1}{3}.1.[1+0]=\frac{1}{3}<1$$

Therefore, all conditions of Theorem 2.1 are satisfied hence T has a unique fixed point .

We introduce a new type of generalized metric space, which we call as an extended (θ,ρ)-metric space.

Definition 2.4. Let X be a non empty set and $\theta: X\times X\to[1,\infty)$, $\rho: X\times X\to[1,\infty)$. A function d_θ is called an extended (θ,ρ)-metric if for all $x,y,z\in X$ it satisfies:

(d_1) $d_\theta(x,y)=0$ if and only if $x=y$,

(d_2) $d_\theta(x,y)=d_\theta(y,x)$

(d_3) $d_\theta(x,y)\le\theta(x,y)(d_\theta(x,z)+d_\theta(z,y))+\rho(x,y)d_\theta(x,z)d_\theta(z,y)$

The pair (X,d_θ) is called an extended (θ,ρ)-metric.

Remark 2.5. *if* $\theta(x,y)=1$, *and* $\rho(x,y)=s$ *for some* $s\ge 1$ *then we obtain the definition of a suprametric metric space.*

Remark 2.6. *if* $\rho(x,y)=0$, *then we obtain the definition of a extended b-metric space.*

Remark 2.7. *if $\rho(x,y) = s$ $for some$ $s \geq 1$, then we obtain the definition of a extended θ-metric space.*

Example 2.8. Let $X = [1,\infty[$, define $d : X \times X \to [0,\infty)$ by:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2}(x+y)^2 & \text{if } x \neq y, \end{cases}$$

, $\theta : X \times X \to [1,\infty)$ by $\theta(x,y) = \frac{x+y+1}{x+y}$ and $\rho : X \times X \to [1,\infty)$ by $\rho(x,y) = 4xy+1$ .Then (X,d) is an extended (θ,ρ)-metric space.

Lemma 2.9. *Let (X,d_θ) be an extended (θ,ρ)-metric space, $T : X \to X$ be a mapping , Assume there exists $K \in [0,1)$ such that:*

$$d_\theta(Tx,Ty) \leq Kd_\theta(x,y) \qquad for all \quad x,y \in X \tag{4}$$

and for each $x_0 \in X$, define the sequence x_n by $x_n = T^n x_0$, $for all$ $n \in \mathbb{N}$.
Then for all $m,n \in \mathbb{N}$,with $m \geq n$:

$$d_\theta(x_n,x_m) \leq \theta(x_n,x_m)K^n d(x_0,x_1) + K^n d_\theta(x_0,x_1) \sum_{i=1}^{m-n-1} \theta(x_{n+i},x_m)K^i \prod_{\eta=0}^{i-1} (\theta(x_{n+\eta},x_m) + \rho K^{n+\eta} d_\theta(x_0,x_1))$$

Proof. By induction:

If:

$$d_n^m = d(x_n,x_m) \quad , \quad \theta_n^m = \theta(x_n,x_m) \quad , \quad \rho_n^m = \rho(x_n,x_m)$$

then it is natural to put

$$B_n^m = \theta_n^m K^n d_0 \quad and \quad A_n^m = \theta_n^m + \rho_n^m K^n d_0 \quad for all\, n < m$$

Then let's show by induction on $k = m-n+1$ that:

$$d_n^m \leq B_n^m + A_n^m B_{n+1}^m + A_n^m A_{n+1}^m B_{n+2}^m + ... + A_n^m A_{n+1}^m ... A_{m-2}^m B_{m-1}^m$$

For $k = 0 \quad (i.e) \quad j = m-1$ we have:

$$d_\theta(x_{m-1},x_m) = d_{m-1}^m \leq K^{m-1} d_0$$

$$\leq \theta(x_{m-1},x_m)K^{m-1}d_0 = B^m_{m-1}$$

then it is clear that the inequality is verified for $k = 0$.
Assume yhat this inequality holds for any $1 \leq k \leq m-n$ and and show that the inequality is verified for $k = m-n+1$
By hypothesis,

$$d^m_{m-k+1} \leq B^m_{m-k+1} + A^m_{m-k+1}B^m_{m-k+2} + A^m_{m-k+1}A^m_{m-k+2}B^m_{m-k+3} + ... + A^m_{m-k+1}A^m_{m-k+2}...A^m_{m-2}B^m_{m-1}$$

for any $1 \leq k \leq m-n$

then:

$$\begin{aligned} d^m_n &\leq B^m_n + A^m_n d^m_{n+1} \\ &\leq B^m_n + A^m_n(B^m_{n+1} + A^m_{n+1}B^m_{n+2} + A^m_{n+1}A^m_{n+2}B^m_{n+3} + ... + A^m_{n+1}A^m_{n+2}...A^m_{m-2}B^m_{m-1}) \\ &\leq B^m_n + A^m_n B^m_{n+1} + A^m_n A^m_{n+1}B^m_{n+2} + A^m_n A^m_{n+1}A^m_{n+2}B^m_{n+3} + A^m_n A^m_{n+1}A^m_{n+2}...A^m_{m-2}B^m_{m-1} \end{aligned}$$

Finally the inequality is verified for $k = m-n+1$.

□

Our second theorem is an analogue of Banach contraction principle in the setting of extended (θ,ρ)-metric space.

Theorem 2.10. *Let (X,d_θ) be a complete extended (θ,ρ)-metric space such that d_θ is a continuous functional, and $T : X \to X$ be a mapping. Assume there exists $K \in [0,1)$ such that:*

$$d_\theta(Tx,Ty) \leq Kd_\theta(x,y) \qquad for\,all \quad x,y \in X \tag{5}$$

where $K \in [0,1)$ such that for each $x_0 \in X$,

$$\lim_{n,m\to\infty} K\theta(x_n,x_m)\theta(x_{n+1},x_m)\rho(x_n,x_m) < 1,$$

where $x_n = T^n x_0$, $n = 1,2,3,....$
Then, T has precisely one fixed point ξ. Moreover for each $y \in X$, $T^n y \to \xi$.

Proof. We choose any $x_0 \in X$ be arbitrary, define the iterative sequence $\{x_n\}$ by:

$$x_0\,, \quad x_1 = Tx_0\,, \quad x_2 = Tx_1 \quad ,..., \quad x_n = T^n x_0$$

By triangular inequality and, for $m > n$ we have:

$$
\begin{aligned}
& d_\theta(x_n,x_m) \\
& \leq \theta(x_n,x_m)d_\theta(x_n,x_{n+1}) + \theta(x_n,x_m)d_\theta(x_{n+1},x_m) + \rho(x_n,x_m)d_\theta(x_n,x_{n+1})d_\theta(x_{n+1},x_m) \\
& \leq \theta(x_n,x_m)d_\theta(x_n,x_{n+1}) + (\theta(x_n,x_m) + \rho(x_n,x_m)d_\theta(x_n,x_{n+1}))d_\theta(x_{n+1},x_m)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& d_\theta(x_{n+1},x_m) \\
& \leq \theta(x_{n+1},x_m)d_\theta(x_{n+1},x_{n+2}) \\
& + \theta(x_{n+1},x_m)d(x_{n+2},x_m) + \rho(x_{n+1},x_m)d_\theta(x_{n+1},x_{n+2})d(x_{n+2},x_m) \\
& \leq \theta(x_{n+1},x_m)d_\theta(x_{n+1},x_{n+2}) + (\theta(x_{n+1},x_m) + \rho(x_{n+1},x_m)d_\theta(x_{n+1},x_{n+2}))d(x_{n+2},x_m)
\end{aligned}
$$

then:

$$
\begin{aligned}
d_\theta(x_n,x_m) \leq \theta(x_n,x_m)K^n d_\theta(x_0,x_1) & \\
& + (\theta(x_n,x_m) + \rho(x_n,x_m)K^n d_\theta(x_0,x_1))\theta(x_{n+1},x_m)K^{n+1}d(x_0,x_1) \\
& + (\theta(x_n,x_m) + \rho(x_n,x_m)K^n d_\theta(x_0,x_1))(\theta(x_{n+1},x_m) \\
& + \rho(x_{n+1},x_m)K^{n+1}d_\theta(x_0,x_1))d_\theta(x_{n+2},x_m)
\end{aligned}
$$

by lemma 2.9 we obtain:

$$
\begin{aligned}
d_\theta(x_n,x_m) \leq \theta(x_n,x_m)K^n d_\theta(x_0,x_1) & \\
& + K^n d_\theta(x_0,x_1) \sum_{i=1}^{m-n-1} \theta(x_{n+i},x_m)K^i \prod_{\eta=0}^{i-1} \big(\theta(x_{n+\eta},x_m) + \rho(x_{n+\eta},x_m)K^{n+\eta}d_\theta(x_0,x_1)\big)
\end{aligned}
$$

since $K \in [0,1)$, it follows that:

$$
\begin{aligned}
d_\theta(x_n,x_m) \leq \theta(x_n,x_m)K^n d_\theta(x_0,x_1) & \\
& + K^n d_\theta(x_0,x_1) \sum_{i=1}^{m-n-1} \theta(x_{n+i},x_m)K^i \prod_{\eta=0}^{i-1} \big(\theta(x_{n+\eta},x_m) + \rho(x_{n+\eta},x_m)K^{\eta}d_\theta(x_0,x_1)\big)
\end{aligned}
$$

Let:

$$
U_i = \theta(x_{n+i},x_m)K^i \prod_{\eta=0}^{i-1} (\theta(x_{n+\eta},x_m) + \rho(x_{n+\eta},x_m)K^{\eta}d_\theta(x_0,x_1)) \quad and \quad U_0 = \theta(x_n,x_m)K^n d(x_0,x_1)
$$

By Ratio test $\sum_{i=0}^{\infty} U_i$ is converges, since

$$\lim_{i\to\infty}\left|\frac{U_{i+1}}{U_i}\right| \leq \lim_{i\to\infty} K\theta(x_{n+i+1},x_m)(\theta(x_{n+i},x_m)+\rho(x_{n+i},x_m)K^i d(x_0,x_1)) < 1$$

in fact , if

$$\lim_{n,m\to\infty} K\theta(x_n,x_m)\theta(x_{n+1},x_m)\rho(x_n,x_m) < 1$$

then

$$\lim_{i\to\infty} K\theta(x_{n+i+1},x_m)\rho(x_{n+i},x_m)K^i d(x_0,x_1) = 0$$

it follows that:

$$\begin{aligned}
\lim_{i\to\infty}\left|\frac{U_{i+1}}{U_i}\right| &\leq \lim_{i\to\infty} K\theta(x_{n+i+1},x_m)(\theta(x_{n+i},x_m)+\rho(x_{n+i},x_m)K^i d(x_0,x_1)) \\
&= \lim_{n,m\to\infty} K\theta(x_n,x_m)\theta(x_{n+1},x_m) \\
&< \lim_{n,m\to\infty} K\theta(x_n,x_m)\theta(x_{n+1},x_m)\rho(x_n,x_m) \\
&< 1
\end{aligned}$$

We can deduce that, $d_\theta(x_n,x_m)$ tends to zero as n,m tend to infinity, that suggests the sequence $\{x_n\}$ is Cauchy. Therefore by completeness of X, as a result of this $\{x_n\}$ converges to some $x \in X$.

We have:

$$d(Tx_n,Tx) \leq Kd(x_n,x)$$

Therefore as $n \to \infty$, thus, we conclude $x = Tx$.

To prove uniqueness, let us assume x_a and x_b are two fixed points of T,

$$\begin{aligned}
d(x_a,x_b) &= d(Tx_a,Tx_b) \\
&\leq Kd(x_a,x_b) \\
&< d(x_a,x_b)\,,\ a\, contradiction
\end{aligned}$$

Hence $x_a = x_b$, This completes the proof of the theorem.

□

Example 2.11. Let $X = [0,1]$, define: $d : X \times X \to [0,\infty)$ by:
$d(x,y) = \frac{1}{4}(e^{|x-y|} - 1)$ and $\theta : X \times X \to [1,\infty)$ by $\theta(x,y) = \frac{1}{4}(x^2+y^2)+1$, and $\rho : X \times X \to [1,\infty)$ by $\rho(x,y) = 4xy+1$. Then (X,d) be a complete extended (θ,ρ)-metric space .
Define $T : X \to X$ by $Tx = \frac{x}{3}$. We have:

$$\begin{aligned} d(Tx,Ty) &= \frac{1}{4}(e^{|\frac{x}{3}-\frac{y}{3}|} - 1) \\ &\leq \frac{1}{12}(e^{|x-y|} - 1) \\ &\leq \frac{1}{3}.\frac{1}{4}(e^{|x-y|} - 1) \\ &\leq \frac{1}{3}d(x,y) \end{aligned}$$

Note that for each $x \in X$, $T^n x = \frac{x}{3^n}$. Then we obtain:

$$\lim_{n,m\to\infty} \theta(x_n,x_m) = \frac{1}{4}(\frac{x^2}{3^{2m}} + \frac{x^2}{3^{2n}}) + 1 = 1$$

$$\lim_{n,m\to\infty} \rho(x_n,x_m) = 4.\frac{x}{3^n}.\frac{x}{3^m} + 1 = 1$$

therefore:

$$\lim_{n,m\to\infty} K\theta(x_n,x_m)\theta(x_{n+1},x_m)\rho(x_n,x_m) = \frac{1}{3}.1.1.1 = \frac{1}{3} < 1$$

Therefore, all conditions of Theorem 2.10 are satisfied hence T has a unique fixed point.

We immediately derive the following corollary:

Corollary 2.12. *Let (X,d_θ) be a complete extended (θ,ρ)-metric space such that d_θ is a continuous functional, and $T : X \to X$ be a mapping, Assume there exists $\psi \in \mathbb{M}_b$ such that:*

$$d_\theta(Tx,Ty) \leq \psi(d_\theta(x,y)) \qquad for\,all \quad x,y \in X \tag{6}$$

such that for each $x_0 \in X$

$$\lim_{n,m\to\infty} \theta(x_n,x_m)\theta(x_{n+1},x_m)\rho(x_n,x_m)\psi^m(d_0) < b$$

, where $x_n = T^n x_0$, $n = 1,2,3,....$
Then, T has precisely one fixed point ξ. Moreover for each $y \in X$, $T^n y \to \xi$.

Proof. The same as for the previous theorem, modifying K by $\psi(d_0)$. □

If $\psi \in \mathbb{M}_b$ then we can drive this results as in [2]:

Theorem 2.13. *Let (X,d_θ) be a complete extended θ-metric space such that d_θ is a continuous functional, and $T : X \to X$ be a mapping, Assume there exists $\psi \in \mathbb{M}_b$ such that:*

$$d_\theta(Tx,Ty) \leq \psi(d_\theta(x,y)) \qquad for all \quad x,y \in X \tag{7}$$

such that, $\limsup_{i\to+\infty} \theta(x_{i+1},x_m)\theta(x_i,x_m) < b \quad for all\ m \in \mathbb{N}$
Then, T has precisely one fixed point ξ. Moreover for each $y \in X$, $T^n y \to \xi$.

Proof. We choose any $x_0 \in X$ be arbitrary, define the iterative sequence $\{x_n\}$ by:

$$x_0, \quad x_1 = Tx_0\,, \quad x_2 = Tx_1 \quad ,..., \quad x_n = T^n x_0$$

By triangular inequality and, for $m > n$ we have:

$$\begin{aligned}
d_\theta(x_n,x_m) &\leq \theta(x_n,x_m)d_\theta(x_n,x_{n+1}) + \theta(x_n,x_m)d(x_{n+1},x_m)\\
&\quad + \rho d_\theta(x_n,x_{n+1})d(x_{n+1},x_m)\\
&\leq \theta(x_n,x_m)d_\theta(x_n,x_{n+1}) + (\theta(x_n,x_m) + \rho d_\theta(x_n,x_{n+1}))d(x_{n+1},x_m)
\end{aligned}$$

Similarly,

$$\begin{aligned}
d_\theta(x_{n+1},x_m) &\leq \theta(x_{n+1},x_m)d_\theta(x_{n+1},x_{n+2}) + \theta(x_n,x_m)d(x_{n+2},x_m)\\
&\quad + \rho d_\theta(x_{n+1},x_{n+2})d(x_{n+2},x_m)\\
&\leq \theta(x_{n+1},x_m)d_\theta(x_{n+1},x_{n+2})\\
&\quad + (\theta(x_{n+1},x_m) + \rho d_\theta(x_{n+1},x_{n+2}))d(x_{n+2},x_m)
\end{aligned}$$

then:

$$\begin{aligned}
d_\theta(x_n,x_m) &\leq \theta(x_n,x_m)\psi^n(d_\theta(x_0,x_1))\\
&\quad + (\theta(x_n,x_m) + \rho\psi^n(d_\theta(x_0,x_1)))\theta(x_{n+1},x_m)\psi^{n+1}(d(x_0,x_1))\\
&\quad + (\theta(x_n,x_m) + \rho\psi^n(d_\theta(x_0,x_1)))(\theta(x_{n+1},x_m)\\
&\quad + \rho\psi^{n+1}(d_\theta(x_0,x_1)))d_\theta(x_{n+2},x_m)
\end{aligned}$$

Performing this process repeatedly we obtain:

$$d_\theta(x_n,x_m) \le \sum_{i=n}^{m} \theta(x_i,x_m)\psi^i(d_\theta(x_0,x_1)) \prod_{j=n}^{i-1}(\theta(x_j,x_m)+\rho\psi^j(d_\theta(x_0,x_1)))$$

Let: $U_i = \theta(x_i,x_m)\psi^i(d_\theta(x_0,x_1))\prod_{j=n}^{i-1}(\theta(x_j,x_m)+\rho\psi^j(d_\theta(x_0,x_1)))$

To this end, observe that from $\psi^{i+1}(d_0) \to 0$ as $i \to \infty$, we obtain

$$\lim_{i\to+\infty} \frac{u_{i+1}}{u_i} \le \limsup_{i\to+\infty} \frac{\theta(x_{i+1},x_m)\psi^{i+1}(d_0)(\theta(x_i,x_m)+\rho\psi^i(d(x_0,x_1)))}{\psi^i(d_0)} < 1$$

which implies that the series $\sum_{i=0}^{\infty} u_i$ converges, so $d_{n,m}$ tends to zero as n,m tend to infinity. Hence, the sequence $\{x_n\}$ is Cauchy, and by completeness of (X,d) we conclude that $\{x_n\}$ converges to some $x \in X$. We next show that x is a fixed point of T

$$\begin{aligned} d(x_*,fx_*) &\le \theta(x_*,fx_*)(d(x_*,x_{k+1})+d(x_{k+1},fx_*))+\rho d(x_*,x_{k+1})d(x_{k+1},fx_*) \\ &= \theta(x_*,fx_*)(d(x_*,x_{k+1})+d(fx_k,fx_*))+\rho d(x_*,fx_k)d(fx_k,fx_*) \\ &\le \theta(x_*,fx_*)d(x_*,x_{k+1})+\theta(x_*,fx_*)\psi(d(x_k,x_*))+\rho d(x_*,fx_k)\psi(d(x_k,x_*)) \\ &\le \theta(x_*,fx_*)d(x_*,x_{k+1})+\theta(x_*,fx_*)d(x_k,x_*)+\rho d(x_*,fx_k)d(x_k,x_*) \end{aligned}$$

Thus, as k tends to infinity, we deduce that $x_* = Tx_*$. Finally, the uniqueness of the fixed point follows immediately □

Immediatly we have the following corollary:

Corollary 2.14. *Let (X,d_θ) be a complete extended b-metric space such that d_θ is a continuous functional, and $T : X \to X$ be a mapping, Assume there exists $\psi \in \mathbb{M}_b$ such that:*

$$d_\theta(Tx,Ty) \le \psi(d_\theta(x,y)) \qquad for\,all \quad x,y \in X \tag{8}$$

such that, $\limsup_{i\to+\infty} \theta(x_i,x_m) < b \quad for\,all\; m \in \mathbb{N}$
Then, T has precisely one fixed point ξ. Moreover for each $y \in X$, $T^n y \to \xi$.

Proof. The same as for the previous theorem, modifying K by $\psi(d_0)$ and using the same technique as for the theorem 2 in [1]. □

As immediate consequences, we obtain the following propositions.

Proposition 2.15. *Theorem* 2.12 *generalizes Theorem* 2.1 *and Theorem* 1.5.

Proposition 2.16. *Theorem* 2.10 *generalizes Theorem* 2.1.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] T. Kamran, M. Samreen, Q. UL Ain, A Generalization of B-Metric Space and Some Fixed Point Theorems, Mathematics 5 (2017), 19. https://doi.org/10.3390/math5020019.

[2] M. Berzig, Nonlinear Contraction in B-Suprametric Spaces, J. Anal. 32 (2024), 2401–2414. https://doi.org/10.1007/s41478-024-00732-5.

[3] S. Matthews, Partial Metric Spaces, in: The 2008 Summer Conference, Universitaria de la UNAM, Mexico City, 2008.

[4] D. Bugajewski, P. Maćkowiak, R. Wang, On Compactness and Fixed Point Theorems in Partial Metric Spaces, arXiv:2005.06327 (2020). http://arxiv.org/abs/2005.06327v2.

[5] V. Gregori, J. Miñana, D. Miravet, Fuzzy Partial Metric Spaces, Int. J. Gen. Syst. 48 (2018), 260–279. https://doi.org/10.1080/03081079.2018.1552687.

[6] S. Kumar, S. Luambano, On Some Fixed Point Theorems for Multivalued F-Contractions in Partial Metric Spaces, Demonstr. Math. 54 (2021), 151–161. https://doi.org/10.1515/dema-2021-0012.

[7] N.H. Hoc, L.V. Nguyen, Existence of Minima of Functions in Partial Metric Spaces and Applications to Fixed Point Theory, Acta Math. Hung. 168 (2022), 345–362. https://doi.org/10.1007/s10474-022-01279-2.

[8] S. Czerwik, Contraction Mappings in b-Metric Spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11. http://dml.cz/dmlcz/120469.

[9] S. Kajántó, A. Lukács, A Note on the Paper ”Contraction Mappings in B-Metric Spaces” by Czerwik, Acta Univ. Sapient. Math. 10 (2018), 85–89. https://doi.org/10.2478/ausm-2018-0007.

[10] H. Afshari, E. Karapınar, A Discussion on the Existence of Positive Solutions of the Boundary Value Problems via ψ-Hilfer Fractional Derivative on b-Metric Spaces, Adv. Differ. Equ. 2020 (2020), 616. https://doi.org/10.1186/s13662-020-03076-z.

[11] M. Berzig, I. Kédim, A. Mannai, Multivalued Fixed Point Theorem in B-Metric Spaces and Its Application to Differential Inclusions, Filomat 32 (2018), 2963–2976. https://doi.org/10.2298/fil1808963b.

[12] M. Boriceanu, M. Bota, A. Petruşel, Multivalued Fractals in b-Metric Spaces, Open Math. 8 (2010), 367–377. https://doi.org/10.2478/s11533-010-0009-4.

[13] J.E. Lazreg, S. Abbas, M. Benchohra, E. Karapınar, Impulsive Caputo-Fabrizio Fractional Differential Equations In b-Metric Spaces, Open Math. 19 (2021), 363–372. https://doi.org/10.1515/math-2021-0040.

[14] V. Gregori, J. Miñana, D. Miravet, Fuzzy Partial Metric Spaces, Int. J. Gen. Syst. 48 (2018), 260–279. https://doi.org/10.1080/03081079.2018.1552687.

[15] P. Baradol, D. Gopal, S. Radenović, Computational Fixed Points in Graphical Rectangular Metric Spaces with Application, J. Comput. Appl. Math. 375 (2020), 112805. https://doi.org/10.1016/j.cam.2020.112805.

[16] L. Budhia, H. Aydi, A.H. Ansari, D. Gopal, Some New Fixed Point Results in Rectangular Metric Spaces with an Application to Fractional-Order Functional Differential Equations, Nonlinear Anal.: Model. Control. 25 (2020), 580–597. https://doi.org/10.15388/namc.2020.25.17928.

[17] M. Turinici, Ekeland Variational Principles in 2-Local Branciari Metric Spaces, Springer, Cham, 2020. https://doi.org/10.1007/978-3-030-61732-5_23.

[18] J. Patil, B. Hardan, A.A. Hamoud, A. Bachhav, H. Emadifar, A New Result on Branciari Metric Space Using (α, γ)-Contractive Mappings, Topol. Algebr. Appl. 10 (2022), 103–112. https://doi.org/10.1515/taa-2022-0117.

[19] J.R.R. Roshan, V. Parvaneh, Z. Kadelburg, New Fixed Point Results in B-Rectangular Metric Spaces, Nonlinear Anal.: Model. Control. 21 (2016), 614–634. https://doi.org/10.15388/na.2016.5.4.

[20] S. Shukla, Partial Rectangular Metric Spaces and Fixed Point Theorems, Sci. World J. 2014 (2014), 756298. https://doi.org/10.1155/2014/756298.

[21] M. Younis, D. Singh, I. Altun, V. Chauhan, Graphical Structure of Extended b-Metric Spaces: An Application to the Transverse Oscillations of a Homogeneous Bar, Int. J. Nonlinear Sci. Numer. Simul. 23 (2021), 1239–1252. https://doi.org/10.1515/ijnsns-2020-0126.

[22] P. Gautam, V. Narayan Mishra, R. Ali, S. Verma, Interpolative Chatterjea and Cyclic Chatterjea Contraction on Quasi-Partial b-Metric Space, AIMS Math. 6 (2021), 1727–1742. https://doi.org/10.3934/math.2021103.

[23] T. Abdeljawad, E. Karapınar, S.K. Panda, N. Mlaiki, Solutions of Boundary Value Problems on Extended-Branciari b-Distance, J. Inequal. Appl. 2020 (2020), 103. https://doi.org/10.1186/s13660-020-02373-1.

[24] M. Berzig, First Results in Suprametric Spaces with Applications, Mediterr. J. Math. 19 (2022), 226. https://doi.org/10.1007/s00009-022-02148-6.

[25] J. Matkowski, Integrable Solutions of Functional Equations, Instytut Matematyczny Polskiej Akademi Nauk, Warszawa, 1975. http://eudml.org/doc/268343.

[26] S.K. Panda, R.P. Agarwal, E. Karapínar, Extended Suprametric Spaces and Stone-Type Theorem, AIMS Math. 8 (2023), 23183–23199. https://doi.org/10.3934/math.20231179.