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COMMON FIXED POINT RESULTS FOR WEAK COMMUTATIVE MAPPINGS

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Abstract. In this paper, we prove common fixed point theorems for weakly commuting mappings satisfying a generalized ϕ-weak contraction condition that involves distance function of a complete metric space and generalize the results of Murthy and Prasad [5] and Jain et al. [4]. We also provide an example in support of our result.

Keywords: fixed point, ϕ-weak contraction, weakly commuting mappings, cubic terms of metric functions.

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1. Introduction and Preliminaries

In 1922, the Polish mathematician, Banach proved a fixed point theorem, which is the basic tool to show the existence and uniqueness of a fixed point under appropriate conditions. This result is known as Banach contraction principle, which states that

Let (X,d) be a complete metric space. If $\mathscr{K}$ satisfies $d(\mathscr{K}\hbar,\mathscr{K}\vartheta) \leq kd(\hbar,\vartheta)$ for each $\hbar,\vartheta \in X$, where $0 \leq k < 1$, then $\mathscr{K}$ has a unique fixed point in X.

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This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. The generalisations of Banach contraction principle gave new direction to researchers in the field of fixed point theory.

In 1969, Boyd and Wong [2] replaced the constant k in Banach contractive condition by an upper semi-continuous function ψ as follows:

Let (X,d) be a complete metric space and $\mathscr{K}$ be a self map on X satisfying the following:

$$d(\mathscr{K}\hbar, \mathscr{K}\vartheta) \leq \psi d(\hbar, \vartheta)$$

for all $\hbar, \vartheta \in X$, where, ψ: $[0,\infty) \to [0,\infty)$ be upper semi continuous from the right such that $0 \leq \psi < t$ for all $t > 0$. Then $\mathscr{K}$ has a unique fixed point $\hbar \in X$ and $\{\mathscr{K}^n(\hbar)\}$ converges to $\hbar$ for all $\hbar \in X$.

In 1997, Alber and Gueree-Delabriere [1] introduced the concept of weak contraction in Hilbert spaces.

Later on, Rhoades [6] had shown that the results of Alber and Gueree-Delabriere [1] equally hold good in complete metric spaces.

A map $\mathscr{K} : X \to X$ is said to be weak contraction if for each $\hbar, \vartheta \in X$, there exists a function $\phi : [0,\infty) \to [0,\infty)$, $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$ such that

$$d(\mathscr{K}\hbar, \mathscr{K}\vartheta) \leq d(\hbar, \vartheta) - \phi(d(\hbar, \vartheta)).$$

In 2013, Murthy and Prasad [5] introduced a new type of inequality involving cubic terms of $d(\hbar, \vartheta)$ that extended and generalized the results of Alber and Gueree-Delabriere [1] and others cited in the literature of fixed point theory.

In this paper, we generalize the results of Murthy and Prasad [5] and Jain et al. [4] for pairs of weakly commuting mappings satisfying generalized weak contractive condition involving various combinations of metric functions.

2. Main Results

In this section, first, we give some basic definitions and results that are useful for proving our main results.

The notion of commutative mappings in fixed point theory was used by Jungck [3] to obtain a generalization of Banach's fixed point theorem for a pair of mappings. This result was further generalized, extended and unified by using various types of minimal commutative mappings.

Definition 2.1. [3] Two self mappings f and g of a metric space (X,d) are said to be commuting if $fg\hbar = gf\hbar$ for all $\hbar \in X$.

The first ever attempt to relax the commutativity of mappings to weak commutative was initiated by Sessa [7] as follows:

Definition 2.2. [7] Two self mappings f and g of a metric space (X,d) are said to be weakly commuting if $d(fg\hbar, gf\hbar) \leq d(g\hbar, f\hbar)$ for all $\hbar \in X$.

Remark 2.3. Commutative mappings are weak commutative mappings, but the converse may not be true.

In 2013, Murthy and Prasad [5] proved the following result:

Theorem 2.4. [5, Theorem 3] *Let $\mathscr{K}$ be a self-map of a complete metric space (X,d) satisfying the following:*

$$
\begin{aligned}
&[1+pd(\hbar,\vartheta)]d^2(\mathscr{K}\hbar,\mathscr{K}\vartheta) \leq \\
&\qquad p\max\left\{\frac{1}{2}\left[d^2(\hbar,\mathscr{K}\hbar)d(\vartheta,\mathscr{K}\vartheta)+d(\hbar,\mathscr{K}\hbar)d^2(\vartheta,\mathscr{K}\vartheta)\right],\right.\\
&\qquad \left. d(\hbar,\mathscr{K}\hbar)d(\hbar,\mathscr{K}\vartheta)d(\vartheta,\mathscr{K}\hbar), d(\hbar,\mathscr{K}\vartheta)d(\vartheta,\mathscr{K}\hbar)d(\vartheta,\mathscr{K}\vartheta)\right\}+\\
&\qquad m(\hbar,\vartheta)-\phi(m(\hbar,\vartheta)), \qquad (1)
\end{aligned}
$$

where

$$
m(\hbar,\vartheta) = \max\left\{\begin{array}{c} d^2(\hbar,\vartheta), d(\hbar,\mathscr{K}\hbar)d(\vartheta,\mathscr{K}\vartheta), d(\hbar,\mathscr{K}\vartheta)d(\vartheta,\mathscr{K}\hbar), \\ \frac{1}{2}\left[d(\hbar,\mathscr{K}\hbar)d(\hbar,\mathscr{K}\vartheta)+d(\vartheta,\mathscr{K}\hbar)d(\vartheta,\mathscr{K}\vartheta)\right] \end{array}\right\}, \qquad (2)
$$

$p \geq 0$ is a real number and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0 \Leftrightarrow t = 0$ and $\phi(t) > 0$ for each $t > 0$.

Then $\mathscr{K}$ has a unique fixed point in X.

Now, we extend and generalize Theorem 2.4 for a pair of weakly commuting mappings as follows:

Theorem 2.5. *Let $\mathfrak{S}$ and $\mathscr{K}$ be mappings of a complete metric space (X,d) into itself satisfying the following conditions:*

(C_1) $\mathscr{K}(X) \subset \mathfrak{S}(X)$,

(C_2) $\mathfrak{S}$ *is continuous,*

$$[1+pd(\mathfrak{S}\hbar,\mathfrak{S}\vartheta)]d^2(\mathscr{K}\hbar,\mathscr{K}\vartheta) \leq$$

$$p\max\left\{\frac{1}{2}[d^2(\mathfrak{S}\hbar,\mathscr{K}\hbar)d(\mathfrak{S}\vartheta,\mathscr{K}\vartheta)+d(\mathfrak{S}\hbar,\mathscr{K}\hbar)d^2(\mathfrak{S}\vartheta,\mathscr{K}\vartheta)],\right.$$

$$\left. d(\mathfrak{S}\hbar,\mathscr{K}\hbar)d(\mathfrak{S}\hbar,\mathscr{K}\vartheta)d(\mathfrak{S}\vartheta,\mathscr{K}\hbar), d(\mathfrak{S}\hbar,\mathscr{K}\vartheta)d(\mathfrak{S}\vartheta,\mathscr{K}\hbar)d(\mathfrak{S}\vartheta,\mathscr{K}\vartheta)\right\}+$$

$$m(\mathfrak{S}\hbar,\mathfrak{S}\vartheta)-\phi(m(\mathfrak{S}\hbar,\mathfrak{S}\vartheta)), \tag{3}$$

where

$$m(\mathfrak{S}\hbar,\mathfrak{S}\vartheta) = \max\left\{\begin{array}{r} d^2(\mathfrak{S}\hbar,\mathfrak{S}\vartheta), d(\mathfrak{S}\hbar,\mathscr{K}\hbar)d(\mathfrak{S}\vartheta,\mathscr{K}\vartheta), \\ d(\mathfrak{S}\hbar,\mathscr{K}\vartheta)d(\mathfrak{S}\vartheta,\mathscr{K}\hbar), \\ \frac{1}{2}\Big[d(\mathfrak{S}\hbar,\mathscr{K}\hbar)d(\mathfrak{S}\hbar,\mathscr{K}\vartheta)+d(\mathfrak{S}\vartheta,\mathscr{K}\hbar)d(\mathfrak{S}\vartheta,\mathscr{K}\vartheta)\Big] \end{array}\right\}, \tag{4}$$

$p \geq \frac{1}{3}$ *is a real number and* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function with* $\phi(t) = 0 \Leftrightarrow t = 0$ *and* $\phi(t) > 0$ *for each* $t > 0$.

Then $\mathfrak{S}$ and $\mathscr{K}$ have a unique common fixed point in X provided $\mathfrak{S}$ and $\mathscr{K}$ weakly commute on X.

Proof. Let $\hbar_0 \in X$ be an arbitrary point. From (C_1), we can find $\hbar_1$ such that $\mathfrak{S}(\hbar_1) = \mathscr{K}(\hbar_0)$.

For this $\hbar_1$, we can find $\hbar_2 \in X$ such that $\mathfrak{S}(\hbar_2) = \mathscr{K}(\hbar_1)$. In general, one can choose $\{\hbar_n\}$ in X such that

$$\mathfrak{S}(\hbar_{n+1}) = \mathscr{K}(\hbar_n), \quad n = 0,1,2... \tag{5}$$

First we will prove that $\{\mathfrak{S}(\hbar_n)\}$ is a Cauchy sequence in X. We may assume that $\mathfrak{S}(\hbar_n) \neq \mathfrak{S}(\hbar_{n+1})$ $\forall$ n. If $\exists$ n such that

$$\mathfrak{S}(\hbar_n) = \mathfrak{S}(\hbar_{n+1}),$$

then $\mathfrak{S}(\hbar_n) = \mathfrak{S}(\hbar_{n+1}) = \mathscr{K}(\hbar_n)$, yields $\mathfrak{S}$ and $\mathscr{K}$ have a fixed point.

We write $\gamma_n = d(\mathfrak{S}\hbar_n, \mathfrak{S}\hbar_{n+1})$.

Firstly we prove that $\{\gamma_n\}$ is non increasing sequence and converges to 0.

If n is even, taking $\hbar = \hbar_{2n}$ and $\vartheta = \hbar_{2n+1}$ in (3), we get

$$\begin{aligned}
&\left[1 + pd(\mathfrak{S}\hbar_{2n}, \mathfrak{S}\hbar_{2n+1})\right] d^2(\mathscr{K}\hbar_{2n}, \mathscr{K}\hbar_{2n+1}) \leq \\
&\qquad p \max\left\{ \frac{1}{2}\left[d^2(\mathfrak{S}\hbar_{2n}, \mathscr{K}\hbar_{2n}) d(\mathfrak{S}\hbar_{2n+1}, \mathscr{K}\hbar_{2n+1}) + \right.\right. \\
&\qquad \left. d(\mathfrak{S}\hbar_{2n}, \mathscr{K}\hbar_{2n}) d^2(\mathfrak{S}\hbar_{2n+1}, \mathscr{K}\hbar_{2n+1}) \right], \\
&\qquad d(\mathfrak{S}\hbar_{2n}, \mathscr{K}\hbar_{2n}) d(\mathfrak{S}\hbar_{2n}, \mathscr{K}\hbar_{2n+1}) d(\mathfrak{S}\hbar_{2n+1}, \mathscr{K}\hbar_{2n}), \\
&\qquad \left. d(\mathfrak{S}\hbar_{2n}, \mathscr{K}\hbar_{2n+1}) d(\mathfrak{S}\hbar_{2n+1}, \mathscr{K}\hbar_{2n}) d(\mathfrak{S}\hbar_{2n+1}, \mathscr{K}\hbar_{2n+1}) \right\} + \\
&\qquad m(\mathfrak{S}\hbar_{2n}, \mathfrak{S}\hbar_{2n+1}) - \phi(m(\mathfrak{S}\hbar_{2n}, \mathfrak{S}\hbar_{2n+1})),
\end{aligned}$$

where

$$m(\mathfrak{S}\hbar_{2n}, \mathfrak{S}\hbar_{2n+1}) = \max \left\{ \begin{array}{r} d^2(\mathfrak{S}\hbar_{2n}, \mathfrak{S}\hbar_{2n+1}), \\ d(\mathfrak{S}\hbar_{2n}, \mathscr{K}\hbar_{2n}) d(\mathfrak{S}\hbar_{2n+1}, \mathscr{K}\hbar_{2n+1}), \\ d(\mathfrak{S}\hbar_{2n}, \mathscr{K}\hbar_{2n+1}) d(\mathfrak{S}\hbar_{2n+1}, \mathscr{K}\hbar_{2n}), \\ \frac{1}{2}\Big[d(\mathfrak{S}\hbar_{2n}, \mathscr{K}\hbar_{2n}) d(S\hbar_{2n}, \mathscr{K}\hbar_{2n+1}) + \\ d(\mathfrak{S}\hbar_{2n+1}, \mathscr{K}\hbar_{2n}) d(\mathfrak{S}\hbar_{2n+1}, \mathscr{K}\hbar_{2n+1}) \Big] \end{array} \right\}.$$

Using (5), we get

$$\begin{aligned}
&\left[1 + pd(\mathfrak{S}\hbar_{2n}, \mathfrak{S}\hbar_{2n+1})\right] d^2(\mathfrak{S}\hbar_{2n+1}, \mathfrak{S}\hbar_{2n+2}) \leq \\
&\qquad p \max\left\{ \frac{1}{2}\left[d^2(\mathfrak{S}\hbar_{2n}, \mathfrak{S}\hbar_{2n+1}) d(\mathfrak{S}\hbar_{2n+1}, \mathfrak{S}\hbar_{2n+2}) + \right.\right. \\
&\qquad \left. d(\mathfrak{S}\hbar_{2n}, \mathfrak{S}\hbar_{2n+1}) d^2(\mathfrak{S}\hbar_{2n+1}, \mathfrak{S}\hbar_{2n+2}) \right], \\
&\qquad d(\mathfrak{S}\hbar_{2n}, \mathfrak{S}\hbar_{2n+1}) d(\mathfrak{S}\hbar_{2n}, \mathfrak{S}\hbar_{2n+2}) d(\mathfrak{S}\hbar_{2n+1}, \mathfrak{S}\hbar_{2n+1}),
\end{aligned}$$

$$d(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+2})d(\mathfrak{S}\hbar_{2n+1},\mathfrak{S}\hbar_{2n+1})d(\mathfrak{S}\hbar_{2n+1},\mathfrak{S}\hbar_{2n+2})\Big\}+$$

$$m(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1})-\phi(m(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1})),$$

where

$$m(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1})=\max\left\{\begin{array}{r} d^2(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1}),\\ d(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1})d(\mathfrak{S}\hbar_{2n+1},\mathfrak{S}\hbar_{2n+2}),\\ d(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+2})d(\mathfrak{S}\hbar_{2n+1},\mathfrak{S}\hbar_{2n+1}),\\ \frac{1}{2}\Big[d(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1})d(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+2})+\\ d(\mathfrak{S}\hbar_{2n+1},\mathfrak{S}\hbar_{2n+1})d(\mathfrak{S}\hbar_{2n+1},\mathfrak{S}\hbar_{2n+2})\Big]\end{array}\right\}.$$

Now consider $\gamma_{2n}=d(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1})$, then we have

$$[1+p\gamma_{2n}]\gamma_{2n+1}^2\leq p\max\left\{\frac{1}{2}[\gamma_{2n}^2\gamma_{2n+1}+\gamma_{2n}\gamma_{2n+1}^2],0,0\right\}+ \quad (6)$$

$$m(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1})-\phi(m(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1})),$$

where

$$m(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1})=\max\left\{\gamma_{2n}^2,\gamma_{2n}\gamma_{2n+1},0,\frac{1}{2}[\gamma_{2n}d(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+2})+0]\right\},$$

and the use of triangle inequality in the right side of the above, gives

$$d(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+2})\leq d(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1}+d(\mathfrak{S}\hbar_{2n+1},\mathfrak{S}\hbar_{2n+2})$$

$$=\gamma_{2n}+\gamma_{2n+1}.$$

Thus,

$$m(\mathfrak{S}\hbar_{2n},\mathfrak{S}\hbar_{2n+1})\leq\max\left\{\gamma_{2n}^2,\gamma_{2n}\gamma_{2n+1},0,\frac{1}{2}[\gamma_{2n}(\gamma_{2n}+\gamma_{2n+1})+0]\right\}.$$

If $\gamma_{2n}<\gamma_{2n+1}$, then (6) reduces to

$$p\gamma_{2n+1}^2\leq p\gamma_{2n+1}^2-\phi(\gamma_{2n+1}^2),$$

which is a contradiction.

Therefore, $\gamma_{2n+1}^2\leq\gamma_{2n}^2$ implies that $\gamma_{2n+1}\leq\gamma_{2n}$.

In a similar way, if n is odd, then we can obtain $\gamma_{2n+2}\leq\gamma_{2n+1}$.

It follows that the sequence $\{\gamma_n\}$ is decreasing.

Let $\lim_{n\to\infty} \gamma_n = r$, for some $r \geq 0$. Suppose $r > 0$; then from inequality (3), we have

$$\begin{aligned}
\big[1+pd(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})\big]d^2(\mathscr{K}\hbar_n,\mathscr{K}\hbar_{n+1}) \leq &\\
& p\max\Bigg\{\frac{1}{2}\Bigg[d^2(\mathfrak{S}\hbar_n,\mathscr{K}\hbar_n)d(\mathfrak{S}\hbar_{n+1},\mathscr{K}\hbar_{n+1})+\\
& d(\mathfrak{S}\hbar_n,\mathscr{K}\hbar_n)d^2(\mathfrak{S}\hbar_{n+1},\mathscr{K}\hbar_{n+1})\Bigg],\\
& d(\mathfrak{S}\hbar_n,\mathscr{K}\hbar_n)d(\mathfrak{S}\hbar_n,\mathscr{K}\hbar_{n+1})d(\mathfrak{S}\hbar_{n+1},\mathscr{K}\hbar_n),\\
& d(\mathfrak{S}\hbar_n,\mathscr{K}\hbar_{n+1})d(\mathfrak{S}\hbar_{n+1},\mathscr{K}\hbar_n)d(\mathfrak{S}\hbar_{n+1},\mathscr{K}\hbar_{n+1})\Bigg\}+\\
& m(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})-\phi(m(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})),
\end{aligned}$$

where

$$m(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1}) = \max\left\{\begin{array}{r} d^2(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1}), d(\mathfrak{S}\hbar_n,\mathscr{K}\hbar_n)d(\mathfrak{S}\hbar_{n+1},\mathscr{K}\hbar_{n+1}),\\ d(\mathfrak{S}\hbar_n,\mathscr{K}\hbar_{n+1})d(\mathfrak{S}\hbar_{n+1},\mathscr{K}\hbar_n),\\ \frac{1}{2}\Big[d(\mathfrak{S}\hbar_n,\mathscr{K}\hbar_n)d(\mathfrak{S}\hbar_n,\mathscr{K}\hbar_{n+1})+\\ d(\mathfrak{S}\hbar_{n+1},\mathscr{K}\hbar_n)d(\mathfrak{S}\hbar_{n+1},\mathscr{K}\hbar_{n+1})\Big] \end{array}\right\}.$$

Now using (5) we get,

$$\begin{aligned}
\big[1+pd(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})\big]d^2(\mathfrak{S}\hbar_{n+1},\mathfrak{S}\hbar_{n+2}) \leq &\\
& p\max\Bigg\{\frac{1}{2}\Bigg[d^2(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})d(\mathfrak{S}\hbar_{n+1},\mathfrak{S}\hbar_{n+2})+\\
& d(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})d^2(\mathfrak{S}\hbar_{n+1},\mathfrak{S}\hbar_{n+2})\Bigg],\\
& d(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})d(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+2})d(\mathfrak{S}\hbar_{n+1},\mathfrak{S}\hbar_{n+1}),\\
& d(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+2})d(\mathfrak{S}\hbar_{n+1},\mathfrak{S}\hbar_{n+1})d(\mathfrak{S}\hbar_{n+1},\mathfrak{S}\hbar_{n+2})\Bigg\}+\\
& m(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})-\phi(m(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})),
\end{aligned}$$

where

$$m(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1}) = \max\left\{\begin{array}{r} d^2(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1}), d(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})d(\mathfrak{S}\hbar_{n+1},\mathfrak{S}\hbar_{n+2}),\\ d(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+2})d(\mathfrak{S}\hbar_{n+1},\mathfrak{S}\hbar_{n+1}),\\ \frac{1}{2}\Big[d(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1})d(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+2})+\\ d(\mathfrak{S}\hbar_{n+1},\mathfrak{S}\hbar_{n+1})d(\mathfrak{S}\hbar_{n+1},\mathfrak{S}\hbar_{n+2})\Big] \end{array}\right\}.$$

Using triangular inequality and property of ϕ and taking limits $n \to \infty$, we get

$$[1+pr]r^2 \le pr^3 + r^2 - \phi(r^2)$$

Then $\phi(r^2) \le 0$, since r is positive, then by property of ϕ, we get $r = 0$, we conclude that

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} d(\mathfrak{S}\hbar_n,\mathfrak{S}\hbar_{n+1}) = r = 0. \tag{7}$$

Now we show that $\{\mathfrak{S}\hbar_n\}$ is a Cauchy sequence. For given $\varepsilon > 0$ we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that

$$d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)}) \ge \varepsilon, \quad d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k-1)}) < \varepsilon \tag{8}$$

and $n(k) > m(k) > k$.

Now

$$\varepsilon \le d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)} \le d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k-1)}) + d(\mathfrak{S}\hbar_{n(k-1)},\mathfrak{S}\hbar_{n(k)}). \tag{9}$$

Letting $k \to \infty$, we get

$$\lim_{n\to\infty} d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)}) = \varepsilon. \tag{10}$$

Now from the triangular inequality, we have

$$|d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{m(k)+1} - d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)})| \le d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{m(k)+1}). \tag{11}$$

Taking limits as $k \to \infty$ and using (7) and (10) we have,

$$\lim_{k\to\infty} d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{m(k)+1}) = \varepsilon. \tag{12}$$

Again from triangular inequality, we have

$$|d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)+1}) - d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)})| \le d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{n(k)+1}). \tag{13}$$

Taking limits as $k \to \infty$ and using (7) and (10) we have,

$$\lim_{k\to\infty} d(\mathfrak{S}\hbar_{m(k)}, \mathfrak{S}\hbar_{n(k)+1}) = \varepsilon. \tag{14}$$

Similarly on using triangular inequality, we have

$$\begin{aligned} |d(\mathfrak{S}\hbar_{m(k)}, \mathfrak{S}\hbar_{n(k)+1}) - d(\mathfrak{S}\hbar_{m(k)}, \mathfrak{S}\hbar_{n(k)})| \leq \Big[& d(\mathfrak{S}\hbar_{m(k)}, \mathfrak{S}\hbar_{m(k)+1}) \\ & + d(\mathfrak{S}\hbar_{n(k)}, \mathfrak{S}\hbar_{n(k)+1}) \Big]. \end{aligned} \tag{15}$$

Taking limits as $k \to \infty$ in the above inequality and using (7) and (10) we have,

$$\lim_{k\to\infty} d(\mathfrak{S}\hbar_{n(k)+1}, \mathfrak{S}\hbar_{m(k)+1}) = \varepsilon.$$

On putting $\hbar = \hbar_{m(k)}$ and $\vartheta = \hbar_{n(k)}$ in (3), we get

$$\begin{aligned} \big[1 + pd(\mathfrak{S}\hbar_{m(k)}, \mathfrak{S}\hbar_{n(k)})\big] & d^2(\mathscr{K}\hbar_{m(k)}, \mathscr{K}\hbar_{n(k)}) \leq \\ & p \max\Bigg\{ \frac{1}{2}\Big[d^2(\mathfrak{S}\hbar_{m(k)}, \mathscr{K}\hbar_{m(k)}) d(\mathfrak{S}\hbar_{n(k)}, \mathscr{K}\hbar_{n(k)}) + \\ & d(\mathfrak{S}\hbar_{m(k)}, \mathscr{K}\hbar_{m(k)}) d^2(\mathfrak{S}\hbar_{n(k)}, \mathscr{K}\hbar_{n(k)}) \Big], \\ & d(\mathfrak{S}\hbar_{m(k)}, \mathscr{K}\hbar_{m(k)}) d(\mathfrak{S}\hbar_{m(k)}, \mathscr{K}\hbar_{n(k)}) d(\mathfrak{S}\hbar_{n(k)}, \mathscr{K}\hbar_{m(k)}), \\ & d(\mathfrak{S}\hbar_{m(k)}, \mathscr{K}\hbar_{n(k)}) d(\mathfrak{S}\hbar_{n(k)}, \mathscr{K}\hbar_{m(k)}) d(\mathfrak{S}\hbar_{n(k)}, \mathscr{K}\hbar_{n(k)}) \Bigg\} + \\ & m(\mathfrak{S}\hbar_{m(k)}, \mathfrak{S}\hbar_{n(k)}) - \phi(m(\mathfrak{S}\hbar_{m(k)}, \mathfrak{S}\hbar_{n(k)})), \end{aligned}$$

where

$$m(\mathfrak{S}\hbar_{m(k)}, \mathfrak{S}\hbar_{n(k)}) = \max \left\{ \begin{array}{r} d^2(\mathfrak{S}\hbar_{m(k)}, \mathfrak{S}\hbar_{n(k)}), \\ d(\mathfrak{S}\hbar_{m(k)}, \mathscr{K}\hbar_{m(k)}) d(\mathfrak{S}\hbar_{n(k)}, \mathscr{K}\hbar_{n(k)}), \\ d(\mathfrak{S}\hbar_{m(k)}, \mathscr{K}\hbar_{n(k)}) d(\mathfrak{S}\hbar_{n(k)}, \mathscr{K}\hbar_{m(k)}), \\ \frac{1}{2}\Big[d(\mathfrak{S}\hbar_{m(k)}, \mathscr{K}\hbar_{m(k)}) d(\mathfrak{S}\hbar_{m(k)}, \mathscr{K}\hbar_{n(k)}) + \\ d(\mathfrak{S}\hbar_{n(k)}, \mathscr{K}\hbar_{m(k)}) d(\mathfrak{S}\hbar_{n(k)}, \mathscr{K}\hbar_{n(k)}) \Big] \end{array} \right\}.$$

Now using (5) we obtain,

$$\big[1 + pd(\mathfrak{S}\hbar_{m(k)}, \mathfrak{S}\hbar_{n(k)})\big] d^2(\mathfrak{S}\hbar_{m(k)+1}, \mathfrak{S}\hbar_{n(k)+1}) \leq$$

$$p\max\Bigg\{\frac{1}{2}\Big[d^2(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{m(k)+1})d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{n(k)+1})+$$

$$d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{m(k)+1})d^2(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{n(k)+1})\Big],$$

$$d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{m(k)+1})d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)+1})d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{m(k)+1}),$$

$$d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)+1})d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{m(k)+1})d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{n(k)+1})\Bigg\}+$$

$$m(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)})-\phi(m(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)})),$$

where

$$m(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)})=\max\left\{\begin{array}{r} d^2(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)}),\\ d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{m(k)+1})d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{n(k)+1}),\\ d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)+1})d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{m(k)+1}),\\ \frac{1}{2}\Big[d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{m(k)+1})d(\mathfrak{S}\hbar_{m(k)},\mathfrak{S}\hbar_{n(k)+1})+\\ d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{m(k)+1})d(\mathfrak{S}\hbar_{n(k)},\mathfrak{S}\hbar_{n(k)+1})\Big]\end{array}\right\}.$$

Letting $k\to\infty$, we get

$$\begin{aligned}[1+p\varepsilon]\varepsilon^2 &\le p\max\{\frac{1}{2}[0+0],0,0\}+\varepsilon^2-\phi(\varepsilon^2)\\ &=\varepsilon^2-\phi(\varepsilon^2),\end{aligned}$$

which is a contradiction. Thus $\{\mathfrak{S}\hbar_n\}$ is a Cauchy sequence in X.

Since (X,d) is a complete metric space, therefore, $\{\mathfrak{S}\hbar_n\}$ converges to a point η and $\mathfrak{S}(\hbar_{n+1})=\mathscr{K}(\hbar_n)$ also converges to the same point η.

Now $\{\mathfrak{S}\hbar_n\}$ converges to a point η, therefore $\{\mathfrak{S}\mathfrak{S}\hbar_n\}$ converges to $\mathfrak{S}\eta$ as $\mathfrak{S}$ is continuous. Also $\{\mathscr{K}\hbar_n\}$ converges to η implies that $\{\mathfrak{S}\mathscr{K}\hbar_n\}$ converges to $\mathfrak{S}\eta$ as $\mathfrak{S}$ is continuous.

However, since $\mathfrak{S}$ and $\mathscr{K}$ are weakly commuting on X therefore,

$$d(\mathfrak{S}\mathscr{K}\hbar_n,\mathscr{K}\mathfrak{S}\hbar_n)\le d(\mathscr{K}\hbar_n,\mathfrak{S}\hbar_n).$$

Taking limit, we have

$$\lim_{n\to\infty} d(\mathfrak{S}\mathscr{K}\hbar_n,\mathscr{K}\mathfrak{S}\hbar_n)\le \lim_{n\to\infty} d(\mathscr{K}\hbar_n,\mathfrak{S}\hbar_n)=d(\eta,\eta)=0.$$

It follows that $\{\mathscr{K}\mathfrak{S}\hbar_n\}$ also converges to $\mathfrak{S}\eta$.

Now we show that $\mathfrak{S}(\eta) = \mathscr{K}(\eta)$.

Let us take

$$\begin{aligned}\big[1+pd(\mathfrak{S}\mathfrak{S}\hbar_n,\mathfrak{S}\eta)\big]d^2(\mathscr{K}\mathfrak{S}\hbar_n,\mathscr{K}\eta) \leq \\ & p\max\Bigg\{\frac{1}{2}\Bigg[d^2(\mathfrak{S}\mathfrak{S}\hbar_n,\mathscr{K}\mathfrak{S}\hbar_n)d(\mathfrak{S}\eta,\mathscr{K}\eta)+ \\ & d(\mathfrak{S}\mathfrak{S}\hbar_n,\mathscr{K}\mathfrak{S}\hbar_n)d^2(\mathfrak{S}\eta,\mathscr{K}\eta)\Bigg], \\ & d(\mathfrak{S}\mathfrak{S}\hbar_n,\mathscr{K}\mathfrak{S}\hbar_n)d(\mathfrak{S}\mathfrak{S}\hbar_n,\mathscr{K}\eta)d(\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\hbar_n), \\ & d(\mathfrak{S}\mathfrak{S}\hbar_n,\mathscr{K}\eta)d(\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\hbar_n)d(\mathfrak{S}\eta,\mathscr{K}\eta)\Bigg\}+ \\ & m(\mathfrak{S}\mathfrak{S}\hbar_n,\mathfrak{S}\eta)-\phi(m(\mathfrak{S}\mathfrak{S}\hbar_n,\mathfrak{S}\eta)),\end{aligned}$$

where

$$m(\mathfrak{S}\mathfrak{S}\hbar_n,\mathfrak{S}\eta) = \max\left\{\begin{array}{r} d^2(\mathfrak{S}\mathfrak{S}\hbar_n,\mathfrak{S}\eta), d(\mathfrak{S}\mathfrak{S}\hbar_n,\mathscr{K}\mathfrak{S}\hbar_n)d(\mathfrak{S}\eta,\mathscr{K}\eta), \\ d(\mathfrak{S}\mathfrak{S}\hbar_n,\mathscr{K}\eta)d(\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\hbar_n), \\ \frac{1}{2}\Big[d(\mathfrak{S}\mathfrak{S}\hbar_n,\mathscr{K}\mathfrak{S}\hbar_n)d(\mathfrak{S}\mathfrak{S}\hbar_n,\mathscr{K}\eta)+ \\ d(\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\hbar_n)d(\mathfrak{S}\eta,\mathscr{K}\eta)\Big] \end{array}\right\}.$$

Letting $n\to\infty$, we have

$$m(\mathfrak{S}\mathfrak{S}\hbar_n,\mathfrak{S}\eta) = \max\{0,0,0\} = 0.$$

Also we get

$$[1+pd(\mathfrak{S}\eta,\mathfrak{S}\eta)]d^2(\mathfrak{S}\eta,\mathscr{K}\eta) \leq p\max\{0,0,0\}+0-\phi(0).$$

Therefore,

$$d^2(\mathfrak{S}\eta,\mathscr{K}\eta) \leq 0$$

and hence $\mathfrak{S}\eta = \mathscr{K}\eta$. Moreover, $\mathfrak{S}\mathfrak{S}(\eta) = \mathfrak{S}\mathscr{K}(\eta)$.

Now put $\hbar = \mathfrak{S}\eta$ and $\vartheta = \eta$ in (3) we get

$$\big[1+pd(\mathfrak{S}\mathfrak{S}\eta,\mathfrak{S}\eta)\big]d^2(\mathscr{K}\mathfrak{S}\eta,\mathscr{K}\eta) \leq$$

$$p\max\Bigg\{\frac{1}{2}\Bigg[d^2(\mathfrak{S}\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\eta)d(\mathfrak{S}\eta,\mathscr{K}\eta)+ d(\mathfrak{S}\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\eta)d^2(\mathfrak{S}\eta,\mathscr{K}\eta)\Bigg], d(\mathfrak{S}\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\eta)d(\mathfrak{S}\mathfrak{S}\eta,\mathscr{K}\eta)d(\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\eta), d(\mathfrak{S}\mathfrak{S}\eta,\mathscr{K}\eta)d(\mathfrak{S}\eta,T\mathfrak{S}\eta)d(\mathfrak{S}\eta,\mathscr{K}\eta)\Bigg\}+ m(\mathfrak{S}\mathfrak{S}\eta,\mathfrak{S}\eta)-\phi(m(\mathfrak{S}\mathfrak{S}\eta,\mathfrak{S}\eta)),$$

where

$$m(\mathfrak{S}\mathfrak{S}\eta,\mathfrak{S}\eta)=\max\left\{\begin{array}{r} d^2(\mathfrak{S}\mathfrak{S}\eta,\mathfrak{S}\eta), d(\mathfrak{S}\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\eta)d(\mathfrak{S}\eta,\mathscr{K}\eta),\\ d(\mathfrak{S}\mathfrak{S}\eta,\mathscr{K}\eta)d(\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\eta),\\ \frac{1}{2}\Big[d(\mathfrak{S}\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\eta)d(\mathfrak{S}\mathfrak{S}\eta,\mathscr{K}\eta)+\\ d(\mathfrak{S}\eta,\mathscr{K}\mathfrak{S}\eta)d(\mathfrak{S}\eta,\mathscr{K}\eta)\Big]\end{array}\right\}.$$

Now by using $\mathfrak{S}\mathfrak{S}\eta=\mathfrak{S}\mathscr{K}\eta=\mathscr{K}\mathfrak{S}\eta$ and $\mathfrak{S}\eta=\mathscr{K}\eta$, we have

$$[1+pd(\mathfrak{S}\mathfrak{S}\eta,\mathfrak{S}\eta)]d^2(\mathfrak{S}\mathfrak{S}\eta,\mathscr{K}\eta)\le p\max\{0,0,0\}+m(\mathfrak{S}\mathfrak{S}\eta,\mathfrak{S}\eta)-\phi(m(\mathfrak{S}\mathfrak{S}\eta,\mathfrak{S}\eta)),$$

i.e., $d^2(\mathfrak{S}\mathfrak{S}\eta,\mathfrak{S}\eta)=0$ and $\mathfrak{S}\mathfrak{S}\eta=\mathfrak{S}\eta$. So, we have $\mathfrak{S}\eta=\mathfrak{S}\mathfrak{S}\eta=\mathfrak{S}\mathscr{K}\eta=\mathscr{K}\mathfrak{S}\eta$. Therefore, $\mathfrak{S}\eta$ is a common fixed point of $\mathfrak{S}$ and $\mathscr{K}$.

Uniqueness:

Suppose $\hbar\neq\vartheta$ be two fixed point of $\mathfrak{S}$ and $\mathscr{K}$. Therefore $\hbar=\mathfrak{S}\hbar=\mathscr{K}\hbar$ and $\vartheta=\mathfrak{S}\vartheta=\mathscr{K}\vartheta$. From (3) we have

$$[1+pd(\hbar,\vartheta)]d^2(\hbar,\vartheta)\le p\max\{0,0,0\}+m(\hbar,\vartheta)-\phi(m(\hbar,\vartheta))$$

i.e., $d^2(\hbar,\vartheta)\le 0$ i.e., $d^2(\hbar,\vartheta)=0$. Therefore, $\hbar=\vartheta$. This completes the proof.

If we put $p=0$ in Theorem 2.5 we have the required result of Rhoades [6].

Corollary 2.6. *Let $\mathfrak{S}$ and $\mathscr{K}$ be mappings of a complete metric space (X,d) into itself satisfying (C_1), (C_2) and the following conditions:*

$$d^2(\mathscr{K}\hbar,\mathscr{K}\vartheta)\le m(\mathfrak{S}\hbar,\mathfrak{S}\vartheta)-\phi(m(\mathfrak{S}\hbar,\mathfrak{S}\vartheta)),$$

where

$$m(\mathfrak{S}\hbar,\mathfrak{S}\vartheta)=\max\left\{\begin{array}{r} d^2(\mathfrak{S}\hbar,\mathfrak{S}\vartheta), d(\mathfrak{S}\hbar,\mathscr{K}\hbar)d(\mathfrak{S}\vartheta,\mathscr{K}\vartheta),\\ d(\mathfrak{S}\hbar,\mathscr{K}\vartheta)d(\mathfrak{S}\vartheta,\mathscr{K}\hbar),\\ \frac{1}{2}\Big[d(\mathfrak{S}\hbar,\mathscr{K}\hbar)d(\mathfrak{S}\hbar,\mathscr{K}\vartheta)+\\ d(\mathfrak{S}\vartheta,\mathscr{K}\hbar)d(\mathfrak{S}\vartheta,\mathscr{K}\vartheta)\Big] \end{array}\right\}, \tag{16}$$

$\hbar,\vartheta\in X$ and $\phi:[0,\infty)\to[0,\infty)$ is a continuous function with $\phi(t)=0\Leftrightarrow t=0$ and $\phi(t)>0$ for each $t>0$. Then $\mathfrak{S}$ and $\mathscr{K}$ have a unique common fixed point in X provided $\mathfrak{S}$ and $\mathscr{K}$ weakly commute on X.

Remark 2.7. On putting $\mathfrak{S}=I$ (Identity map) in Theorem 2.5, we get the result of Murthy and Prasad [5, Theorem 3] and since commutative mappings are weak commutative mappings but not conversely, our result generalize the result of Jain et al. [4, Theorem 2.2].

Example 2.8. Let $X=\{0,1,2,3\}$ with usual metric space $d(\hbar,\vartheta)=|\hbar-\vartheta|$ and define $\mathscr{K},\mathfrak{S}:X\to X$ by

$$\mathfrak{S}(\hbar)=\begin{cases}1, & \hbar=0,\\ 2, & \hbar=1\ \ or\ 2\\ 3, & \hbar=3,\end{cases}\qquad\text{and}\qquad \mathscr{K}(\hbar)=\begin{cases}2, & \hbar=0\ \ or\ 3\\ 1, & \hbar=1\ \ or\ 2\end{cases}$$

and define $\phi:[0,\infty)\to[0,\infty)$ by $\phi(t)=\frac{t}{3}$. For value of $p\geq\frac{1}{3}$ and $\hbar,\vartheta\in X$, it is easy to verify the inequality (3). Hence the Theorem 2.5 holds well.

We further generalize the above result for two pairs of weakly commuting mappings:

Theorem 2.9. *Let (X,d) be a complete metric space. Let $\mathfrak{S}$, $\mathscr{K}$, $\mathscr{P}$ and $\mathscr{Q}$ be four mappings of a complete metric space (X,d) into itself satisfying the following conditions:*

(C_3) $\mathfrak{S}(X)\subset\mathscr{Q}(X)$, $\mathscr{K}(X)\subset\mathscr{P}(X)$;

(C_4) *The pairs $(\mathscr{P},\mathfrak{S})$ and $(\mathscr{Q},\mathscr{K})$ are weakly commuting;*

(C_5) *One of $\mathfrak{S}$, $\mathscr{K}$, $\mathscr{P}$ and $\mathscr{Q}$ is continuous;*

(C_6)

$$[1+pd(\mathscr{P}\hbar,\mathscr{Q}\vartheta)]d^2(\mathfrak{S}\hbar,\mathscr{K}\vartheta)\leq$$

$$p\max\Big\{\frac{1}{2}\Big[d^2(\mathscr{P}\hbar,\mathfrak{S}\hbar)d(\mathscr{Q}\vartheta,\mathscr{K}\vartheta)+d(\mathscr{P}\hbar,\mathfrak{S}\hbar)d^2(\mathscr{Q}\vartheta,\mathscr{K}\vartheta)\Big],$$

$$d(\mathscr{P}\hbar,\mathfrak{S}\hbar)d(\mathscr{P}\hbar,\mathscr{K}\vartheta)d(\mathscr{Q}\vartheta,\mathfrak{S}\hbar),d(\mathscr{P}\hbar,\mathscr{K}\vartheta)d(\mathscr{Q}\vartheta,\mathfrak{S}\hbar)d(\mathscr{Q}\vartheta,\mathscr{K}\vartheta)\Big\}+$$

$$m(\mathscr{P}\hbar,\mathscr{Q}\vartheta)-\phi(m(\mathscr{P}\hbar,\mathscr{Q}\vartheta)), \tag{17}$$

where

$$m(\mathscr{P}\hbar,\mathscr{Q}\vartheta)=\max\left\{\begin{array}{r} d^2(\mathscr{P}\hbar,\mathscr{Q}\vartheta),d(\mathscr{P}\hbar,\mathfrak{S}\hbar)d(\mathscr{Q}\vartheta,\mathscr{K}\vartheta),\\ d(\mathscr{P}\hbar,\mathscr{K}\vartheta)d(\mathscr{Q}\vartheta,\mathfrak{S}\hbar),\\ \frac{1}{2}\Big[d(\mathscr{P}\hbar,\mathfrak{S}\hbar)d(\mathscr{P}\hbar,\mathscr{K}\vartheta)+d(\mathscr{Q}\vartheta,\mathfrak{S}\hbar)d(\mathscr{Q}\vartheta,\mathscr{K}\vartheta)\Big]\end{array}\right\}, \tag{18}$$

$p\in(0,\frac{1}{3}]$ *is a real number and* $\phi:[0,\infty)\to[0,\infty)$ *is a continuous function with* $\phi(t)=0\Leftrightarrow t=0$ *and* $\phi(t)>0$ *for each* $t>0$.
Then $\mathfrak{S}$, $\mathscr{K}$, $\mathscr{P}$ *and* $\mathscr{Q}$ *have a unique point in* X.

Proof. Let $\hbar_0\in X$ be an arbitrary point. From (C_3) we can find $\hbar_1$ such that $\mathfrak{S}(\hbar_0)=\mathscr{Q}(\hbar_1)=\vartheta_0$. For this $\hbar_1$, we can find $\hbar_2\in X$ such that $\mathscr{K}(\hbar_1)=\mathscr{P}(\hbar_2)=\vartheta_1$. Continuing in this way, one can construct a sequence such that

$$\vartheta_{2n}=\mathfrak{S}(\hbar_{2n})=\mathscr{Q}(\hbar_{2n+1}),\quad \vartheta_{2n+1}=\mathscr{K}(\hbar_{2n+1})=\mathscr{P}(\hbar_{2n+2}),\quad for\ each\ \ n\geq 0. \tag{19}$$

For brevity, we write $\gamma_n=d(\vartheta_n,\vartheta_{n+1})$.
Firstly we prove that $\{\gamma_n\}$ is non increasing sequence and converges to zero.
If n is even, taking $\hbar=\hbar_{2n}$ and $\vartheta=\hbar_{2n+1}$ in (C_6), we get

$$[1+pd(\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})]d^2(\mathfrak{S}\hbar_{2n},\mathscr{K}\hbar_{2n+1})\leq$$

$$p\max\Big\{\frac{1}{2}\Big[d^2(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})+$$

$$d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n}),d^2(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\Big],$$

$$d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\hbar_{2n}),$$

$$d(\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\Big\}+$$

$$m(\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})-\phi(m(\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})),$$

where

$$m(\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})=\max\left\{\begin{array}{r} d^2(\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1}),d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1}),\\ d(\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\hbar_{2n}),\\ \frac{1}{2}\Big[d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})+\\ d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\Big] \end{array}\right\}.$$

Using (19), we have

$$\begin{aligned}[1+pd(\vartheta_{2n-1},\vartheta_{2n})]d^2(\vartheta_{2n},\vartheta_{2n+1})\leq&\\ & p\max\left\{\frac{1}{2}\Big[d^2(\vartheta_{2n-1},\vartheta_{2n})d(\vartheta_{2n},\vartheta_{2n+1})\right.\\ &+d(\vartheta_{2n-1},\vartheta_{2n})d^2(\vartheta_{2n},\vartheta_{2n+1})\Big],\\ &d(\vartheta_{2n-1},\vartheta_{2n})d(\vartheta_{2n-1},\vartheta_{2n+1})d(\vartheta_{2n},\vartheta_{2n}),\\ &\left.d(\vartheta_{2n-1},\vartheta_{2n+1})d(\vartheta_{2n},\vartheta_{2n})d(\vartheta_{2n},\vartheta_{2n+1})\right\}+\\ &m(\vartheta_{2n-1},\vartheta_{2n})-\phi(m(\vartheta_{2n-1},\vartheta_{2n})),\end{aligned}$$

where

$$m(\vartheta_{2n-1},\vartheta_{2n})=\max\left\{\begin{array}{r} d^2(\vartheta_{2n-1},\vartheta_{2n}),d(\vartheta_{2n-1},\vartheta_{2n})d(\vartheta_{2n},\vartheta_{2n+1}),\\ d(\vartheta_{2n-1},\vartheta_{2n+1})d(\vartheta_{2n},\vartheta_{2n}),\\ \frac{1}{2}\Big[d(\vartheta_{2n-1},\vartheta_{2n})d(\vartheta_{2n-1},\vartheta_{2n+1})+\\ d(\vartheta_{2n},\vartheta_{2n})d(\vartheta_{2n},\vartheta_{2n+1})\Big] \end{array}\right\}.$$

On using $\gamma_{2n}=d(\vartheta_{2n},\vartheta_{2n+1})$, we have

$$\begin{aligned}(20)\qquad [1+p\gamma_{2n-1}]\gamma_{2n}^2\leq p\max\left\{\frac{1}{2}\Big[\gamma_{2n-1}^2\gamma_{2n}+\gamma_{2n-1}\gamma_{2n}^2\Big],0,0\right\}+&\\ m(\vartheta_{2n-1},\vartheta_{2n})-\phi(m(\vartheta_{2n-1},\vartheta_{2n})),&\end{aligned}$$

where

$$m(\vartheta_{2n-1},\vartheta_{2n}) = \max\left\{\gamma_{2n-1}^2, \gamma_{2n-1}\gamma_{2n}, 0, \frac{1}{2}\left[\gamma_{2n-1}d(\vartheta_{2n-1},\vartheta_{2n+1})+0\right]\right\}. \tag{21}$$

By the triangular inequality, we get

$$\begin{aligned} d(\vartheta_{2n-1},\vartheta_{2n+1}) &\leq d(\vartheta_{2n-1},\vartheta_{2n}) + d(\vartheta_{2n},\vartheta_{2n+1}) \\ &= \gamma_{2n-1} + \gamma_{2n} \end{aligned}$$

and putting in (21), we have

$$m(\vartheta_{2n-1},\vartheta_{2n}) \leq m(\hbar,\vartheta) = \max\left\{\gamma_{2n-1}^2, \gamma_{2n-1}\gamma_{2n}, 0, \frac{1}{2}\left[\gamma_{2n-1}(\gamma_{2n-1}+\gamma_{2n}), 0\right]\right\}.$$

If $\gamma_{2n-1} < \gamma_{2n}$, then (20) reduces to

$$p\gamma_{2n}^2 < p\gamma_{2n}^2 - \phi\gamma_{2n}^2,$$

which is a contradiction.

Therefore, $\gamma_{2n}^2 \leq \gamma_{2n-1}^2$ implies that $\gamma_{2n} \leq \gamma_{2n-1}$.

In a similar way, if n is odd, then we obtain $\gamma_{2n+1} \leq \gamma_{2n}$.

It follows that the sequence $\{\gamma_n\}$ is decreasing.

Let $\lim_{n\to\infty}\gamma_{2n} = r$, for some $r \geq 0$.

Suppose $r > 0$; then from inequality (C_6), we have

$$\begin{aligned} &[1+pd(\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})]d^2(\mathfrak{S}\hbar_{2n},\mathscr{K}\hbar_{2n+1}) \leq \\ &\qquad p\max\left\{\frac{1}{2}\left[d^2(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})+\right.\right. \\ &\qquad \left. d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d^2(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\right], \\ &\qquad d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\hbar_{2n}), \\ &\qquad \left. d(\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\right\}+ \\ &\qquad m(\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1}) - \phi(m(\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})), \end{aligned}$$

where

$$m(\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1}) = \max\left\{\begin{array}{r} d^2(\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1}), d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1}), \\ d(\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\hbar_{2n}), \\ \frac{1}{2}\Big[d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})+ \\ d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\Big] \end{array}\right\}.$$

By using (19), triangular inequality, property of ϕ and taking limits $n \to \infty$, we get

$$[1+pr]r^2 \leq pr^3 + r^2 - \phi(r^2).$$

Then $\phi(r^2) \leq 0$, since r is positive, then by property of ϕ, we get $r = 0$, we conclude that

$$\lim_{n\to\infty} \gamma_{2n} = \lim_{n\to\infty} d(\vartheta_{2n},\vartheta_{2n+1}) = r = 0. \tag{22}$$

Claim: $\{\vartheta_n\}$ is a Cauchy sequence.

Let $\{\vartheta_n\}$ be not a Cauchy sequence. For given $\varepsilon > 0$, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$, such that

$$d(\vartheta_{m(k)},\vartheta_{n(k)}) \geq \varepsilon, \quad d(\vartheta_{m(k)},\vartheta_{n(k)-1)}) < \varepsilon.$$

Now

$$\varepsilon \leq d(\vartheta_{m(k)},\vartheta_{n(k)}) \leq d(\vartheta_{m(k)},\vartheta_{n(k)-1)}) + d(\vartheta_{n(k)-1)},\vartheta_{n(k)}).$$

Letting $k \to \infty$, we get

$$\lim_{n\to\infty} d(\vartheta_{m(k)},\vartheta_{n(k)}) = \varepsilon. \tag{23}$$

Now from the triangular inequality we have,

$$|d(\vartheta_{n(k)},\vartheta_{m(k)+1}) - d(\vartheta_{m(k)},\vartheta_{n(k)})| \leq d(\vartheta_{m(k)},\vartheta_{m(k)+1}).$$

Taking limits as $k \to \infty$ and using (22) and (23), we have

$$\lim_{n\to\infty} d(y_{m(k)},\vartheta_{m(k)+1}) = \varepsilon. \tag{24}$$

Again from the triangular inequality, we have

$$|d(\vartheta_{m(k)},\vartheta_{n(k)+1}) - d(\vartheta_{m(k)},\vartheta_{n(k)})| \leq d(\vartheta_{n(k)},\vartheta_{n(k)+1}).$$

Taking limits as $k \to \infty$ and using (22) and (23), we have

$$\lim_{n\to\infty} d(\vartheta_{m(k)}, \vartheta_{n(k)+1}) = \varepsilon. \tag{25}$$

Similarly using triangular inequality, we have

$$|d(\vartheta_{m(k)+1}, \vartheta_{n(k)+1}) - d(\vartheta_{m(k)}, \vartheta_{n(k)})| \leq d(\vartheta_{m(k)}, \vartheta_{m(k)+1}) + d(\vartheta_{n(k)}, \vartheta_{n(k)+1}).$$

Taking limits as $k \to \infty$ in the above inequality and using (22) and (23), we have

$$\lim_{n\to\infty} d(\vartheta_{n(k)+1}, \vartheta_{m(k)+1}) = \varepsilon. \tag{26}$$

On putting $\hbar = \hbar_{m(k)}$ and $\vartheta = \hbar_{n(k)}$ in inequality (C_6), we get

$$\begin{aligned}\big[1 + pd(\mathscr{P}\hbar_{m(k)}, \mathscr{Q}\hbar_{n(k)})\big] d^2(\mathfrak{S}\hbar_{m(k)}, \mathscr{K}\hbar_{n(k)}) \leq \\ p \max\Bigg\{\frac{1}{2}\Bigg[& d^2(\mathscr{P}\hbar_{m(k)}, \mathfrak{S}\hbar_{m(k)}) d(\mathscr{Q}\hbar_{n(k)}, \mathscr{K}\hbar_{n(k)}) + \\ & d(\mathscr{P}\hbar_{m(k)}, \mathfrak{S}\hbar_{m(k)}) d^2(\mathscr{Q}\hbar_{n(k)}, \mathscr{K}\hbar_{n(k)})\Bigg], \\ & d(\mathscr{P}\hbar_{m(k)}, \mathfrak{S}\hbar_{m(k)}) d(\mathscr{P}\hbar_{m(k)}, \mathscr{K}\hbar_{n(k)}) d(\mathscr{Q}\hbar_{n(k)}, \mathfrak{S}\hbar_{m(k)}), \\ & d(\mathscr{P}\hbar_{m(k)}, \mathscr{K}\hbar_{n(k)}) d(\mathscr{Q}\hbar_{n(k)}, \mathfrak{S}\hbar_{m(k)}) d(\mathscr{Q}\hbar_{n(k)}, \mathscr{K}\hbar_{n(k)})\Bigg\} + \\ & m(\mathscr{P}\hbar_{m(k)}, \mathscr{Q}\hbar_{n(k)}) - \phi(m(\mathscr{P}\hbar_{m(k)}, \mathscr{Q}\hbar_{n(k)})),\end{aligned}$$

where

$$m(\mathscr{P}\hbar_{m(k)}, \mathscr{Q}\hbar_{n(k)}) = \max\left\{\begin{array}{r} d^2(\mathscr{P}\hbar_{m(k)}, \mathscr{Q}\hbar_{n(k)}), \\ d(\mathscr{P}\hbar_{m(k)}, \mathfrak{S}\hbar_{m(k)}) d(\mathscr{Q}\hbar_{n(k)}, \mathscr{K}\hbar_{n(k)}), \\ d(\mathscr{P}\hbar_{m(k)}, \mathscr{K}\hbar_{n(k)}) d(\mathscr{Q}\hbar_{n(k)}, \mathfrak{S}\hbar_{m(k)}), \\ \frac{1}{2}\Big[d(\mathscr{P}\hbar_{m(k)}, \mathfrak{S}\hbar_{m(k)}) d(\mathscr{P}\hbar_{m(k)}, \mathscr{K}\hbar_{n(k)}) + \\ d(\mathscr{Q}\hbar_{n(k)}, \mathfrak{S}\hbar_{m(k)}) d(\mathscr{Q}\hbar_{n(k)}, \mathscr{K}\hbar_{n(k)})\Big] \end{array}\right\}.$$

Now using (19) we obtain,

$$\begin{aligned}\big[1 + pd(\vartheta_{m(k)-1}, \vartheta_{n(k)-1})\big] d^2(\vartheta_{m(k)}, \vartheta_{n(k)}) \leq \\ p \max\Bigg\{\frac{1}{2}\Bigg[& d^2(\vartheta_{m(k)-1}, \vartheta_{m(k)}) d(\vartheta_{n(k)-1}, \vartheta_{n(k)}) +\end{aligned}$$

$$d(\vartheta_{m(k)-1},\vartheta_{m(k)})d^2(\vartheta_{n(k)-1},\vartheta_{n(k)})\Big],$$

$$d(\vartheta_{m(k)-1},\vartheta_{m(k)})d(\vartheta_{m(k)-1},\vartheta_{n(k)})d(\vartheta_{n(k)-1},\vartheta_{m(k)}),$$

$$d(\vartheta_{m(k)-1},\vartheta_{n(k)})d(\vartheta_{n(k)-1},\vartheta_{m(k)})d(\vartheta_{n(k)-1},\vartheta_{n(k)})\Big\}+$$

$$m(\mathscr{P}\hbar_{m(k)},\mathscr{Q}\hbar_{n(k)})-\phi(m(\mathscr{P}\hbar_{m(k)},\mathscr{Q}\hbar_{n(k)})),$$

where

$$m(\mathscr{P}\hbar_{m(k)},\mathscr{Q}\hbar_{n(k)})=\max\left\{\begin{array}{r} d^2(\vartheta_{m(k)-1},\vartheta_{n(k)-1}),\\ d(\vartheta_{m(k)-1},\vartheta_{m(k)})d(\vartheta_{n(k)-1},\vartheta_{n(k)}),\\ d(\vartheta_{m(k)-1},\vartheta_{n(k)})d(\vartheta_{n(k)-1},\vartheta_{m(k)}),\\ \frac{1}{2}\Big[d(\vartheta_{m(k)-1},\vartheta_{m(k)})d(\vartheta_{m(k)-1},\vartheta_{n(k)})+\\ d(\vartheta_{n(k)-1},\vartheta_{m(k)})d(\vartheta_{n(k)-1},\vartheta_{n(k)})\Big] \end{array}\right\}.$$

Letting $k\to\infty$, we get

$$\begin{aligned}[1+p\varepsilon]\varepsilon^2 &\le p\max\{\frac{1}{2}[0+0],0,0\}+\varepsilon^2-\phi(\varepsilon^2)\\ &=\varepsilon^2-\phi(\varepsilon^2),\end{aligned}$$

a contradiction. Thus $\{\vartheta_n\}$ is a Cauchy sequence in X. Since (X,d) is a complete metric space. Therefore, $\{\vartheta_n\}$ converges to a point η as $n\to\infty$. Consequently the subsequences $\{\mathfrak{S}\hbar_{2n}\},\{\mathscr{P}\hbar_{2n}\},\{\mathscr{K}\hbar_{2n+1}\}$ and $\{\mathscr{Q}\hbar_{2n+1}\}$ also converges to the same point η.

Case1. Suppose that $\mathscr{P}$ is continuous. Then $\{\mathscr{P}\mathscr{P}\hbar_{2n}\}$ and $\{\mathscr{P}\mathfrak{S}\hbar_{2n}\}$ converges to $\mathscr{P}\eta$ as $n\to\infty$. Since the mappings $\mathscr{P}$ and $\mathfrak{S}$ are weakly commuting on X, then

$$d(\mathfrak{S}\mathscr{P}\hbar_{2n},\mathscr{P}\mathfrak{S}\hbar_{2n})=d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathscr{P}\hbar_{2n})\le d(\mathfrak{S}\hbar_{2n},\mathscr{P}\hbar_{2n}).$$

Let $n\to\infty$, we get $\lim\limits_{n\to\infty}d(\mathfrak{S}\mathscr{P}\hbar_{2n},\mathscr{P}\eta)\le d(\eta,\eta)=0, i.e., \lim\limits_{n\to\infty}\mathfrak{S}\mathscr{P}\hbar_{2n}=\mathscr{P}\eta$.

Now we prove that $\eta=\mathscr{P}\eta$. On putting $\hbar=\mathscr{P}\hbar_{2n}$ and $\vartheta=\hbar_{2n+1}$ in (C_6),

we get

$$\big[1+pd(\mathscr{P}\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})\big]d^2(\mathfrak{S}\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})\le$$

$$p\max\left\{\frac{1}{2}\Big[d^2(\mathscr{P}\mathscr{P}\hbar_{2n},\mathfrak{S}\mathscr{P}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})+\right.$$

$$d(\mathscr{P}\mathscr{P}\hbar_{2n},\mathfrak{S}\mathscr{P}\hbar_{2n})d^2(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\Big],$$

$$d(\mathscr{P}\mathscr{P}\hbar_{2n},\mathfrak{S}\mathscr{P}\hbar_{2n})d(\mathscr{P}\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\mathscr{P}\hbar_{2n}),$$

$$d(\mathscr{P}\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\mathscr{P}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\Big\}+$$

$$m(\mathscr{P}\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})-\phi(m(\mathscr{P}\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})),$$

where

$$m(\mathscr{P}\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})=\max\left\{\begin{array}{r} d^2(\mathscr{P}\mathscr{P}\hbar_{2n},\mathscr{Q}\hbar_{2n+1}),\\ d(\mathscr{P}\mathscr{P}\hbar_{2n},\mathfrak{S}\mathscr{P}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1}),\\ d(\mathscr{P}\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\mathscr{P}\hbar_{2n}),\\ \frac{1}{2}\Big[d(\mathscr{P}\mathscr{P}\hbar_{2n},\mathfrak{S}\mathscr{P}\hbar_{2n})d(\mathscr{P}\mathscr{P}\hbar_{2n},\mathscr{K}\hbar_{2n+1})+\\ d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\mathscr{P}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\Big]\end{array}\right\}.$$

Letting $n\to\infty$, we get

$$[1+pd(\mathscr{P}\eta,\eta)]d^2(\mathscr{P}\eta,\eta)\leq p\max\{\frac{1}{2}[0+0],0,0\}+d^2(\mathscr{P}\eta,\eta)-\phi d^2(\mathscr{P}\eta,\eta)$$
$$=d^2(\mathscr{P}\eta,\eta)-\phi d^2(\mathscr{P}\eta,\eta)$$

Thus we get $d^2(\mathscr{P}\eta,\eta)=0$ and hence $\mathscr{P}\eta=\eta$.

Next we will show that $\mathfrak{S}\eta=\eta$. On putting $\hbar=\eta$ and $\vartheta=\hbar_{2n+1}$ in (C_6),

$$\big[1+pd(\mathscr{P}\eta,\mathscr{Q}\hbar_{2n+1})\big]d^2(\mathfrak{S}\eta,\mathscr{K}\hbar_{2n+1})\leq$$

$$p\max\left\{\frac{1}{2}\Big[d^2(\mathscr{P}\eta,\mathfrak{S}\eta)d(\eta,\eta)+d(\mathscr{P}\eta,\mathfrak{S}\eta)d^2(\eta,\eta)\Big],\right.$$

$$\left.d(\mathscr{P}\eta,\mathfrak{S}\eta)d(\mathscr{P}\eta,\eta)d(\eta,\mathfrak{S}\eta),d(\mathscr{P}\eta,\eta)d(\eta,\mathfrak{S}\eta)d(\eta,\eta)\right\}+$$

$$m(\mathscr{P}\eta,\eta)-\phi(m(\mathscr{P}\eta,\eta)),$$

where

$$m(\mathscr{P}\eta,\eta)=\max\left\{\begin{array}{r} d^2(\mathscr{P}\eta,\eta),d(\mathscr{P}\eta,\mathfrak{S}\eta)d(\eta,\eta),d(\mathscr{P}\eta,\eta)d(\eta,\mathfrak{S}\eta),\\ \frac{1}{2}\Big[d(\mathscr{P}\eta,\mathfrak{S}v)d(\mathscr{P}\eta,\eta)+d(\eta,\mathfrak{S}\eta)d(\eta,\eta)\Big]\end{array}\right\}=0.$$

Therefore, by letting $n \to \infty$

$$[1+pd(\eta,\eta)]d^2(\mathfrak{S}\eta,\eta) \leq p\max\{\frac{1}{2}[0+0],0,0\}+0-\phi(0).$$

Thus $d^2(\mathfrak{S}\eta,\eta)=0$ giving $\mathfrak{S}\eta=\eta$. Since $\mathfrak{S}(X)\subset\mathscr{Q}(X)$, there exists a point $u\in X$ such that $\eta=S\eta=\mathscr{Q}u$. Now we show that $\eta=\mathscr{K}u$. Taking $\hbar=\eta$ and $\vartheta=u$ in (C_6), we get

$$\begin{aligned}\big[1+pd(\mathscr{P}\eta,\mathscr{Q}u)\big]d^2(\mathfrak{S}\eta,\mathscr{K}u) \leq &\\ &p\max\left\{\frac{1}{2}\left[d^2(\mathscr{P}\eta,\mathfrak{S}\eta)d(\mathscr{Q}u,\mathscr{K}u)+d(\mathscr{P}\eta,\mathfrak{S}\eta),d^2(\mathscr{Q}u,\mathscr{K}u)\right],\right.\\ &\left.d(\mathscr{P}\eta,\mathfrak{S}\eta)d(\mathscr{P}\eta,\mathscr{K}u)d(\mathscr{Q}u,\mathfrak{S}\eta),d(\mathscr{P}\eta,\mathscr{K}u)d(\mathscr{Q}u,\mathfrak{S}\eta)d(\mathscr{Q}u,\mathscr{K}u)\right\}+\\ &m(\mathscr{P}\eta,\mathscr{Q}u)-\phi(m(\mathscr{P}\eta,\mathscr{Q}u)),\end{aligned}$$

where

$$m(\mathscr{P}\eta,\mathscr{Q}u)=\max\left\{\begin{array}{r}d^2(\mathscr{P}\eta,\mathscr{Q}u),\\ d(\mathscr{P}\eta,\mathfrak{S}\eta)d(\mathscr{Q}u,\mathscr{K}u),d(\mathscr{P}\eta,\mathscr{K}u)d(\mathscr{Q}u,\mathfrak{S}\eta),\\ \frac{1}{2}\Big[d(\mathscr{P}\eta,\mathfrak{S}\eta)d(\mathscr{P}\eta,\mathscr{K}u)+d(\mathscr{Q}u,\mathfrak{S}\eta)d(\mathscr{Q}u,\mathscr{K}u)\Big]\end{array}\right\}$$

$$=\max\left\{\begin{array}{r}d^2(\eta,\eta),d(\eta,\eta)d(\eta,\mathscr{K}u),\\ d(\eta,\mathscr{K}u)d(\eta,\eta),\\ \frac{1}{2}\Big[d(\eta,\eta)d(\mathscr{P}\eta,\mathscr{K}u)+d(\eta,\eta)d(\eta,\mathscr{K}u)\Big]\end{array}\right\}=0.$$

On solving, we get

$$\begin{aligned}\big[1+pd(\eta,\eta)\big]d^2(\eta,\mathscr{K}u) \leq &\\ &p\max\left\{\frac{1}{2}\left[d^2(\eta,\eta)d(\eta,\mathscr{K}u)+d(\eta,\eta),d^2(\eta,\mathscr{K}u)\right],\right.\\ &\left.d(\eta,\eta)d(\eta,\mathscr{K}u)d(\eta,\eta),d(\eta,\mathscr{K}u)d(\eta,\eta)d(\eta,\mathscr{K}u)\right\}+0-\phi(0).\end{aligned}$$

This implies that $\eta=\mathscr{K}u$. Since the pair $(\mathscr{Q},\mathscr{K})$ is weak commutative, then

$$d(\mathscr{Q}\eta,\mathscr{K}\eta)=d(\mathscr{Q}\mathscr{K}u,\mathscr{K}\mathscr{Q}u)\leq d(\mathscr{Q}u,\mathscr{K}u)=d(\eta,\eta)=0.$$

So $\mathscr{Q}\eta=\mathscr{K}\eta$.

Also we have

$$\begin{aligned}&\big[1+pd(\mathscr{P}\eta,\mathscr{Q}\eta)\big]d^2(\mathfrak{S}\eta,\mathscr{K}\eta)\leq\\&\qquad p\max\Bigg\{\frac{1}{2}\Bigg[d^2(\mathscr{P}\eta,\mathfrak{S}\eta)d(\mathscr{Q}\eta,\mathscr{K}\eta)+d(\mathscr{P}\eta,\mathfrak{S}\eta)d^2(\mathscr{Q}\eta,\mathscr{K}\eta)\Bigg],\\&\qquad d(\mathscr{P}\eta,\mathfrak{S}\eta)d(\mathscr{P}\eta,\mathscr{K}\eta)d(\mathscr{Q}\eta,\mathfrak{S}\eta),d(\mathscr{P}\eta,\mathscr{K}\eta)d(\mathscr{Q}\eta,\mathfrak{S}\eta)d(\mathscr{Q}\eta,\mathscr{K}\eta)\Bigg\}+\\&\qquad m(\mathscr{P}\eta,\mathscr{Q}\eta)-\phi(m(\mathscr{P}\eta,\mathscr{Q}\eta)),\end{aligned}$$

where

$$m(\mathscr{P}\eta,\mathscr{Q}\eta)=\max\left\{\begin{array}{r}d^2(\mathscr{P}\eta,\mathscr{Q}\eta),d(\mathscr{P}\eta,\mathfrak{S}\eta)d(\mathscr{Q}\eta,\mathscr{K}\eta),\\ d(\mathscr{P}\eta,\mathscr{K}\eta)d(\mathscr{Q}\eta,\mathscr{K}\eta),\\ \frac{1}{2}\Big[d(\mathscr{P}\eta,\mathfrak{S}\eta)d(\mathscr{P}\eta,\mathscr{K}\eta)+\\ d(\mathscr{Q}\eta,\mathfrak{S}\eta)d(\mathscr{Q}\eta,\mathscr{K}\eta)\Big]\end{array}\right\}=d^2(\eta,\mathscr{K}\eta).$$

Therefore, we get

$$[1+pd(\eta,\mathscr{K}\eta)]d^2(\eta,\mathscr{K}\eta)\leq p\max\{\frac{1}{2}[0+0],0,0\}+d^2(\eta,\mathscr{K}\eta)-\phi(d^2(\eta,\mathscr{K}\eta)).$$

This implies that $\eta=\mathscr{K}\eta$.

Case 2. Suppose that $\mathscr{Q}$ is continuous, we can obtain the same result by the way of Case 1.

Case 3. Suppose that $\mathfrak{S}$ is continuous. Then $\{\mathfrak{S}\mathfrak{S}\hbar_{2n}\}$ and $\{\mathfrak{S}\mathscr{P}\hbar_{2n}\}$ converges to $\mathfrak{S}\eta$ as $n\to\infty$. Since the mappings $\mathscr{P}$ and $\mathfrak{S}$ are weakly commuting on X, then

$$d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathscr{P}\hbar_{2n})\leq d(\mathfrak{S}\hbar_{2n},\mathscr{P}\hbar_{2n}).$$

Let $n\to\infty$, we get $\lim_{n\to\infty}d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\eta)\leq d(\eta,\eta)=0$, i.e., $\lim_{n\to\infty}\mathscr{P}\mathfrak{S}\hbar_{2n}=\mathfrak{S}\eta$.

Now we prove that $\eta=\mathfrak{S}\eta$. For this put $\hbar=\mathfrak{S}\hbar_{2n}$ and $\vartheta=\hbar_{2n+1}$ in inequality (C_6), we get

$$\begin{aligned}&\big[1+pd(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})\big]d^2(\mathfrak{S}\mathfrak{S}\hbar_{2n},\mathscr{K}\hbar_{2n+1})\leq\\&\qquad p\max\Bigg\{\frac{1}{2}\Bigg[d^2(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})+\\&\qquad d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathfrak{S}\hbar_{2n}),d^2(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\Bigg],\\&\qquad d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\mathfrak{S}\hbar_{2n}),\end{aligned}$$

$$d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\Big\}+$$

$$m(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})-\phi(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{Q}\hbar_{2n+1}),$$

where

$$m(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{Q}\hbar_{2n+1})=\max\left\{\begin{array}{r} d^2(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{Q}\hbar_{2n+1}),d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1}),\\ d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{K}\hbar_{2n+1})d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\mathfrak{S}\hbar_{2n}),\\ \frac{1}{2}\Big[d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{K}\hbar_{2n+1})+\\ d(\mathscr{Q}\hbar_{2n+1},\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar_{2n+1},\mathscr{K}\hbar_{2n+1})\Big]\end{array}\right\}.$$

Letting $n\to\infty$, we get

$$m(\mathfrak{S}\eta,\eta)=\max\left\{\begin{array}{c} d^2(\mathfrak{S}\eta,\eta),d(\mathfrak{S}\eta,\eta)d(\eta,\eta),d(\mathfrak{S}\eta,\eta)d(\eta,\mathfrak{S}\eta),\\ \frac{1}{2}\Big[d(\mathfrak{S}\eta,\mathfrak{S}\eta)d(\mathfrak{S}\eta,\eta)+d(\eta,\mathfrak{S}\eta)d(\eta,\eta)\Big]\end{array}\right\}=d^2(\mathfrak{S}\eta,\eta).$$

Therefore, we get

$$\begin{aligned}[1+pd(\mathfrak{S}\eta,\eta)]d^2(S\eta,\eta)&\le p\max\{\frac{1}{2}[0+0],0,0\}+d^2(\mathfrak{S}\eta,\eta)-\phi(d^2(\mathfrak{S}\eta,\eta))\\ &=d^2(\mathfrak{S}\eta,\eta)-\phi(d^2(\mathfrak{S}\eta,\eta)).\end{aligned}$$

Thus we get $d^2(\mathfrak{S}\eta,\eta)=0$ implies that $\mathfrak{S}\eta=\eta$. Since $\mathfrak{S}(X)\subset\mathscr{Q}(X)$ and hence there exists a point $v\in X$ such that $\eta=\mathfrak{S}\eta=\mathscr{Q}v$.

We claim that $\eta=\mathscr{K}v$.

For this we put $\hbar=\mathfrak{S}\hbar_{2n}$ and $\vartheta=v$ in (C_6) we get

$$\begin{aligned}\big[1+pd(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{Q}v)\big]d^2(\mathfrak{S}\mathfrak{S}\hbar_{2n},\mathscr{K}v)\le&\\ &p\max\Big\{\frac{1}{2}\Big[d^2(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{Q}v,\mathscr{K}v)+\\ &d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathfrak{S}\hbar_{2n})d^2(\mathscr{Q}v,\mathscr{K}v)\Big],\\ &d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{K}v)d(\mathscr{Q}v,\mathfrak{S}\mathfrak{S}\hbar_{2n}),\\ &d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{K}v)d(\mathscr{Q}v,\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{Q}v,\mathscr{K}v)\Big\}+\\ &m(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{Q}v)-\phi(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{Q}v),\end{aligned}$$

where

$$m(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{Q}v)=\max\left\{\begin{array}{r} d^2(\mathscr{P}\mathfrak{S}\hbar_{2n},Bv),d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{Q}v,\mathscr{K}v),\\ d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{K}v)d(\mathscr{Q}v,\mathfrak{S}\mathfrak{S}\hbar_{2n}),\\ \frac{1}{2}\Big[d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{P}\mathfrak{S}\hbar_{2n},\mathscr{K}v)+\\ d(\mathscr{Q}v,\mathfrak{S}\mathfrak{S}\hbar_{2n})d(\mathscr{Q}v,\mathscr{K}v)\Big]\end{array}\right\}.$$

Letting $n\to\infty$, we get

$$m(\eta,\mathscr{Q}v)=\max\left\{\begin{array}{c} d^2(\eta,\eta),d(\eta,\eta)d(\eta,\mathscr{K}v),d(\eta,\mathscr{K}v)d(\eta,\eta),\\ \frac{1}{2}\Big[d(\eta,\eta)d(\eta,\mathscr{K}v)+d(\eta,\eta)d(\eta,\mathscr{K}v)\Big]\end{array}\right\}=0.$$

Therefore, on solving

$$\begin{aligned}[1+pd(\eta,\eta)]d^2(\eta,\mathscr{K}v)\le p\max\Bigg\{&\frac{1}{2}\Big[d^2(\eta,\eta)d(\eta,\mathscr{K}v)+d(\eta,\eta)d^2(\eta,\mathscr{K}v)\Big],\\ &d(\eta,\eta)d(\eta,\mathscr{K}v)d(\eta,\eta),d(\eta,\mathscr{K}v)d(\eta,\eta)d(\eta,\mathscr{K}v)\Bigg\}+0-\phi(0).\end{aligned}$$

This implies that $\eta=\mathscr{K}v$. Since the pair $(\mathscr{Q},\mathscr{K})$ is weakly commuting on X, then

$$d(\mathscr{K}\mathscr{Q}v,\mathscr{Q}\mathscr{K}v)\le d(\mathscr{K}v,\mathscr{Q}v)=d(\eta,\eta)=0.$$

So $\mathscr{Q}\eta=\mathscr{K}\eta$.

Now put $\hbar=\hbar_{2n}$ and $\vartheta=\eta$ in (C_6)

$$\begin{aligned}[1+pd(\mathscr{P}\hbar_{2n},\mathscr{Q}\eta)]d^2(\mathfrak{S}\hbar_{2n},\mathscr{K}\eta)\le p\max\Bigg\{&\frac{1}{2}\Big[d^2(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\hbar,\mathscr{K}\eta)\\ &+d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d^2(\mathscr{Q}\eta,\mathscr{K}\eta)\Big],\\ &d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{P}\hbar_{2n},\mathscr{K}\eta)d(\mathscr{Q}\eta,\mathfrak{S}\hbar_{2n}),\\ &d(\mathscr{P}\hbar_{2n},\mathscr{K}\eta)d(\mathscr{Q}\eta,\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\eta,\mathscr{K}\eta)\Bigg\}\\ &+m(\mathscr{P}\hbar_{2n},\mathscr{Q}\eta)-\phi(m(\mathscr{P}\hbar_{2n},\mathscr{Q}\eta)),\end{aligned}$$

where

$$m(\mathscr{P}\hbar_{2n},\mathscr{Q}\eta)=\max\left\{\begin{array}{r} d^2(\mathscr{P}\hbar_{2n},\mathscr{Q}\eta),d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\eta,\mathscr{K}\eta),\\ d(\mathscr{P}\hbar_{2n},\mathscr{K}\eta)d(\mathscr{Q}\eta,\mathfrak{S}\hbar_{2n}),\\ \frac{1}{2}\Big[d(\mathscr{P}\hbar_{2n},\mathfrak{S}\hbar_{2n})d(\mathscr{P}\hbar_{2n},\mathscr{K}\eta)+\\ d(\mathscr{Q}\eta,\mathfrak{S}\hbar_{2n})d(\mathscr{Q}\eta,\mathscr{K}\eta)\Big] \end{array}\right\}=d^2(\eta,\mathscr{K}\eta).$$

Therefore, we get

$$[1+pd(\eta,\mathscr{K}\eta)]d^2(\eta,\mathscr{K}\eta)\leq p\max\{\frac{1}{2}[0,0],0,\}+d^2(\eta,\mathscr{K}\eta)-\phi(d^2(\eta,\mathscr{K}\eta)).$$

This gives $\eta=\mathscr{K}v$. Since $\mathscr{K}(X)\subset\mathscr{P}(X)$ and hence there exists a point $w\in X$ such that $\eta=\mathscr{K}\eta=\mathscr{P}w$.

We claim that $\eta=\mathfrak{S}w$.

For this we put $\hbar=w$ and $\vartheta=\eta$ in (C_6) we get

$$\begin{aligned}&[1+pd(\mathscr{P}w,\mathscr{Q}\eta)]d^2(\mathfrak{S}w,\mathscr{K}\eta)\\ &\leq p\max\left\{\frac{1}{2}\left[d^2(\mathscr{P}w,\mathfrak{S}w)d(\mathscr{Q}\eta,\mathscr{K}\eta)+d(\mathscr{P}w,\mathfrak{S}w)d^2(\mathscr{Q}\eta,\mathscr{K}\eta)\right],\right.\\ &\left.d(\mathscr{P}w,\mathfrak{S}w)d(\mathscr{P}w,\mathscr{K}\eta)d(\mathscr{Q}\eta,\mathfrak{S}w),d(\mathscr{P}w,\mathscr{K}\eta)d(\mathscr{Q}\eta,\mathfrak{S}w)d(\mathscr{Q}\eta,\mathscr{K}\eta)\right\}\\ &+m(\mathscr{P}w,\mathscr{Q}\eta)-\phi(m(\mathscr{P}w,\mathscr{Q}\eta)),\end{aligned}$$

where

$$m(\mathscr{P}w,\mathscr{Q}\eta)=\max\left\{\begin{array}{r} d^2(\mathscr{P}w,\mathscr{Q}\eta),d(\mathscr{P}w,\mathfrak{S}w)d(\mathscr{Q}\eta,\mathscr{K}\eta),\\ d(\mathscr{P}w,\mathscr{K}\eta)d(\mathscr{Q}\eta,\mathfrak{S}w),\\ \frac{1}{2}\Big[d(\mathscr{P}w,\mathfrak{S}w)d(\mathscr{P}w,\mathscr{K}\eta)+\\ d(\mathscr{Q}\eta,\mathfrak{S}w)d(\mathscr{Q}\eta,\mathscr{K}\eta)\Big] \end{array}\right\}$$

$$=\max\left\{\begin{array}{r} d^2(\eta,\eta),d(\eta,\mathfrak{S}w)d(\mathscr{K}\eta,\mathscr{K}\eta),\\ d(\eta,\eta)d(\eta,\mathfrak{S}w),\\ \frac{1}{2}\Big[d(\eta,\mathfrak{S}w)d(\eta,\eta)+\\ d(\eta,\mathfrak{S}w)d(\mathscr{K}\eta,\mathscr{K}\eta)\Big] \end{array}\right\}=0.$$

Therefore, we obtain

$$[1+pd(\eta,\eta)]d^2(\mathfrak{S}w,\eta) \leq p\max\left\{\frac{1}{2}\left[d^2(\eta,\mathfrak{S}w)d(\eta,\eta)+d(z,\mathfrak{S}w)d^2(\eta,\eta)\right],\right.$$
$$\left. d(\eta,\mathfrak{S}w)d(\eta,\eta)d(\eta,\mathfrak{S}w), d(\eta,\eta)d(\eta,\mathfrak{S}w)d(\eta,\eta)\right\}$$
$$+0-\phi(0).$$

This implies that $\mathfrak{S}w=\eta$. Since the pair $(\mathfrak{S},\mathscr{P})$ is weakly commuting on X,

$$d(\mathscr{P}\mathfrak{S}w,\mathfrak{S}\mathscr{P}w) \leq d(\mathfrak{S}w,\mathscr{P}w) = d(\eta,\eta) = 0.$$

So, $\mathscr{P}\eta = \mathfrak{S}\eta$.

That is, $\eta = \mathscr{P}\eta = \mathfrak{S}\eta = \mathscr{Q}\eta = \mathscr{K}\eta$. Therefore, η is a common fixed point of $\mathfrak{S},\mathscr{K},\mathscr{P}$ and $\mathscr{Q}$.

Case 4. Suppose that $\mathscr{K}$ is continuous, we can obtain the same way of case 3.

Uniqueness: Suppose $\eta \neq w$ be two common fixed points of $\mathfrak{S},\mathscr{K},\mathscr{P}$ and $\mathscr{Q}$.

Put $\hbar = \eta$ and $\vartheta = w$ in (C_6), we get

$$[1+pd(\mathscr{P}\eta,\mathscr{Q}w)]d^2(\mathfrak{S}\eta,\mathscr{K}w) \leq p\max\{0,0,0\} + m(\mathscr{P}\eta,\mathscr{Q}w) - \phi m(\mathscr{P}\eta,\mathscr{Q}w)$$

$$[1+pd(\mathscr{P}\eta,\mathscr{Q}w)]d^2(\mathfrak{S}\eta,\mathscr{K}w) \leq p\max\{0,0,0\} + d^2(\mathfrak{S}\eta,\mathscr{K}w) - \phi d^2(\mathfrak{S}\eta,\mathscr{K}w)$$

$$\implies d^2(\eta,w) = 0 \implies \eta = w.$$

This completes the proof.

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