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SOME FIXED CIRCLE THEOREMS IN MULTIPLICATIVE METRIC SPACES

SREEREKHA PARAPPILLY[1,*], SHAINI PULICKAKUNNEL[2]

[1]Department of Mathematics, Govt. Victoria College, Palakkad, Kerala, India
[2]Department of Mathematics, Central University of Kerala, Kasaragod, Kerala, India

Abstract. The fixed point theory and its relevance to diverse scientific domains are widely recognized. The field of fixed point theory has incorporated a novel geometric concept known as the fixed circle problem. This paper establishes the existence and uniqueness theorems concerning fixed circles of self mappings on multiplicative metric spaces. We introduced the Caristi type contraction in the framework of multiplicative metric spaces and utilizing this concept, established theorems guaranteeing the existence of fixed circles. We validate our findings through illustrative examples.

Keywords: fixed point; fixed circle; multiplicative metric space; Caristi type multiplicative contraction.

2020 AMS Subject Classification: 47H09, 47H10, 37E10.

1. Introduction

Fixed point theory has rapidly become a cornerstone of nonlinear analysis, experiencing exponential growth in research interest. Its versatility and applicability across diverse domains have rendered it an indispensable and powerful tool in nonlinear analysis. Indeed, numerous

*Corresponding author
E-mail address: sreerekha.2300907002@cukerala.ac.in
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problems across various branches of mathematics can be reduced to fixed point problems, underscoring the fundamental role of this theory. Moreover, the utility of fixed point theory extends far beyond mathematics, permeating fields such as engineering sciences, medical science, economics, neural networking, and beyond. A self mapping g on a non empty set X possesses a fixed point x if $g(x) = x$, indicating that x remains unchanged under the mapping. If no such x exists, g is termed a fixed point-free map. Conditions on both g and X that ensure the existence of fixed points constitute a fixed point theorem. Fixed point theory, a prominent branch of mathematics, primarily concerns itself with establishing the existence of fixed point solutions. Rooted in three pivotal theorems—the Banach Contraction Principle, Brouwer's Fixed Point Theorem, and Tarski's Fixed Point Theorem—fixed point theory encompasses Metric Fixed Point Theory, Topological Fixed Point Theory, and Discrete Fixed Point Theory. This paper delves into the metric dimensions of fixed point theory.

The Banach Fixed Point Theorem[1] provides a general criterion for ensuring the existence of a fixed point for contraction mappings in complete metric spaces. It also states that the Picard iteration converges to the fixed point of the map. Banach contraction principle has been broadened mainly in two ways: by relaxing or generalizing the contraction condition and exploring different types of spaces.The field of fixed point theory is rich with variations on the concept of metric spaces, leading to new fixed point theorems. Examples include partial metric spaces, b-metric spaces, s-metric spaces, geodesic spaces, and spaces with metric values in vectors, complex numbers, or even elements from a c* algebra. The O-metric space is a recent addition to this list, introduced in [2].

Multiplicative metric spaces are one such example. Grossman and Katz[3] introduced the concept of multiplicative calculus in 1972 which later inspired Bashirov et al.[4] to develop the notion of multiplicative metric spaces.The field of fixed point theory has seen significant advancements through the use of multiplicative metric spaces. Pioneering work by Ozavsar and Cevikel[5] explored the topological properties of these spaces and established fixed point theorems for specific types of contractive mappings within multiplicative metric spaces. Building upon this foundation, researchers like S. Jiping, L. Tianqi, L. Lei [6] investigated the existence of common fixed points for multiplicative contractions. Abbas et al.[7] further extended the

theory by studying common fixed points of locally contractive mappings. Additionally, Aminato. Ige, Hallowedo. Olaoluwa, Johnson O. Olaleru[8] explored fixed points for multivalued maps. Gu and Cho [9] contributed by establishing common fixed point results for mappings satisfying a ϕ-contractive condition. More recently, Verma et al [10] investigated the existence of common fixed points for weakly commuting mapping on a multiplicative b-metric space.
However, a recent study by Özgür and Taş [11] takes a fresh approach with a geometric twist: the "fixed-circle problem" in metric spaces. Özgür and Taş[11] started this research by proving fixed circle theorems for self-mappings in metric spaces. They provided geometric interpretations and examples to support their findings. The fixed circle problem has become a popular topic due to its rapid theoretical development and applications in various areas of mathematics, including neural networks. Researchers actively seek new solutions using diverse approaches and contractive conditions in metric spaces and their more general forms.The concept of fixed points in self mappings is gaining increasing attention in the field of neural networks. Research by [12] suggests that the fixed points of an activation function can identify the fixed points of a neural network. Additionally, the network has at least one or two fixed points if the overall input output relationship uses Möbius transformations. For instance, Özdemir et al.[13] used self mappings to develop new activation functions that fix a circle in complex valued neural networks (CVNNs). Recent studies have explored how theoretical fixed circle results can be applied to neural networks [14, 15]. Motivated by these connections, this work delves into fixed circle results within the context of multiplicative metric spaces.

2. Preliminaries

Definition 2.1. [4] *Let X be a nonempty set. Multiplicative metric is a mapping $d : X \times X \to R$ satisfying the following conditions :*

(1) $d(x,y) \geq 1$, $\forall x,y \in X$ *and* $d(x,y) = 1$ *iff* $x = y$
(2) $d(x,y) = d(y,x)$, $\forall x,y \in X$
(3) $d(x,z) \leq d(x,y)\, d(y,z)$

(X,d) *is known as a Multiplicative metric space.*

Definition 2.2. [5]*Semi-multiplicative Continuity: Let*(X,d) *be a multiplicative metric space,* (Y,d) *be a metric space and* $f : X \to Y$ *be a function. If* f *holds the requirement that, for every* $\varepsilon > 0$*, there exists* $\delta > 1$ *such that* $f(B_\delta(x)) \subset B_\varepsilon(f(x))$*, then we call* f *semi-multiplicative continuous at* $x \in X$.

Definition 2.3. [5] *Let* (X,d) *be a multiplicative metric space. A mapping* $T : X \to X$ *is called a multiplicative contraction mapping if there exist a real number* $\alpha \in [0,1)$ *such that for all* $x,y \in X$

$$d(Tx,Ty) \leq d(x,y)^\alpha$$

Theorem 2.1. [5] *Let* (X,d) *be a multiplicative metric space and let* $T : X \to X$ *be a multiplicative contraction. If* (X,d) *is complete, then* T *has a unique fixed point.*

Definition 2.4. [11] *For a self mapping* $T : X \to X$ *if* $Tx = x$ *for all* $x \in C_{x_0,r} = \{x \in X : d(x_0,x) = r\}$ *, then we call the circle* $C_{x_0,r}$ *as the fixed circle of* T

3. Main Results

In this section, we introduce Caristi type multiplicative contractions and a fixed point theorem for such mappings.The classical Caristi theorem [16] for standard metric spaces states that if (X,d) is a complete metric space, $T : X \to X$ and if there exists a lower semi continuous function ϕ mapping X into the non negative real numbers such that $d(x,Tx) \leq \phi(x) - \phi(Tx)$ for all $x \in X$, then T has a fixed point. We present the multiplicative analogue of this theorem.

3.1. Caristi Theorem in Multiplicative Framework.

Definition 3.1. *Let* (X,d) *be a multiplicative metric space. If the self mapping* $T : X \to X$ *satisfies the condition*

$$d(x,Tx) \leq \frac{\phi(x)}{\phi(Tx)},\ \forall x \in X$$

where $\phi : X \to [1,\infty)$ *is a semi multiplicative continuous function, then* T *is called a Caristi type Multiplicative Contraction.*

Proposition 3.1. *Let (X,d) be a complete multiplicative metric space and $\phi : X \to [1,\infty)$ a semi multiplicative continuous function. Suppose that $\{x_n\}$ is a sequence in X such that*

$$d(x_n,x_{n+1}) \leq \frac{\phi(x_n)}{\phi(x_{n+1})},\ \forall n \in N_0 = N \cup \{0\}.$$

Then x_n converges to a point $v \in x$ and $d(x_n,v) \leq \frac{\phi(x_n)}{\phi(v)}$ for all $n \in N_0$

Proof. Since $d(x_n,x_{n+1}) \leq \frac{\phi(x_n)}{\phi(x_{n+1})}$, $\forall n \in N_0$, it follows that $\{\phi(x_n)\}$ is a decreasing sequence.For $m \in N_0$,

$$\begin{aligned}\prod_{n=0}^{m} d(x_n,x_{n+1}) &= d(x_0,x_1)d(x_1,x_2)\cdots d(x_m,x_{m+1}) \\ &\leq \frac{\phi(x_0)}{\phi(x_{m+1})} \\ &\leq \frac{\phi(x_0)}{\inf_{n\in N_0}\phi(x_n)}\end{aligned}$$

Let $m \to \infty$, we have

$$\prod_{n=0}^{\infty} d(x_n,x_{n+1}) < \infty$$

This implies that $\{x_n\}$ is a multiplicative Cauchy sequence in X. Because X is multiplicative complete, there exists $v \in X$ such that $\lim_{n\to\infty} x_n = v$. Let $m,n \in N_0$ with $m > n$. Then

$$\begin{aligned}d(x_n,x_m) &\leq \prod_{i=n}^{m-1} d(x_i,x_{i+1}) \\ &\leq \frac{\phi(x_n)}{\phi(x_m)}.\end{aligned}$$

Letting $m \to \infty$, we obtain

$$\begin{aligned}d(x_n,v) &\leq \frac{\phi(x_n)}{\lim_{m\to\infty}\phi(x_m)} \\ &\leq \frac{\phi(x_n)}{\phi(v)} \text{ for all } n \in N_0\end{aligned}$$

□

Proposition 3.2. *Let X be a complete multiplicative metric space and $\phi : X \to [1,\infty)$a semi multiplicative continuous function. Suppose that, for each $u \in X$ with $\inf_{x\in X}\phi(x) < \phi(u)$,there exists a $v \in X$ such that*

$$u \neq v \text{ and } d(u,v) \leq \frac{\phi(u)}{\phi(v)}.$$

Then there exists an $x_0 \in X$ *such that* $\phi(x_0) = \inf_{x\in X} \phi(x)$.

Proof. Assume that $\inf_{x\in X} \phi(x) < \phi(y)$ for every $y \in X$. Let $u_0 \in X$. If $\inf_{x\in X} \phi(x) = \phi(u_0)$, then we are done. Otherwise $\inf_{x\in X} \phi(x) < \phi(u_0)$, and there exists a $u_1 \in X$ such that $u_0 \neq u_1$ and $d(u_0,u_1) \leq \frac{\phi(u_0)}{\phi(u_1)}$. Suppose $u_{n-1} \in X$ is known. Then choose $u_n \in S_n$, where

$$S_n := \{w \in X : d(u_{n-1},w) \leq \frac{\phi(u_{n-1})}{\phi(w)}\}$$

such that

$$\phi(u_n) \leq \inf_{w\in S_n} \phi(w) \left[\frac{\phi(u_{n-1})}{\inf_{w\in S_n} \phi(w)}\right]^{1/2}. \tag{1}$$

Because $u_n \in S_n$, we get

$$d(u_{n-1},u_n) \leq \frac{\phi(u_{n-1})}{\phi(u_n)},\ n \in N.$$

Proposition 3.1 implies that $u_n \to v \in X$ and $d(u_{n-1},v) \leq \frac{\phi(u_{n-1})}{\phi(v)}$. By hypothesis, there exists a $z \in X$ such that $z \neq v$ and $d(v,z) \leq \frac{\phi(v)}{\phi(z)}$. Note that

$$\begin{aligned}
\phi(z) &\leq \frac{\phi(v)}{d(v,z)} \\
&\leq \frac{\phi(v)}{d(v,z)} \frac{\phi(u_{n-1})}{\phi(v)d(u_{n-1},v)} \\
&\leq \frac{\phi(u_{n-1})}{d(u_{n-1},z)}
\end{aligned}$$

this implies that $z \in S_n$. It follows from inequality (1) that

$$\frac{(\phi(u_n))^2}{\phi(u_{n-1})} \ \leq \ \inf_{w\in S_n} \phi(w) \ \leq \ \phi(z)$$

Thus,

$$\phi(z) < \phi(v) \leq \lim_{n\to\infty} \phi(u_n) \leq \phi(z),$$

a contradiction. Therefore, there exists a point $x_0 \in X$ such that $\phi(x_0) = inf_{x\in X}\phi(x)$ □

We now present the multiplicative analogue of Caristi's fixed point theorem.

Theorem 3.1. *Let* X *be a complete multiplicative metric space and* $\phi : X \to [1,\infty)$ *a semi multiplicative continuous function. Let* $T : X \to X$ *be a mapping such that*

$$d(x,Tx) \leq \frac{\phi(x)}{\phi(Tx)} \textit{ for all } x \in X. \tag{2}$$

Then there exists a point $v \in X$ such that $v = Tv$.

Proof. Let

$$C = \{x \in X : d(u,x) \leq \frac{\phi(u)}{\phi(x)}\}.$$

Then C is a nonempty closed subset of X. We show that C is invariant under T. For each $x \in C$, we have

$$d(u,x) \leq \frac{\phi(u)}{\phi(x)}$$

and hence from (2), we have

$$\begin{aligned}
\phi(Tx) &\leq \frac{\phi(x)}{d(x,Tx)} \\
&\leq \frac{\phi(x)}{d(x,Tx)} \frac{\phi(u)}{\phi(x)d(u,x)} \\
&= \frac{\phi(u)}{d(x,Tx)d(u,x)} \\
&\leq \frac{\phi(u)}{d(u,Tx)},
\end{aligned}$$

and it follows that $Tx \in C$.

Suppose that $x \neq Tx$ for all $x \in C$. Then, for each $x \in C$, there exists $w \in C$ such that

$$x \neq w \text{ and } d(x,w) \leq \frac{\phi(x)}{\phi(w)}.$$

By Proposition 3.2, there exists an $x_0 \in C$ with $\phi(x_0) = \inf_{x \in C} \phi(x)$. Hence for such an $x_0 \in C$, we have

$$\begin{aligned}
1 < d(x_0, Tx_0) &\leq \frac{\phi(x_0)}{\phi(Tx_0)} \\
&\leq \frac{\phi(Tx_0)}{\phi(Tx_0)} \\
&= 1,
\end{aligned}$$

a contradiction. □

3.2. The existence of fixed circles. This section introduces the concept of fixed circle in multiplicative metric spaces. We prove fixed circle theorems that guarantee the existence of such a fixed circle for a self mapping that satisfies certain conditions within multiplicative metric spaces.

Definition 3.2. *Let (X,d) be a multiplicative metric space, $x_0 \in X$ and $r \in [1,\infty)$. Then the circle centered at x_0 and radius r is defined by*

$$C_{x_0,r} = \{x \in X : d(x_0,x) = r\}$$

Example 3.1. *Consider the multiplicative metric space R_+ with $d^*(x,y) = \left|\frac{x}{y}\right|^*$ where*

$$|a|^* = \begin{cases} a, \text{ if } a \geq 1 \\ \frac{1}{a}, \text{ if } a < 1 \end{cases}$$

A circle with center 5 and radius 2 is

$$C_{5,2} = \{x \in R_+ : d^*(x,5) = 2\} = \{\frac{5}{2}, 10\}$$

Example 3.2. *Consider R_+^2 with $d^*(x,y) = \left|\frac{x_1}{y_1}\right|^* \left|\frac{x_2}{y_2}\right|^*$ where $x = (x_1,x_2), y = (y_1,y_2)$. A circle with center $(2,3)$ and radius 5 is*

$$C_{(2,3),5} = \left\{x \in R_+^2 : d^*(x,(2,3)) = 5\right\}$$

$$= \left\{x \in R_+^2 : \left|\frac{x_1}{2}\right|^* \left|\frac{x_2}{3}\right|^* = 5\right\}$$

$$= \left\{x \in R_+^2 : x_1x_2 = 30, 2 \leq x_1 \leq 10\right\} \bigcup \left\{x \in R_+^2 : x_2 = 0.3x_1, 2 \leq x_1 < 10\right\}$$

$$\bigcup \left\{x \in R_+^2 : x_2 = 7.5x_1, 0.4 \leq x_1 < 2\right\} \bigcup \left\{x \in R_+^2 : x_1x_2 = 1.2, 0.4 < x_1 < 2\right\}$$

Definition 3.3. *Let (X,d) be a multiplicative metric space and $T : X \to X$ be a self map. Then the circle $C_{x_0,r} = \{x \in X : d(x_0,x) = r\}$ is a fixed circle of T if $Tx = x$ for all $x \in C_{x_0,r}$.*

Theorem 3.2. *Let (X,d) be a multiplicative metric space and $C_{x_0,r}$ be the multiplicative circle on X with center x_0 and radius r. Let $\phi : X \to [1,\infty)$ be defined by $\phi(x) = d(x,x_0)$ $\forall x \in X$. If there exists a self mapping $T : X \to X$ satisfying*

(1) $d(x,Tx) \leq \frac{\phi(x)}{\phi(Tx)}$ and

(2) $d(Tx,x_0) \geq r$

for each $x \in C_{x_0,r}$, then the circle $C_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C_{x_0,r}$.

$$\begin{aligned} d(x,Tx) &\leq \frac{\phi(x)}{\phi(Tx)} \text{ using (1)} \\ &= \frac{d(x,x_0)}{d(Tx,x_0)} \\ &= \frac{r}{d(Tx,x_0)} \\ &\leq \frac{r}{r} \\ &\leq 1 \end{aligned}$$

and so $d(x,Tx) = 1$ which implies that $Tx = x$. Thus we obtain that $C_{x_0,r}$ is a fixed circle of T. □

Theorem 3.3. *Let (X,d) be a multiplicative metric space and $C_{x_0,r}$ be the multiplicative circle on X with center x_0 and radius r. Let $\phi : X \to [1,\infty)$ be defined by $\phi(x) = d(x,x_0)$ $\forall x \in X$. If there exists a self mapping $T : X \to X$ satisfying*

(1) $d(x,Tx) \leq \frac{\phi(x)\phi(Tx)}{r^2}$ and

(2) $d(Tx,x_0) \leq r$

for each $x \in C_{x_0,r}$, then the circle $C_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C_{x_0,r}$.

$$d(x,Tx) \leq \frac{\phi(x)\phi(Tx)}{r^2} \text{ using (1)}$$

$$
\begin{aligned}
&= \frac{d(x,x_0)\, d(Tx,x_0)}{r^2} \\
&= \frac{r\, d(Tx,x_0)}{r^2} \\
&= \frac{d(Tx,x_0)}{r} \\
&\leq 1
\end{aligned}
$$

and so $d(x,Tx) = 1$ which implies that $Tx = x$. Thus we obtain that $C_{x_0,r}$ is a fixed circle of T. □

Theorem 3.4. *Let (X,d) be a multiplicative metric space and $C_{x_0,r}$ be the multiplicative circle on X with center x_0 and radius r. Let $\phi : X \to [1,\infty)$ be defined by $\phi(x) = d(x,x_0)\ \forall x \in X$. If there exists a self mapping $T : X \to X$ satisfying*

(1) $d(x,Tx) \leq \frac{\phi(x)}{\phi(Tx)}$ and

(2) $d(x,Tx)^h d(Tx,x_0) \geq r$ for each $x \in C_{x_0,r}$ and some $h \in [0,1)$ then the circle $C_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C_{x_0,r}$.

$$
\begin{aligned}
d(x,Tx) &\leq \frac{\phi(x)}{\phi(Tx)} \quad \text{using (1)} \\
&= \frac{d(x,x_0)}{d(Tx,x_0)} \\
&= \frac{r}{d(Tx,x_0)} \\
&\leq \frac{d(x,Tx)^h\, d(Tx,x_0)}{d(Tx,x_0)} \\
&\leq d(x,Tx)^h \quad \text{where } h \in [0,1)
\end{aligned}
$$

and so $d(x,Tx) = 1$ which implies that $Tx = x$. Thus we obtain that $C_{x_0,r}$ is a fixed circle of T. □

Example 3.3. *Let (X,d) be a multiplicative metric space and $C_{x_0,r}$ be the multiplicative circle. Define $T : X \to X$ such that*

$$Tx = \frac{r\,x}{r_x}, \text{ where } r_x = d(x,x_0)$$

Example 3.4. *Let (X,d) be a multiplicative metric space and $C_{x_0,r}$ be the multiplicative circle. Let $y_0 \in X$ be such that $d(y_0,x_0) > r$. Define $T : X \to X$ as*

$$Tx = \begin{cases} x, & \text{if } x \in C_{x_0,r} \\ y_0, & \text{otherwise} \end{cases}$$

Clearly $C_{x_0,r}$ is a fixed circle of T in both cases.

The application of discontinuous activation functions in neural networks has been investigated in [17]. The following example illustrates a function of this type.

Example 3.5. *Consider the multiplicative metric space R_+ discussed in example 3.1 and the multiplicative circle $C_{5,2}$ Define $T : X \to X$ as*

$$Tx = \begin{cases} 2, & 0 < x < 1 \\ x+1, & 1 \leq x \leq 2 \\ -x+5, & 2 < x \leq 3 \\ 10, & 3 < x < \infty \end{cases}$$

The mapping T satisfies conditions of Theorem 3.2 for the circle $C_{5,2} = \{5/2, 10\}$ and hence fixes the circle $C_{5,2}$.This can also be verified from the definition of T.

Theorem 3.5. *Let (X,d) be a multiplicative metric space and $C_{x_0,r}$ be any circle on X. Define the mapping $\phi : X \to [1,\infty),\ \phi(x) = d(x,x_0)\ \forall x \in X$. If there exists a self mapping $T : X \to X$ satisfies the condition*

$$d(x,Tx) \leq \left[\frac{\phi(x)}{\phi(Tx)}\right]^{\frac{1}{h}} \quad \forall\, x \in X \text{ and some } h > 1,$$

then $T = I_x$ and $C_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in X$ and $Tx \neq x$. Then

$$\begin{aligned} \Big[d(x,Tx)\Big]^h &\leq \frac{\phi(x)}{\phi(Tx)} \\ &= \frac{d(x,x_0)}{d(Tx,x_0)} \end{aligned}$$

$$\leq \frac{d(x,Tx)\, d(Tx,x_0)}{d(Tx,x_0)}$$

$$= d(x,Tx)$$

$$\Big[d(x,Tx)\Big]^{h-1} \leq 1$$

which is a contradiction, since $h > 1$ Therefore $Tx = x\ \forall x \in X$ and $T = I_X$ and $C_{x_0,r}$ is a fixed circle of T. ∎

3.3. The uniqueness of fixed circles.

In this part,we discuss the uniqueness of fixed circles. Note that fixed circles given by theorems in section 3.2 may not be unique.

Proposition 3.3. *Let (X,d) be a multiplicative metric space and $C_{x_1,r_1}, C_{x_2,r_2}, ..., C_{x_n,r_n}$ be any given circles in X.. Then, there exists at least one self mapping T of X such that T fixes all the circles $C_{x_1,r_1}, C_{x_2,r_2}, ..., C_{x_n,r_n}$.*

Proof. Define the self mapping $T : X \to X$ as

$$Tx = \begin{cases} x, & x \in \bigcup_{i=1}^{n} C_{x_i,r_i} \\ y_0, & x \notin \bigcup_{i=1}^{n} C_{x_i,r_i} \end{cases}$$

where $y_0 \in X$ satisfies $d(y_0,x_i) \neq r_i$. Define the mapping $\phi_i : X \to [1,\infty)$ as

$$\phi_i(x) = d(x,x_i)$$

for $i = 1,2,...,n$. Then it is easy to verify that conditions 1 and 2 in Theorem 3.2 are satisfied by T for all the given circles. Consequently $C_{x_1,r_1}, C_{x_2,r_2}, ..., C_{x_n,r_n}$ are fixed circles of T. ∎

The following theorem provides some uniqueness conditions.

Theorem 3.6. *Let (X,d) be a multiplicative metric space and $T : X \to X$ be a self mapping having a fixed circle $C_{x_0,r}$. If T satisfies any one of the following contraction conditions, then the fixed circle $C_{x_0,r}$ is unique.*

i) $d(Tx,Ty) \leq d(x,y)^h$ *for all* $x \in C_{x_0,r},\ y \in X \backslash C_{x_0,r}$ *and some* $h \in [0,1)$

ii) $d(Tx,Ty) \leq \big[d(x,Tx)d(y,Ty)\big]^h,$ *for all* $x \in C_{x_0,r},\ y \in X \backslash C_{x_0,r}$ *and some* $h \in [0,1)$

iii) $d(Tx,Ty) < max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$ *for all* $x \in C_{x_0,r},\ y \in X \setminus C_{x_0,r}$

Proof. Assume that the fixed circle is not unique. That is there are two fixed circles C_{x_0,r_0} and C_{x_1,r_1} of the self mapping T. Let $x \in C_{x_0,r_0}$ and $y \in C_{x_1,r_1}$ be arbitrary points.Then $Tx = x,\ Ty = y$. So $d(x,Tx) = d(y,Ty) = 1$. Also $d(x,Ty) = d(y,Tx) = d(x,y)$.

i) Using the contraction condition we get

$$d(x,y) = d(Tx,Ty) \leq d(x,y)^h$$

where $h \in [0,1)$. So $d(x,y) = 1$ and $x = y$ Thus the fixed circle is unique.

ii) The contraction condition gives

$$1 < d(x,y) = d(Tx,Ty) \leq \left[d(x,Tx)d(y,Ty)\right]^h = 1$$

and the fixed circle is unique.

iii) Using the contraction condition we get

$$d(x,y) = d(Tx,Ty) < max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$$

$$= d(x,y), \text{ a contradiction.}$$

Thus the fixed circle is unique.

□

4. Conclusion

In this article, we have examined the existence and uniqueness of fixed circles for self mappings under certain conditions within multiplicative metric spaces. Additionally, we have presented examples to illustrate the applicability of our theoretical findings. As this work introduces a novel area in the study of multiplicative metric spaces,these results will contribute and inspire further research in the field.

Conflict of Interests

The authors declare that there is no conflict of interests.

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