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FIXED POINT APPLICATIONS OF TWISTED (α,β)-φ-CONTRACTIVE MAPPINGS IN C^*-ALGEBRA VALUED S_b-METRIC SPACES

D. HARITHA[1], K. RAMA KOTESWARA RAO[1], B. SRINUVASA RAO[2], N. MANGAPATHI[3,*]

[1]Department of Mathematics, GITAM School of Science, GITAM Deemed to be University, Hyderabad, Rudraram-502329, Telangana, India

[2]Department of Mathematics, Dr. B.R. Ambedkar University, Srikakulam-532410, Andhra Pradesh, India

[3]Department of Mathematics, B V Raju Institute of Technology, Narsapur, Medak-502313,Telangana, India

Abstract: This article explores the concept of generalized twisted(α,β)-φ-contractive type mappings and establishes fixed point theorems for these mappings within complete C^*-algebra valued S_b-metric spaces. We introduce twisted (α,β)-admissible mappings and demonstrate their utility in finding fixed points. Our results extend and enhance previous theorems in the literature. Additionally, we provide examples and applications related to integral equations and homotopy to highlight the key findings and innovations of our work.

Keywords: generalized twisted (α,β)-φ-contractive type mappings; twisted (α,β)-admissible mapping; C^*-algebra valued S_b- metric spaces and fixed point.

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1. Introduction

A metric space (MS) is ideal for individuals engaged in analysis, mathematical physics, and applied sciences. Numerous extensions of MS have been investigated, leading to various results concerning the existence of fixed points (FP) (see [1]-[18]).

*Corresponding author
E-mail address: nmp.maths@gmail.com
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In 2023, Razavi *et al.*[12] introduced the concept of C^*-algebra valued S_b-metric spaces ($\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$), and they established several common fixed point results in this context [13]. Subsequently, N. Narsimha *et al.* [14] successfully derived unique common fixed point theorems within $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$.

This study aims to establish fixed point theorems (FPT) using generalized twisted (α,β)-φ-type contraction mappings through twisted (α,β)-admissible mappings within the framework of $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$. Additionally, we will provide examples that are pertinent and applicable to both integral equations and homotopy.

What follows is in our subsequent conversations, we compile a few suitable definitions.

2. Preliminaries

This section provides a brief introduction to some fundamental aspects of C^*-algebra theory [19]. Let $\mathscr{A}$ be a unital C^*-algebra with the unit element $1_{\mathscr{A}}$. Define $\mathscr{A}_h = \{\mathfrak{e} \in \mathscr{A} : \mathfrak{e} = \mathfrak{e}^*\}$. An element $\mathfrak{e} \in \mathscr{A}$ is considered positive, denoted as $\mathfrak{e} \succeq 0_{\mathscr{A}}$, if $\mathfrak{e} = \mathfrak{e}^*$ and its spectrum $\eta(\mathfrak{e}) \subseteq [0,\infty)$. Here, $0_{\mathscr{A}}$ in $\mathscr{A}$ represents the zero element in $\mathscr{A}$, and $\eta(\mathfrak{e})$ denotes the spectrum of $\mathfrak{e}$. On $\mathscr{A}_h$, a natural partial ordering is defined by $\mathfrak{s} \preceq \mathfrak{v}$ if and only if $\mathfrak{v} - \mathfrak{s} \succeq 0_{\mathscr{A}}$. We denote $\mathscr{A}_+ = \{\mathfrak{e} \in \mathscr{A} : \mathfrak{e} \succeq 0_{\mathscr{A}}\}$ and $\mathscr{A}' = \{\mathfrak{e} \in \mathscr{A} : \mathfrak{e}\mathfrak{d} = \mathfrak{d}\mathfrak{e}\ \forall \mathfrak{d} \in \mathscr{A}\}$.

Definition 2.1:([12]) Given $\varsigma \in \mathscr{A}'$ and $||\varsigma|| \geq 1$, let $\pounds \neq \emptyset$. Assume that there is a mapping $\rho_\varsigma : \pounds^3 \to \mathscr{A}$ that meets the requirements listed below:

(S_{b_1}) $\rho_\varsigma(\mathfrak{s},\mathfrak{v},\mathfrak{e}) \succeq 0_{\mathscr{A}}$ for all $\mathfrak{s},\mathfrak{v},\mathfrak{e} \in \pounds$,

(S_{b_2}) $\rho_\varsigma(\mathfrak{s},\mathfrak{v},\mathfrak{e}) = 0_{\mathscr{A}} \Leftrightarrow \mathfrak{s} = \mathfrak{v} = \mathfrak{e}$,

(S_{b_3}) $\rho_\varsigma(\mathfrak{s},\mathfrak{v},\mathfrak{e}) \preceq \varsigma(\rho_\varsigma(\mathfrak{s},\mathfrak{s},\theta) + \rho_\varsigma(\mathfrak{v},\mathfrak{v},\theta) + \rho_\varsigma(\mathfrak{e},\mathfrak{e},\theta))$ for all $\mathfrak{s},\mathfrak{v},\mathfrak{e},\theta \in \pounds$.

Then the function ρ_ς is called a C^*-algebra valued S_b-metric on $\pounds$ and the pair $(\pounds,\mathscr{A},\rho_\varsigma)$ is called a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with a coefficient ς.

Definition 2.2:([12]) Let $(\pounds,\mathscr{A},\rho_\varsigma)$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ and $\{\mathfrak{e}_a\}$ be a sequence in $\pounds$:

(1) If for all $\mathfrak{v} \in \mathbb{N}$, $||\rho_\varsigma(\mathfrak{e}_{a+\mathfrak{v}},\mathfrak{e}_{a+\mathfrak{v}},\mathfrak{e}_a)|| \to 0$, where $a \to \infty$, then $\{\mathfrak{e}_a\}$ is a Cauchy sequence (CS) in $\pounds$.

(2) If $||\rho_\varsigma(\mathfrak{e}_a,\mathfrak{e}_a,\mathfrak{e})|| \to 0$, where $a \to \infty$, then $\{\mathfrak{e}_a\}$ converges to $\mathfrak{e}$, and we present it with $\lim\limits_{a\to\infty} \mathfrak{e}_a = \mathfrak{e}$.

(3) If every CS is convergent in $\pounds$, then $(\pounds,\mathscr{A},\rho_\varsigma)$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$.

Definition 2.3:([12]) Suppose that $(\pounds_1,\mathscr{A}_1,\rho_{\varsigma_1})$ and $(\pounds_2,\mathscr{A}_2,\rho_{\varsigma_2})$ are $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$, and let $\Gamma:(\pounds_1,\mathscr{A}_1,\rho_{\varsigma_1})\to(\pounds_2,\mathscr{A}_2,\rho_{\varsigma_2})$ be a function. Then, Γ is continuous at a point $\mathfrak{s}\in\pounds_1$ if, for every sequence, $\{\mathfrak{s}_a\}$ in $\pounds_1$, $\rho_\varsigma(\mathfrak{s}_a,\mathfrak{s}_a,\mathfrak{s})\to 0_{\mathscr{A}}$, $(a\to\infty)$ implies $\rho_\varsigma(\Gamma(\mathfrak{s}_a),\Gamma(\mathfrak{s}_a),\Gamma(\mathfrak{s}))\to 0_{\mathscr{A}'}$ where $a\to\infty$. A function Γ is continuous at $\pounds_1$ iff it is continuous at all $\mathfrak{s}\in\pounds_1$.

Definition 2.4:([17]) Let $\pounds\neq\emptyset$, $\Gamma:\pounds\to\pounds$ and $\alpha,\beta:\pounds\times\pounds\to\mathscr{A}_+$ be a functions. We say that Γ is a twisted (α,β)-admissible mappings, if for all $\mathfrak{s},\mathfrak{e}\in\pounds$

$$\begin{cases}\alpha(\mathfrak{s},\mathfrak{e})\succeq 1_{\mathscr{A}}\\ \beta(\mathfrak{s},\mathfrak{e})\succeq 1_{\mathscr{A}}\end{cases}\Rightarrow\begin{cases}\alpha(\Gamma\mathfrak{s},\Gamma\mathfrak{e})\succeq 1_{\mathscr{A}}\\ \beta(\Gamma\mathfrak{s},\Gamma\mathfrak{e})\succeq 1_{\mathscr{A}}\end{cases}$$

Lemma 2.5:([20]) Suppose that $\mathscr{A}$ is a unital C^*-algebra with a unit $1_{\mathscr{A}}$:

(1) If $\{\mathfrak{s}_a\}_{a=1}^\infty\subseteq\mathscr{A}$ and $\lim\limits_{a\to\infty}\mathfrak{s}_a=0_{\mathscr{A}}$, then for any $\mathfrak{s}\in\mathscr{A}$, $\lim\limits_{a\to\infty}\mathfrak{s}^*\mathfrak{s}_a\mathfrak{s}=0_{\mathscr{A}}$

(2) If $\mathfrak{s},\mathfrak{e}\in\mathscr{A}_h$ and $s\in\mathscr{A}_+'$ then $\mathfrak{s}\preceq\mathfrak{e}$ yields $s\mathfrak{s}\preceq s\mathfrak{e}$ in which $\mathscr{A}_+'=\mathscr{A}_+\cap\mathscr{A}'$.

(3) If $\mathfrak{s}\in\mathscr{A}_+$ with $||\mathfrak{s}||<\frac{1}{2}$ then $1_{\mathscr{A}}-\mathfrak{s}$ is invertible,and $||\mathfrak{s}(1_{\mathscr{A}}-\mathfrak{s})^{-1}||<1$.

(4) If $\mathfrak{s},\mathfrak{e}\in\mathscr{A}_+$ such that $\mathfrak{s}\mathfrak{e}=\mathfrak{e}\mathfrak{s}$, then $\mathfrak{s}\mathfrak{e}\succeq 0_{\mathscr{A}}$.

Now we prove our main result.

3. Main Results

The concepts of a twisted (α,β)-φ-contractive type mapping and a twisted (α,β)-adm are introduced at the beginning of this section, after which we establish our main result.

In this manuscript we indicate:

$$\mathfrak{D}=\left\{\begin{array}{c}\varphi:\mathscr{A}_+\to\mathscr{A}_+/\varphi\text{ is non-decreasing, continuous and}\\ \varphi(a)\prec a\text{ for all }a\succ 0_{\mathscr{A}}\text{ and }\varphi(a)=0_{\mathscr{A}}\iff a=0_{\mathscr{A}}\end{array}\right\}.$$

Definition 3.1: Let $(\pounds,\mathscr{A},\rho_\varsigma)$ be a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$, $\Gamma:\pounds\to\pounds$ and $\alpha,\beta:\pounds^3\to\mathscr{A}_+$ be a functions then Γ is called a twisted (α,β)-adm, if for all $\mathfrak{s},\mathfrak{e}\in\pounds$

$$\begin{cases}\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\succeq 1_{\mathscr{A}}\\ \beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})\succeq 1_{\mathscr{A}}\end{cases}\Rightarrow\begin{cases}\alpha(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e})\succeq 1_{\mathscr{A}}\\ \beta(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e})\succeq 1_{\mathscr{A}}\end{cases}$$

Definition 3.2: Let $(\pounds,\mathscr{A},\rho_\varsigma)$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with $||\varsigma||\geq 1$, $\Gamma:\pounds\to\pounds$ be a continuous generalized twisted (α,β)-φ-contractive mapping of type (i) or (ii) or (iii), if there exist two functions $\alpha,\beta : \pounds\times\pounds\times\pounds \to \mathscr{A}_{+}$ and $\varphi\in\Omega$ such that for all $\mathfrak{s},\mathfrak{e}\in\pounds$, $i=1,2,3,4,5$.

(A) Generalized twisted (α,β)-φ-contractive mapping of type (i), if there exist $a\in\mathscr{A}$ with $||a||<1$ such that

$$\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})\rho_{\varsigma}(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e}) \preceq \frac{1}{\varsigma^2}\varphi\left(a^{\star}\mathbb{M}_b^i(\mathfrak{s},\mathfrak{e})a\right)$$

(B) Generalized twisted (α,β)-φ-contractive mapping of type (ii), if there exist $a,b\in\mathscr{A}$ with $||a||<1$ and $0<||b||\leq 1$ such that

$$\left(\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})+b\right)^{\rho_{\varsigma}(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e})} \preceq (1_{\mathscr{A}}+b)^{\frac{1}{\varsigma^2}\left(a^{\star}\mathbb{M}_b^i(\mathfrak{s},\mathfrak{e})a\right)}$$

(C) Generalized twisted (α,β)-φ-contractive mapping of type (iii), if there exist $a,b\in\mathscr{A}$ with $||a||<1$ and $||b||\geq 1$ such that

$$\left(\rho_{\varsigma}(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e})+b\right)^{\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})} \preceq \frac{1}{\varsigma^2}\varphi\left(a^{\star}\mathbb{M}_b^i(\mathfrak{s},\mathfrak{e})a\right)+b$$

where

$$\mathbb{M}_b^1(\mathfrak{s},\mathfrak{e}) = \rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e})$$
$$\mathbb{M}_b^2(\mathfrak{s},\mathfrak{e}) = \max\left\{ \rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e}), \frac{\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\Gamma\mathfrak{s})+\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})}{2\varsigma^4}, \right\}$$
$$\mathbb{M}_b^3(\mathfrak{s},\mathfrak{e}) = \max\left\{ \begin{array}{c} \rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e}), \frac{\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\Gamma\mathfrak{s})+\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})}{2\varsigma^4}, \\ \frac{\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\Gamma\mathfrak{e})+\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{s})}{2\varsigma^4} \end{array}\right\}$$
$$\mathbb{M}_b^4(\mathfrak{s},\mathfrak{e}) = \max\left\{ \begin{array}{c} \rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e}), \rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\Gamma\mathfrak{s}), \\ \rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e}), \frac{\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\Gamma\mathfrak{e})+\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{s})}{2\varsigma^4} \end{array}\right\}$$
$$\mathbb{M}_b^5(\mathfrak{s},\mathfrak{e}) = \max\left\{ \begin{array}{c} \rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e}), \rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\Gamma\mathfrak{s}), \rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e}), \frac{\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\Gamma\mathfrak{s}).\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})}{2\varsigma^4\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e})}, \\ \frac{\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\Gamma\mathfrak{s})\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\Gamma\mathfrak{e})+\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{s})\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})}{2\varsigma^4\left(\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{s})+\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\Gamma\mathfrak{e})\right)} \end{array}\right\}.$$

We start our work by proving the following one crucial Lemma.

Lemma 3.3: In a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$, we have

$$\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \preceq \varsigma\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\mathfrak{s}) \text{ and } \rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\mathfrak{s}) \preceq \varsigma\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \;\forall\; \mathfrak{s},\mathfrak{e}\in\pounds.$$

Proof Using condition (S_{b_3}) of Definition-2.1, we have

$$\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \preceq \varsigma\left(2\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{s})+\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\mathfrak{s})\right) \preceq \varsigma\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\mathfrak{s})$$

and similarly

$$\rho_\varsigma(\mathfrak{e},\mathfrak{e},\mathfrak{s}) \preceq \varsigma\left(2\rho_\varsigma(\mathfrak{e},\mathfrak{e},\mathfrak{e})+\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{e})\right) \preceq \varsigma\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{e})$$

Lemma 3.4: Let $(\pounds,\mathscr{A},\rho_\varsigma)$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with $||\varsigma|| \geq 1$, then we have

$$\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \preceq 2\varsigma\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{v})+\varsigma^2\rho_\varsigma(\mathfrak{v},\mathfrak{v},\mathfrak{e})\ \forall\ \mathfrak{s},\mathfrak{e},\mathfrak{v}\in\pounds$$

Proof Using condition (S_{b_3}) of Definition-2.1 and Lemma-3.3, we have

$$\begin{aligned}\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{e}) &\preceq \varsigma\left(\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{v})+\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{v})+\rho_\varsigma(\mathfrak{e},\mathfrak{e},\mathfrak{v})\right)\\ &\preceq 2\varsigma\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{v})+\varsigma^2\rho_\varsigma(\mathfrak{v},\mathfrak{v},\mathfrak{e})\end{aligned}$$

Lemma 3.5: If $(\pounds,\mathscr{A},\rho_\varsigma)$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with $||\varsigma|| \geq 1$ and suppose that $\{\mathfrak{e}_\mathfrak{z}\}$ is a C^*-AV-S_b-convergent to $\mathfrak{s}$, then we have
$\frac{1}{2\varsigma}\rho_\varsigma(\mathfrak{v},\mathfrak{v},\mathfrak{s}) \preceq \lim_{\mathfrak{z}\to\infty}\inf\ \rho_\varsigma(\mathfrak{v},\mathfrak{v},\mathfrak{e}_\mathfrak{z}) \preceq \lim_{a\to\infty}\sup\rho_\varsigma(\mathfrak{v},\mathfrak{v},\mathfrak{e}_\mathfrak{z}) \preceq 2\varsigma\rho_\varsigma(\mathfrak{v},\mathfrak{v},\mathfrak{s})$
for all $\mathfrak{v}\in\pounds$. In particular, if $\mathfrak{s}=\mathfrak{v}$, then we have $\lim_{\mathfrak{z}\to\infty}\rho_\varsigma(\mathfrak{s}_\mathfrak{z},\mathfrak{s}_\mathfrak{z},\mathfrak{v}) = 0_\mathscr{A}$.

Proof Using condition (S_{b_3}) of Definition-2.1, we have

$$\rho_\varsigma(\mathfrak{v},\mathfrak{v},\mathfrak{e}_\mathfrak{z}) \preceq 2\varsigma\rho_\varsigma(\mathfrak{v},\mathfrak{v},\mathfrak{s})+\varsigma\rho_\varsigma(\mathfrak{e}_\mathfrak{z},\mathfrak{e}_\mathfrak{z},\mathfrak{s})$$

and

$$\rho_\varsigma(\mathfrak{v},\mathfrak{v},\mathfrak{s}) \preceq 2\varsigma\rho_\varsigma(\mathfrak{v},\mathfrak{v},\mathfrak{e}_\mathfrak{z})+\varsigma\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{e}_\mathfrak{z}).$$

We reach the required conclusion by taking the upper limit in the first inequality as $\mathfrak{z}\to\infty$ and the lower limit in the second inequality as $\mathfrak{z}\to\infty$.

Theorem 3.6: Let $(\pounds,\mathscr{A},\rho_\varsigma)$ be a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with $||\varsigma|| \geq 1$ and suppose $\Gamma:\pounds\to\pounds$ be a continuous generalized twisted (α,β)-φ-contractive mapping of type (i) or (ii) or (iii) with $i=5$. If there exists $\mathfrak{e}_0\in\pounds$ such that $\alpha(\mathfrak{e}_0,\mathfrak{e}_0,\Gamma\mathfrak{e}_0)\succeq 1_\mathscr{A}$ and $\beta(\mathfrak{e}_0,\mathfrak{e}_0,\Gamma\mathfrak{e}_0)\succeq 1_\mathscr{A}$, then Γ has a FP in $\pounds$.

Proof Let $\mathfrak{e}_0\in\pounds$ such that $\alpha(\mathfrak{e}_0,\mathfrak{e}_0,\Gamma\mathfrak{e}_0)\geq 1_\mathscr{A}$, $\beta(\mathfrak{e}_0,\mathfrak{e}_0,\Gamma\mathfrak{e}_0)\geq 1_\mathscr{A}$. We construct a sequence $\{\mathfrak{e}_\mathfrak{v}\}$ by setting $\mathfrak{e}_\mathfrak{v}=\Gamma\mathfrak{e}_{\mathfrak{v}-1}=\Gamma^\mathfrak{v}\mathfrak{e}_0\ \forall\ \mathfrak{v}\in N$. Since Γ is a twisted (α,β)-adm, then $\alpha(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)=\alpha(\mathfrak{e}_0,\mathfrak{e}_0,\Gamma\mathfrak{e}_0)\succeq 1_\mathscr{A}$ then $\alpha(\mathfrak{e}_2,\mathfrak{e}_2,\mathfrak{e}_1)=\alpha(\Gamma\mathfrak{e}_1,\Gamma\mathfrak{e}_1,\Gamma\mathfrak{e}_0)\succeq 1_\mathscr{A}$ which implies that $\alpha(\mathfrak{e}_2,\mathfrak{e}_2,\mathfrak{e}_3)=\alpha(\Gamma\mathfrak{e}_1,\Gamma\mathfrak{e}_1,\Gamma\mathfrak{e}_2)\succeq 1_\mathscr{A}$.

By repeating similar process, we obtain $\alpha(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1}) \succeq 1_{\mathscr{A}}$, $\alpha(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}-1}) \succeq 1_{\mathscr{A}}$ $\forall\, \mathfrak{v} \in \mathbb{N}$.

Similarly, we have $\beta(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1}) \succeq 1_{\mathscr{A}}$, $\beta(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}-1}) \succeq 1_{\mathscr{A}}$ $\forall\, \mathfrak{v} \in \mathbb{N}$.

Now, we distinguish the following cases:

Case(i): Let Γ be a mapping of type (i). Then by condition (A) in definition (3.2), we have

$$\begin{aligned} \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2}) &\preceq \alpha(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})\beta(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})\rho_\varsigma(\Gamma\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}+1}) \\ &\preceq \frac{1}{\varsigma^2}\varphi\left(a^\star \mathbb{M}_b^5(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})a\right) \end{aligned} \tag{1}$$

Now, by simple computations, we have

$$\begin{aligned} \mathbb{M}_b^5(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1}) &= \max\left\{ \begin{array}{c} \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1}), \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}}), \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\Gamma\mathfrak{e}_{\mathfrak{v}+1}), \\ \frac{\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}}).\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\Gamma\mathfrak{e}_{\mathfrak{v}+1})}{2\varsigma^4\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})}, \\ \frac{\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}})\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}+1})+\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\Gamma\mathfrak{e}_{\mathfrak{v}})\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\Gamma\mathfrak{e}_{\mathfrak{v}+1})}{2\varsigma^4\left(\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\Gamma\mathfrak{e}_{\mathfrak{v}})+\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}+1})\right)} \end{array} \right\} \\ &= \max\left\{ \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1}), \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2}) \right\}. \end{aligned}$$

From Eq. (1) and taking in consideration that φ is a nondecreasing function, we get

$$\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2}) \preceq \frac{1}{\varsigma^2}\varphi\left(a^\star \max\left\{ \begin{array}{c} \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1}), \\ \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2}) \end{array} \right\} a\right) \tag{2}$$

If we take $\max\left\{ \begin{array}{c} \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1}), \\ \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2}) \end{array} \right\} = \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2})$, since $||a|| < 1$ then (2) gives a contradiction. Thus, we obtain

$$\begin{aligned} \rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2}) &\preceq \frac{1}{\varsigma^2}a^\star\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})a \\ &\preceq \frac{1}{(\varsigma^2)^2}(a^*)^2\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}})(a)^2 \\ &\;\;\vdots \\ &\preceq \frac{1}{(\varsigma^2)^{\mathfrak{v}+1}}(a^*)^{\mathfrak{v}+1}\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)(a)^{\mathfrak{v}+1} \end{aligned}$$

For each $\mathfrak{v} \in \mathbb{N}$, we may see the following by keeping in mind the property where if $a, b \in \mathscr{A}_h$ then $a \preceq b$ gives $u^*au \preceq u^*bu$.

$$\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1}) \preceq \frac{1}{(\varsigma^2)^{\mathfrak{v}}}(a^*)^{\mathfrak{v}}\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)(a)^{\mathfrak{v}} \tag{3}$$

Case(ii): Let Γ be a mapping of type (ii). Then by condition (B) in definition (3.2), we have

$$\begin{aligned}(1_{\mathscr{A}}+b)^{\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2})} &\preceq (\alpha(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})\beta(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})+b)^{\rho_\varsigma(\Gamma\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}+1})}\\ &\preceq (1_{\mathscr{A}}+b)^{\frac{1}{\varsigma^2}\varphi\left(a^\star\mathbb{M}_b^5(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})a\right)}\end{aligned}$$

Then we have

$$\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2}) \preceq \frac{1}{\varsigma^2}\varphi\left(a^\star\mathbb{M}_b^5(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})a\right).$$

For all $\mathfrak{v}\in\mathbb{N}$, we can deduce by induction from Eq. (1) and Eq. (2), we get

$$\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})\preceq\frac{1}{(\varsigma^2)^{\mathfrak{v}}}(a^*)^{\mathfrak{v}}\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)(a)^{\mathfrak{v}} \tag{4}$$

Case(iii): Let Γ be a mapping of type (iii). Then by condition (C) in definition (3.2), we have

$$\begin{aligned}\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2})+b &\preceq \left(\rho_\varsigma(\Gamma\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}+1})+b\right)^{\alpha(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})\beta(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})}\\ &\preceq \frac{1}{\varsigma^2}\varphi\left(a^\star\mathbb{M}_b^5(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})a\right)+b\end{aligned}$$

Then we have

$$\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+2}) \preceq \frac{1}{\varsigma^2}\varphi\left(a^\star\mathbb{M}_b^5(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})a\right).$$

For all $\mathfrak{v}\in\mathbb{N}$, we can deduce by induction from Eq. (1) and Eq. (2), we get

$$\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})\preceq\frac{1}{(\varsigma^2)^{\mathfrak{v}}}(a^*)^{\mathfrak{v}}\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)(a)^{\mathfrak{v}} \tag{5}$$

From Eq.(3), (4) and Eq.(5) we get that

$$\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1})\preceq\frac{1}{(\varsigma^2)^{\mathfrak{v}}}(a^*)^{\mathfrak{v}}\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)(a)^{\mathfrak{v}}\to 0_{\mathscr{A}} \text{ as } p\to\infty$$

Let $\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)=X_0$ for some $X_0\in\mathscr{A}_+$. For any $\mathfrak{z}\in\mathbb{N}$, we achieve

$$\begin{aligned}\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}}) &\preceq \varsigma\left(\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1})+\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1})+\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1})\right)\\ &\preceq 2\varsigma\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1})+\varsigma\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1})\\ &\preceq 2\varsigma\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1})+\varsigma^2\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1},\mathfrak{e}_{\mathfrak{v}})\\ &\preceq 2\varsigma\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1})+2\varsigma^3\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-2})\\ &\quad+\varsigma^4\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-2},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-2},\mathfrak{e}_{\mathfrak{v}})\end{aligned}$$

$$\begin{aligned}
&\vdots\\
&\preceq 2\varsigma\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1})+2\varsigma^3\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-1},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-2})\\
&\quad +2\varsigma^5\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-2},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-2},\mathfrak{e}_{\mathfrak{v}+\mathfrak{z}-3})+\ldots+2\varsigma^{2\mathfrak{z}-1}\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}})\\
&\preceq \frac{2\varsigma}{(\varsigma^2)^{\mathfrak{v}+\mathfrak{z}-1}}(a^*)^{\mathfrak{v}+\mathfrak{z}-1}\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)(a)^{\mathfrak{v}+\mathfrak{z}-1}\\
&\quad +\frac{2\varsigma^3}{(\varsigma^2)^{\mathfrak{v}+\mathfrak{z}-2}}(a^*)^{\mathfrak{v}+\mathfrak{z}-2}\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)(a)^{\mathfrak{v}+\mathfrak{z}-2}\\
&\quad +\frac{2\varsigma^5}{(\varsigma^2)^{\mathfrak{v}+\mathfrak{z}-3}}(a^*)^{\mathfrak{v}+\mathfrak{z}-3}\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)(a)^{\mathfrak{v}+\mathfrak{z}-3}+\ldots\\
&\quad +\frac{2\varsigma^{2\mathfrak{z}-1}}{(\varsigma^2)^{\mathfrak{v}}}(a^*)^{\mathfrak{v}}\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)(a)^{\mathfrak{v}}\\
&\preceq \frac{2\varsigma}{(\varsigma^2)^{\mathfrak{v}+\mathfrak{z}-1}}\sum_{i=1}^{\mathfrak{z}}\varsigma^{4i-4}(a^*)^{\mathfrak{v}+\mathfrak{z}-i}\rho_\varsigma(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_1)(a)^{\mathfrak{v}+\mathfrak{z}-i}\\
&= \frac{2\varsigma}{(2\varsigma^2)^{\mathfrak{v}+\mathfrak{z}-1}}\sum_{i=1}^{\mathfrak{z}}\varsigma^{4i-4}(a^*)^{\mathfrak{v}+\mathfrak{z}-i}X_0(a)^{\mathfrak{v}+\mathfrak{z}-i}\\
&= \frac{2\varsigma}{(\varsigma^2)^{\mathfrak{v}+\mathfrak{z}-1}}\sum_{i=1}^{\mathfrak{z}}\left((a^*)^{\mathfrak{v}+\mathfrak{z}-i}\varsigma^{\frac{4i-4}{2}}X_0^{\frac{1}{2}}\right)\left(X_0^{\frac{1}{2}}\varsigma^{\frac{4i-4}{2}}(a)^{\mathfrak{v}+\mathfrak{z}-i}\right)\\
&\preceq \frac{2\varsigma}{(\varsigma^2)^{\mathfrak{v}+\mathfrak{z}-1}}\sum_{i=1}^{\mathfrak{z}}\left(X_0^{\frac{1}{2}}\varsigma^{\frac{4i-4}{2}}a^{\mathfrak{v}+\mathfrak{z}-i}\right)^*\left(X_0^{\frac{1}{2}}\varsigma^{\frac{4i-4}{2}}(a)^{\mathfrak{v}+\mathfrak{z}-i}\right)\\
&\leq ||\frac{2\varsigma}{(\varsigma^2)^{\mathfrak{v}+\mathfrak{z}-1}}||\sum_{i=1}^{\mathfrak{z}}||X_0^{\frac{1}{2}}\varsigma^{\frac{4i-4}{2}}a^{\mathfrak{v}+\mathfrak{z}-i}||^2 1_{\mathscr{A}}\\
&\leq ||\frac{2\varsigma}{(\varsigma^2)^{\mathfrak{v}+\mathfrak{z}-1}}||||X_0||\sum_{i=1}^{\mathfrak{z}}||a||^{2(\mathfrak{v}+\mathfrak{z}-i)}||\varsigma||^{4i-4}1_{\mathscr{A}}\\
&\leq ||\frac{2\varsigma}{(\varsigma^2)^{\mathfrak{v}+\mathfrak{z}-1}}||||X_0||\frac{||a||^{2\mathfrak{v}+3}}{||a||-||\varsigma||^4}1_{\mathscr{A}}\to 0 \text{ as } \mathfrak{v},\mathfrak{z}\to\infty.
\end{aligned}$$

in which $1_{\mathscr{A}}$ is the unit element in $\mathscr{A}$. As $\{\mathfrak{e}_{\mathfrak{v}}\}$ is a CS in $\pounds$, and $\pounds$ is complete, there exists $\mathfrak{e}\in\pounds$ such that $\lim_{\mathfrak{v}\to\infty}\mathfrak{e}_{\mathfrak{v}}=\mathfrak{e}=\lim_{\mathfrak{v}\to\infty}\mathfrak{e}_{\mathfrak{v}+1}$ and

$$\lim_{\mathfrak{v}\to\infty}\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e})=\rho_\varsigma(\mathfrak{e},\mathfrak{e},\mathfrak{e})=\lim_{\mathfrak{v},l\to\infty}\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{z}})=0. \tag{6}$$

We will now demonstrate that $\Gamma\mathfrak{e}=\mathfrak{e}$. Assuming Γ is continuous via hypotheses,, then using (6), we have

$$\lim_{\mathfrak{v}\to\infty}\rho_\varsigma(\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e})=\lim_{\mathfrak{v}\to\infty}\rho_\varsigma(\Gamma\mathfrak{e}_{\mathfrak{v}},\Gamma\mathfrak{e}_{\mathfrak{v}},\mathfrak{e})=\rho_\varsigma(\Gamma\mathfrak{e},\Gamma\mathfrak{e},\mathfrak{e})=0.$$

This implies that $\Gamma\mathfrak{e}=\mathfrak{e}$. Hence, Γ has a FP $\mathfrak{e}\in\pounds$.

Similarly, by eliminating the continuity requirement on Γ and using an adjunctive condition on $\pounds$, we can arrive at the same result under a different premise in the subsequent theorem.

Theorem 3.7: Let $(\pounds,\mathscr{A},\rho_\varsigma)$ be a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with $||\varsigma||\geq 1$ and suppose $\Gamma:\pounds\to\pounds$ be a generalized twisted (α,β)-φ-contractive mapping of type (i) or (ii) or (iii) with $i=5$. Also, suppose that the following conditions hold:

(i) if $\exists\ \mathfrak{e}_0\in\pounds$ such that $\alpha\left(\mathfrak{e}_0,\mathfrak{e}_0,\Gamma\mathfrak{e}_0\right)\succeq 1_{\mathscr{A}}$ and $\beta\left(\mathfrak{e}_0,\mathfrak{e}_0,\Gamma\mathfrak{e}_0\right)\succeq 1_{\mathscr{A}}$;

(ii) if $\{\mathfrak{e}_\mathfrak{p}\}$ is a sequence in $\pounds$ such that $\alpha\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e}_{\mathfrak{p}+1}\right)\succeq 1_{\mathscr{A}}$ and $\beta\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e}_{\mathfrak{p}+1}\right)\succeq 1_{\mathscr{A}}$ for all $p\in\mathbb{N}\cup\{0\}$ and $\mathfrak{e}_\mathfrak{p}\to\mathfrak{e}$ as $p\to\infty$ then $\alpha\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e}\right)\succeq 1_{\mathscr{A}}$ and $\beta\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e}\right)\succeq 1_{\mathscr{A}}$ for all $p\in\mathbb{N}\cup\{0\}$.

Then Γ has a fixed point in $\pounds$.

Proof Let $\mathfrak{e}_0\in\pounds$ such that $\alpha\left(\mathfrak{e}_0,\mathfrak{e}_0,\Gamma\mathfrak{e}_0\right)\succeq 1_{\mathscr{A}}$ and $\beta\left(\mathfrak{e}_0,\mathfrak{e}_0,\Gamma\mathfrak{e}_0\right)\succeq 1_{\mathscr{A}}$. Proceeding as in the proof of Theorem 3.6, we know that there is $\mathfrak{e}\in\pounds$ such that $\mathfrak{e}_\mathfrak{p}\to\mathfrak{e}$ as $p\to\infty$,
$\alpha\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e}_{\mathfrak{p}+1}\right)\succeq 1_{\mathscr{A}}$ and $\beta\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e}_{\mathfrak{p}+1}\right)\succeq 1_{\mathscr{A}}$.
Now, we shall prove that $\Gamma\mathfrak{e}=\mathfrak{e}$
Assume to the contrary that $\Gamma\mathfrak{e}\neq\mathfrak{e}$. From (ii), we have $\alpha\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e}\right)\succeq 1_{\mathscr{A}}$ and $\beta\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e}\right)\succeq 1_{\mathscr{A}}$ for all $p\in\mathbb{N}\cup\{0\}$. Now, we distinguish the following cases:
Case(i): Let Γ be a mapping of type (I). Then by (A), in definition (3.2), we have

$$\rho_\varsigma(\Gamma\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e})\preceq\alpha(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e})\beta(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e})\rho_\varsigma(\Gamma\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e})\preceq\frac{1}{\varsigma^2}\varphi\left(a^\star\mathbb{M}_b^5(\mathfrak{e}_\mathfrak{p},\mathfrak{e})a\right)\tag{7}$$

where $\mathbb{M}_b^5\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}\right)=\max\left\{\begin{array}{c}\rho_\varsigma\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e}\right),\rho_\varsigma\left(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e}_\mathfrak{p}\right),\rho_\varsigma\left(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e}\right),\frac{\rho_\varsigma(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e}_\mathfrak{p}).\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})}{2\varsigma^4\rho_\varsigma(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e})},\\ \frac{\rho_\varsigma(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e}_\mathfrak{p})\rho_\varsigma(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e})+\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e}_\mathfrak{p})\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})}{2\varsigma^4\left(\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e}_\mathfrak{p})+\rho_\varsigma(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e})\right)}\end{array}\right\}$.
Case(ii): Let Γ be a mapping of type (ii). Then by condition (B) in definition (3.2), we have

$$\begin{aligned}(1_{\mathscr{A}}+b)^{\rho_\varsigma(\Gamma\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e})} &\preceq (\alpha(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e})\beta(\mathfrak{e}_\mathfrak{p},\mathfrak{e}_\mathfrak{p},\mathfrak{e})+b)^{\rho_\varsigma(\Gamma\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e})}\\ &\preceq (1_{\mathscr{A}}+b)^{\frac{1}{\varsigma^2}\varphi\left(a^\star\mathbb{M}_b^5(\mathfrak{e}_\mathfrak{p},\mathfrak{e})a\right)}\end{aligned}$$

Then we have

$$\rho_\varsigma(\Gamma\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e}_\mathfrak{p},\Gamma\mathfrak{e}) \preceq \frac{1}{\varsigma^2}\varphi\left(a^\star\mathbb{M}_b^5(\mathfrak{e}_\mathfrak{p},\mathfrak{e})a\right).\tag{8}$$

Case(iii): Let Γ be a mapping of type (iii). Then by condition (C) in definition (3.2), we have

$$\begin{aligned}\rho_\varsigma(\Gamma\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e})+b &\preceq \left(\rho_\varsigma(\Gamma\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e})+b\right)^{\alpha(\mathfrak{e}_\mathfrak{v},\mathfrak{e}_\mathfrak{v},\mathfrak{e})\beta(\mathfrak{e}_\mathfrak{v},\mathfrak{e}_\mathfrak{v},\mathfrak{e})}\\ &\preceq \frac{1}{\varsigma^2}\varphi\left(a^\star\mathbb{M}_b^5(\mathfrak{e}_\mathfrak{v},\mathfrak{e})a\right)+b\end{aligned}$$

Then we have

$$\rho_\varsigma(\Gamma\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e}) \preceq \frac{1}{\varsigma^2}\varphi\left(a^\star\mathbb{M}_b^5(\mathfrak{e}_\mathfrak{v},\mathfrak{e})a\right). \tag{9}$$

Therefore, in all cases, by using the triangular inequality, we can write

$$\begin{aligned}\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e}) &\preceq 2\varsigma\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e}_\mathfrak{v})+\varsigma\rho_\varsigma(\Gamma\mathfrak{e},\Gamma\mathfrak{e},\Gamma\mathfrak{e}_\mathfrak{v})\\ &\preceq 2\varsigma\rho_\varsigma(\mathfrak{e},\mathfrak{e},\mathfrak{e}_{\mathfrak{v}+1})+\varsigma^2\rho_\varsigma(\Gamma\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e})\\ &\preceq 2\varsigma\rho_\varsigma(\mathfrak{e},\mathfrak{e},\mathfrak{e}_{\mathfrak{v}+1})+\varphi\left(a^\star\mathbb{M}_b^5(\mathfrak{e}_\mathfrak{v},\mathfrak{e})a\right)\\ &\preceq 2\varsigma\rho_\varsigma(\mathfrak{e},\mathfrak{e},\mathfrak{e}_{\mathfrak{v}+1})\\ &\quad+\varphi\left(a^\star\max\left\{\begin{array}{c}\rho_\varsigma(\mathfrak{e}_\mathfrak{v},\mathfrak{e}_\mathfrak{v},\mathfrak{e}),\rho_\varsigma(\mathfrak{e}_\mathfrak{v},\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e}_\mathfrak{v}),\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e}),\\ \frac{\rho_\varsigma(\mathfrak{e}_\mathfrak{v},\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e}_\mathfrak{v}).\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})}{2\varsigma^4\rho_\varsigma(\mathfrak{e}_\mathfrak{v},\mathfrak{e}_\mathfrak{v},\mathfrak{e})},\\ \frac{\rho_\varsigma(\mathfrak{e}_\mathfrak{v},\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e}_\mathfrak{v})\rho_\varsigma(\mathfrak{e}_\mathfrak{v},\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e})+\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e}_\mathfrak{v})\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})}{2\varsigma^4\left(\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e}_\mathfrak{v})+\rho_\varsigma(\mathfrak{e}_\mathfrak{v},\mathfrak{e}_\mathfrak{v},\Gamma\mathfrak{e})\right)}\end{array}\right\}a\right).\end{aligned}$$

Taking the limit as $p\to\infty$ in above inequality and taking in consideration that φ is a nondecreasing function, we get

$$||\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})|| \le ||a||^2||\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})||$$

a contradiction since $||a||<1$. Thus, we have $||\rho_\varsigma(\mathfrak{e},\mathfrak{e},\Gamma\mathfrak{e})||=0$, that is, $\Gamma\mathfrak{e}=\mathfrak{e}$.

Theorem 3.8: Assume that Theorem 3.6 and Theorem 3.7 are true in their entirety. Including the subsequent requirement:

for all $\mathfrak{s},\mathfrak{e}\in\pounds$ with $\mathfrak{s}\neq\mathfrak{e}$, there exists $\mathfrak{v}\in\pounds$ such that $\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{v})\succeq 1_\mathscr{A}$, $\alpha(\mathfrak{e},\mathfrak{e},\mathfrak{v})\succeq 1_\mathscr{A}$ and $\beta(\mathfrak{s},\mathfrak{s},\mathfrak{v})\succeq 1_\mathscr{A}$, $\beta(\mathfrak{e},\mathfrak{e},\mathfrak{v})\succeq 1_\mathscr{A}$. Then Γ has a UFP in $\pounds$.

Proof Suppose that $\mathfrak{e}$ and $\mathfrak{e}^\star$ are two FP of Γ such that $\mathfrak{e}\neq\mathfrak{e}^\star$. By the hypothesis, there exists $\mathfrak{v}\in\pounds$ such that $\alpha(\mathfrak{e},\mathfrak{e},\mathfrak{v})\succeq 1_\mathscr{A}$, $\alpha(\mathfrak{e}^\star,\mathfrak{e}^\star,\mathfrak{v})\succeq 1_\mathscr{A}$ and $\beta(\mathfrak{e},\mathfrak{e},\mathfrak{v})\succeq 1_\mathscr{A}$, $\beta(\mathfrak{e}^\star,\mathfrak{e}^\star,\mathfrak{v})\succeq 1_\mathscr{A}$.
Therefore, since Γ is a twisted (α,β)-adm, we deduce that $\alpha(\Gamma^\mathfrak{v}\mathfrak{e},\Gamma^\mathfrak{v}\mathfrak{e},\Gamma^\mathfrak{v}\mathfrak{v})\succeq 1_\mathscr{A}$,
$\alpha\left(\Gamma^{\mathfrak{v}-1}\mathfrak{v},\Gamma^{\mathfrak{v}-1}\mathfrak{v},\Gamma^{\mathfrak{v}-1}\mathfrak{e}\right)\succeq 1_\mathscr{A}$ and $\alpha(\Gamma^\mathfrak{v}\mathfrak{e}^\star,\Gamma^\mathfrak{v}\mathfrak{e}^\star,\Gamma^\mathfrak{v}\mathfrak{v})\succeq 1_\mathscr{A}$, $\alpha\left(\Gamma^{\mathfrak{v}-1}\mathfrak{v},\Gamma^{\mathfrak{v}-1}\mathfrak{v},\Gamma^{\mathfrak{v}-1}\mathfrak{e}^\star\right)\succeq 1_\mathscr{A}$.

Similarly,

we get $\beta\left(\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{v}\right)\succeq 1_{\mathscr{A}}$, $\beta\left(\Gamma^{\mathfrak{v}-1}\mathfrak{v},\Gamma^{\mathfrak{v}-1}\mathfrak{v},\Gamma^{\mathfrak{v}-1}\mathfrak{e}\right)\succeq 1_{\mathscr{A}}$ and $\beta\left(\Gamma^{\mathfrak{v}}\mathfrak{e}^{\star},\Gamma^{\mathfrak{v}}\mathfrak{e}^{\star},\Gamma^{\mathfrak{v}}\mathfrak{v}\right)\succeq 1_{\mathscr{A}}$, $\beta\left(\Gamma^{\mathfrak{v}-1}\mathfrak{v},\Gamma^{\mathfrak{v}-1}\mathfrak{v},\Gamma^{\mathfrak{v}-1}\mathfrak{e}^{\star}\right)\succeq 1_{\mathscr{A}}$.

Now, if Γ is a mapping of type (i), then by (A), in definition (3.2), we have

$$\begin{aligned}
\rho_{\varsigma}(\Gamma\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma\Gamma^{\mathfrak{v}}\mathfrak{v}) &\preceq \alpha(\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{v})\beta(\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{v})\rho_{\varsigma}(\Gamma\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma\Gamma^{\mathfrak{v}}\mathfrak{v})\\
&\preceq \frac{1}{\varsigma^2}\varphi\left(a^{\star}\mathbb{M}_b^5(\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{v})a\right)\\
&\preceq \frac{1}{\varsigma^2}\varphi\left(a^{\star}\rho_{\varsigma}\left(\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{v}\right)a\right)
\end{aligned}$$

Therefore,

$$\rho_{\varsigma}(\Gamma^{\mathfrak{v}+1}\mathfrak{e},\Gamma^{\mathfrak{v}+1}\mathfrak{e},\Gamma^{\mathfrak{v}+1}\mathfrak{v}) \preceq \frac{1}{\varsigma^2}a^{\star}\rho_{\varsigma}\left(\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{e},\Gamma^{\mathfrak{v}}\mathfrak{v}\right)a$$

Inductively, we get

$$\rho_{\varsigma}(\Gamma^{\mathfrak{v}+1}\mathfrak{e},\Gamma^{\mathfrak{v}+1}\mathfrak{e},\Gamma^{\mathfrak{v}+1}\mathfrak{v}) \preceq \frac{1}{(\varsigma^2)^{\mathfrak{v}+1}}(a^{*})^{\mathfrak{v}+1}\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\mathfrak{v})(a)^{\mathfrak{v}+1}$$

But, since $\mathfrak{e}$ is the fixed point of Γ, we get

$$\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma^{\mathfrak{v}+1}\mathfrak{v}) \preceq \frac{1}{(\varsigma^2)^{\mathfrak{v}+1}}(a^{*})^{\mathfrak{v}+1}\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\mathfrak{v})(a)^{\mathfrak{v}+1} \tag{10}$$

Likewise, if we assume that Γ is a mapping of type (ii) or (iii), we obtain the same result.
$\lim\limits_{\mathfrak{v}\to\infty}\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma^{\mathfrak{v}+1}\mathfrak{v})=0_{\mathscr{A}}$ is obtained by taking the limit as $p\to\infty$ in Eq. (10). We also obtain $\lim\limits_{\mathfrak{v}\to\infty}\rho_{\varsigma}(\mathfrak{e}^{\star},\mathfrak{e}^{\star},\Gamma^{\mathfrak{v}+1}\mathfrak{v})=0_{\mathscr{A}}$. using the same reasoning as above.
Triangular inequality gives us

$$\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\mathfrak{e}^{\star}) \preceq \lim_{\mathfrak{v}\to\infty}\left(2\varsigma\rho_{\varsigma}(\mathfrak{e},\mathfrak{e},\Gamma^{\mathfrak{v}+1}\mathfrak{v})+\varsigma\rho_{\varsigma}(\mathfrak{e}^{\star},\mathfrak{e}^{\star},\Gamma^{\mathfrak{v}+1}\mathfrak{v})\right)=0_{\mathscr{A}}.$$

Thus, $\mathfrak{e}=\mathfrak{e}^{\star}$ and hence $\mathfrak{e}$ is UFP in Γ.

Corollary 3.9: Let $(\pounds,\mathscr{A},\rho_{\varsigma})$ be a complete $\mathscr{C}^{*}$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with $||\varsigma||\geq 1$ and suppose $\Gamma:\pounds\to\pounds$ be a generalized twisted (α,β)-φ-contractive mapping of type (i) or (ii) or (iii) with $i=4,3,2,1$ and satisfies hypothesis of Theorem 3.6, 3.7, 3.8, then Γ has a unique fixed point in $\pounds$.

Proof If we substitute $\mathbb{M}_b^4(\mathfrak{s},\mathfrak{e})$, $\mathbb{M}_b^3(\mathfrak{s},\mathfrak{e})$, $\mathbb{M}_b^2(\mathfrak{s},\mathfrak{e})$ or $\mathbb{M}_b^1(\mathfrak{s},\mathfrak{e})$ for $\mathbb{M}_b^5(\mathfrak{s},\mathfrak{e})$ in Theorem (3.6), (3.7), (3.8) then follows in a manner similar to Theorem (3.6), (3.7), (3.8) respectively.

Example 3.10: Let $\pounds = [0,\infty)$ and $\mathscr{A} = M_2(\mathbb{R})$ be a all 2×2 matrices whose norm is $||\mathscr{A}|| = \max\{a_1,a_2,a_3,a_4\}$ where $a_i's$ are the entries of $\mathscr{A}$. Then, clearly $(\pounds,\mathscr{A},\rho_\varsigma)$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with $\varsigma = 4$ whenever $\rho_\varsigma : \pounds^3 \to M_2(\mathbb{R})$ be as $\rho_\varsigma(\mathfrak{s},\mathfrak{e},\mathfrak{v}) = \begin{pmatrix}(|\mathfrak{e}+\mathfrak{v}-2\mathfrak{s}|+|\mathfrak{e}-\mathfrak{v}|)^2 & 0\end{pmatrix}$.

Let $\varphi : \mathscr{A}_+ \to \mathscr{A}_+$ defined by $\varphi(\aleph) = \frac{\aleph}{2}$, and $a \in \mathscr{A}$ with $||a|| = \frac{1}{\sqrt{2}} < 1$. We define mappings $\Gamma : \pounds \to \pounds$ as follows $\Gamma(\mathfrak{e}) = \begin{cases} \frac{\mathfrak{e}}{16(\mathfrak{e}+1)} & \text{if } \mathfrak{e} \in [0,1] \\ \ln\mathfrak{e} + |\sin\mathfrak{e}| & \text{if } \mathfrak{e} \in (1,\infty) \end{cases}$

Also, we define $\alpha,\beta : \pounds\times\pounds\times\pounds \to \mathscr{A}_+$ as $\alpha(\aleph,\aleph,\mathfrak{v}) = \beta(\aleph,\aleph,\mathfrak{v}) = \begin{cases} 2.1_{\mathscr{A}} & \text{if } \aleph,\mathfrak{v} \in [0,1] \\ 0_{\mathscr{A}} & \text{Otherwise} \end{cases}$

First, we show that Γ is a twisted (α,β)-adm. Let $\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \succeq 1_{\mathscr{A}}$ for $\mathfrak{s},\mathfrak{e} \in \pounds$, then $\mathfrak{s} \in [0,1]$ and $\mathfrak{e} \in [0,1]$ and so $\Gamma\mathfrak{e} \in [0,1]$ and $\Gamma\mathfrak{s} \in [0,1]$ implies that $\alpha(\Gamma\mathfrak{e},\Gamma\mathfrak{s}) \succeq 1_{\mathscr{A}}$,
similarly $\beta(\Gamma\mathfrak{e},\Gamma\mathfrak{s}) \succeq 1_{\mathscr{A}}$. Obviously, $\alpha(0,\Gamma 0) \succeq 1_{\mathscr{A}}$ and $\beta(0,\Gamma 0) \succeq 1_{\mathscr{A}}$. Now, let $\{\mathfrak{s}_\mathfrak{v}\}$ be a sequence in $\pounds$ such that $\alpha(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1}) \succeq 1_{\mathscr{A}}$ and $\beta(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1}) \succeq 1_{\mathscr{A}}$ for all $p \in \mathbb{N}\cup\{0\}$ and $\mathfrak{s}_\mathfrak{v} \to \mathfrak{s}$ as $p \to \infty$. This implies that $\{\mathfrak{s}_{\mathfrak{v}+1}\} \subseteq [0,1]$ and $\{\mathfrak{s}_\mathfrak{v}\} \subseteq [0,1]$. Thus, $\mathfrak{s} = 0$ and so $\alpha(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}) \succeq 1_{\mathscr{A}}$ and $\beta(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}) \succeq 1_{\mathscr{A}}$ for all $p \in \mathbb{N}\cup\{0\}$. Moreover, for $\mathfrak{s} \in [0,1]$ and $\mathfrak{e} \in [0,1]$, we have

$$\begin{aligned}
\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})\rho_\varsigma(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e})) &= 4.1_{\mathscr{A}}\begin{pmatrix}(|\Gamma\mathfrak{s}+\Gamma\mathfrak{e}-2\Gamma\mathfrak{s}|+|\Gamma\mathfrak{s}-\Gamma\mathfrak{e}|)^2 & 0\end{pmatrix} \\
&= 16\begin{pmatrix}|\frac{\mathfrak{s}}{4^2(1+\mathfrak{s})} - \frac{\mathfrak{e}}{4^2(\mathfrak{e}+1)}|^2 & 0\end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix}|\frac{\mathfrak{s}-\mathfrak{e}}{(1+\mathfrak{s})(1+\mathfrak{e})}|^2 & 0\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&\preceq \frac{1}{16}\begin{pmatrix}|\mathfrak{s}-\mathfrak{e}|^2 & 0\end{pmatrix} \\
&\preceq \frac{1}{(4^2)}\frac{1}{2}\left[\begin{pmatrix}\frac{1}{\sqrt{2}} & 0\end{pmatrix}\begin{pmatrix}(2|\mathfrak{s}-\mathfrak{e}|)^2 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & 0\end{pmatrix}\right] \\
&\preceq \frac{1}{\varsigma^2}\varphi\left(a^*\mathbb{M}_b^i(\mathfrak{s},\mathfrak{e})a\right)
\end{aligned}$$

where $a = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ with $||a|| = \frac{1}{\sqrt{2}} < 1$.

Otherwise, $\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e}) = 0$ and condition (A) trivially holds. Then Γ is a generalized twisted (α,β)-φ-contractive mapping of type (i) and, by Theorem 3.7, Γ has a FP.

Example 3.11: $\pounds$, $\mathscr{A}$, and ρ_ς be as in Example 3.10 and $\Gamma:\pounds\to\pounds$ as follows $\Gamma(\mathfrak{e})=\begin{cases}\frac{-1}{16\sqrt{3}}(\mathfrak{e}+\mathfrak{e}^2) & \text{if } \mathfrak{e}\in[-1,1]\\ \frac{\mathfrak{e}^2-\cos(\mathfrak{e}^5)}{2+\sin(\mathfrak{e})} & \text{if } \mathfrak{e}\in\mathbb{R}\backslash[-1,1]\end{cases}$. Let $\varphi:\mathscr{A}_+\to\mathscr{A}_+$ defined by $\varphi(\aleph)=\frac{\aleph}{2}$, and $a\in\mathscr{A}$ with $||a||=\frac{1}{\sqrt{3}}<1$.

Also, we define $\alpha,\beta:\pounds\times\pounds\times\pounds\to\mathscr{A}_+$ as $\alpha(\aleph,\aleph,\mathfrak{v})=\beta(\aleph,\aleph,\mathfrak{v})=\begin{cases}1_{\mathscr{A}} & \text{if } \aleph,\mathfrak{v}\in[-1,1]\\ 0_{\mathscr{A}} & \text{Otherwise}\end{cases}$.

We demonstrate the applicability of Theorem 3.7 to Γ. We conclude that Γ is a twisted (α,β)-adm and that the criteria (i) and (ii) of Theorem 3.7 hold by following the steps in the proof of Example 3.10. Moreover, if $0<||b||\le 1$ and $\mathfrak{s}\in[0,1]$ and $\mathfrak{e}\in[-1,0]$,

$$\begin{aligned}
&(\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})+b)^{\rho_\varsigma(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e})}=(1_{\mathscr{A}}+b)^{\left((|\Gamma\mathfrak{s}+\Gamma\mathfrak{e}-2\Gamma\mathfrak{s}|+|\Gamma\mathfrak{s}-\Gamma\mathfrak{e}|)^2 \quad 0\right)}\\
=\ &(1_{\mathscr{A}}+b)^{\frac{1}{3(4^3)}\left(|\mathfrak{s}+\mathfrak{s}^2-(\mathfrak{e}+\mathfrak{e}^2)|^2 \quad 0\right)}\\
=\ &(1_{\mathscr{A}}+b)^{\frac{1}{3(4^3)}\left((|\mathfrak{s}-\mathfrak{e}||\mathfrak{s}+\mathfrak{e}+1|)^2 \quad 0\right)}\\
\preceq\ &(1_{\mathscr{A}}+b)^{\frac{8}{3(4^3)}\left(|\mathfrak{s}-\mathfrak{e}|^2 \quad 0\right)}\\
\preceq\ &(1_{\mathscr{A}}+b)^{\frac{1}{3(32)}\left((2|\mathfrak{s}-\mathfrak{e}|)^2 \quad 0\right)}\\
\preceq\ &(1_{\mathscr{A}}+b)^{\frac{1}{(4^2)}\frac{1}{2}\left[(\frac{1}{\sqrt{3}} \quad 0)\left((2|\mathfrak{s}-\mathfrak{e}|)^2 \quad 0\right)(\frac{1}{\sqrt{3}} \quad 0)\right]}\\
\preceq\ &(1_{\mathscr{A}}+b)^{\frac{1}{\varsigma^2}\varphi\left(a^*\mathbb{M}_b^i(\mathfrak{s},\mathfrak{e})a\right)}
\end{aligned}$$

where $a=\begin{bmatrix}\frac{1}{\sqrt{3}} & 0\\ 0 & \frac{1}{\sqrt{3}}\end{bmatrix}$ with $||a||=\frac{1}{\sqrt{3}}<1$.

In the absence of this, $\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})=0$ and (B) is a trivially held condition. Then, by Theorem 3, Γ has a FP and is a generalized twisted (α,β)-φ-contractive mapping of type (ii).

Example 3.12: Let $\pounds=[0,\infty)$ and $\mathscr{A}=M_2(\mathbb{R})$ be all 2×2 matrices whose norm is $||\mathfrak{S}||=\sqrt{\sum_{i,j=1}^{2}|\mathfrak{s}_{ij}|^2}$, where $\mathfrak{S}=(\mathfrak{s}_{ij})\in\mathscr{A}$ and define the mapping $\rho:\pounds^2\to[0,\infty)$ as $\rho(\mathfrak{s},\mathfrak{e})=|\mathfrak{s}-\mathfrak{e}|^2$ for all $\mathfrak{s},\mathfrak{e}\in\pounds$. Then clearly, $(\pounds,\rho)$ is b-metric space with $\varsigma=2$. Let $\rho_\varsigma:\pounds^3\to M_2(\mathbb{R})$ be as $\rho_\varsigma(\mathfrak{s},\mathfrak{t},\mathfrak{r})=\begin{bmatrix}\rho(\mathfrak{s},\mathfrak{t})+\rho(\mathfrak{t},\mathfrak{r})+\rho(\mathfrak{r},\mathfrak{s}) & 0\\ 0 & \rho(\mathfrak{s},\mathfrak{t})+\rho(\mathfrak{t},\mathfrak{r})+\rho(\mathfrak{r},\mathfrak{s})\end{bmatrix}$. Then, clearly $(\pounds,\mathscr{A},\rho_\varsigma)$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with $||\varsigma||=2\ge 1$.

Let $\varphi:\mathscr{A}_+\times\mathscr{A}_+\to\mathscr{A}$ defined by $\varphi(\mathfrak{v})=\frac{\mathfrak{v}}{2}$ and $a\in\mathscr{A}$ with $||a||=\frac{1}{\sqrt{6}}<1$.

We define mappings $\Gamma(\mathfrak{e}) = \begin{cases} \frac{1}{8\sqrt{5}}\mathfrak{e}^2 & \text{if } \mathfrak{e} \in [0,1] \\ \frac{1}{\mathfrak{e}} - \frac{1}{1+\mathfrak{e}} & \text{if } \mathfrak{e} \in (1,\infty) \end{cases}$.

Also, we define $\alpha,\beta : \pounds^3 \to \mathscr{A}_+$ as $\alpha(\aleph,\aleph,\mathfrak{v}) = \beta(\aleph,\aleph,\mathfrak{v}) = \begin{cases} 1_{\mathscr{A}} & \text{if } \aleph,\mathfrak{v} \in [0,1] \\ 0_{\mathscr{A}} & \text{Otherwise} \end{cases}$.

We demonstrate the applicability of Theorem 3.7 to Γ. We conclude that Γ is a twisted (α,β)-adm and that the criteria (i) and (ii) of Theorem 3.7 hold by following the steps in the proof of Example 3.10. Additionally, if $||b|| \geq 1$ and $\mathfrak{s},\mathfrak{e} \in [0,1]$, then we have

$$\begin{aligned}
&\left(\rho_\varsigma(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e}) + b\right)^{\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})} = \left(2|\Gamma\mathfrak{s} - \Gamma\mathfrak{e}|^2 \quad 0\right) + b \\
= \;& \frac{2}{5(4^3)}\left(|\mathfrak{s}^2 - \mathfrak{e}^2|^2 \quad 0\right) + b \\
= \;& \frac{2}{5(4^3)}\left((|\mathfrak{s} - \mathfrak{e}||\mathfrak{s} + \mathfrak{e}|)^2 \quad 0\right) + b \\
\preceq \;& \frac{2}{5(4^3)}\left(|\mathfrak{s} - \mathfrak{e}|^2 \quad 0\right) + b \\
\preceq \;& \frac{1}{(4^2)}\frac{1}{4}\left[\left(\frac{1}{\sqrt{5}} \quad 0\right)\left(2|\mathfrak{s} - \mathfrak{e}|^2 \quad 0\right)\left(\frac{1}{\sqrt{5}} \quad 0\right)\right] + b \\
\preceq \;& \frac{1}{\varsigma^2}\varphi\left(a^*\mathbb{M}_b^i(\mathfrak{s},\mathfrak{e})a\right) + b
\end{aligned}$$

where $a = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix}$ with $||a|| = \frac{1}{\sqrt{5}} < 1$.

In the absence of this, $\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e}) = 0$ and (C) are essentially true. According to Theorem 3.7, Γ has a fixed point and is a generalized twisted (α,β)-φ-contractive mapping of type (iii).

3.1. Application to Integral Equations.

Suppose that $\mathscr{E} = [0,1]$ be a Lebesgue measurable set with $m(\mathscr{E}) < \infty$ such that $\pounds = L^\infty(\mathscr{E})$ and $B(L^2(\mathscr{E}))$ is a set of bounded linear operators on a Hilbert space $L^2(\mathscr{E})$. We equip $\pounds$ with $\rho_\varsigma : \pounds \times \pounds \times \pounds \to B(L^2(\mathscr{E}))$, which is ascertained by $\rho_\varsigma(\mathfrak{e},\text{ß},\text{œ}) = \mathbb{M}_{(|\mathfrak{e}-\text{œ}|+|\text{ß}-\text{œ}|)^{\mathfrak{v}}}$, where $\mathbb{M}_{(|\mathfrak{e}-\text{œ}|+|\text{ß}-\text{œ}|)^{\mathfrak{v}}}$ is the multiplication operator on $L^2(\mathscr{E})$ ascertained by $\mathbb{M}_h(\alpha) = h.\alpha$, $\alpha \in L^2(\mathscr{E})$. Consequently, $(\pounds, B(L^2(\mathscr{E})), \rho_\varsigma)$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with $\varsigma = 2^{2(p-1)}$ where $p > 1$.

Examining the integral equation

$$\mathfrak{s}(\iota)=\mathfrak{f}(\iota)+\int_{\mathscr{E}}\mathfrak{K}(\iota,\varsigma)\mathfrak{g}(\varsigma,\mathfrak{s}(\varsigma))d\varsigma. \tag{11}$$

$$\text{Let } \Gamma:\pounds\to\pounds \text{ as } \Gamma(\mathfrak{s})(\iota)=\mathfrak{f}(\iota)+\int_{\mathscr{E}}\mathfrak{K}(\iota,\varsigma)\mathfrak{g}(\varsigma,\mathfrak{s}(\varsigma))d\varsigma \tag{12}$$

Assume that the following circumstances are met:

(i_0) $\mathfrak{g}:\mathscr{E}\times\mathbb{R}\to\mathbb{R}$ is continuous;

(i_1) $\mathfrak{f}:\mathscr{E}\to\mathbb{R}$ is continuous;

(i_2) $\mathfrak{K}:\mathscr{E}\times\mathbb{R}\to\mathbb{R}$ is continuous;

(i_3) there exists $\varphi\in\mathfrak{D}$, $\eta:\pounds^3\to\mathscr{A}_+$ such that $\eta(a,a,b)\succeq 0_{\mathscr{A}}$ for $a,b\in\pounds$, then for every $\varsigma\in\mathscr{E}$ we have $\lambda\in(0,1)$ such that
$|\mathfrak{g}(\varsigma,\mathfrak{s}(\varsigma))-\mathfrak{g}(\varsigma,\mathfrak{s}(\tau))|\le\frac{\lambda}{4\sqrt{2}}||\mathfrak{s}(\varsigma)-\mathfrak{s}(\tau)||$ for all $\varsigma,\tau\in\mathscr{E}$;

(i_4) $\exists\,\mathfrak{s}_0\in\pounds\ni\eta(\mathfrak{s}_0(\varsigma),\mathfrak{s}_0(\varsigma),\Gamma\mathfrak{s}_0(\varsigma))\succeq 0_{\mathscr{A}}$ for all $\varsigma\in\mathscr{E}$;

(i_5) for each $\varsigma\in\mathscr{E}$ and $\mathfrak{s},\mathfrak{e}\in\pounds$, $\eta(\mathfrak{s}(\varsigma),\mathfrak{s}(\varsigma),\mathfrak{e}(\varsigma))\succ 0_{\mathscr{A}}\Rightarrow\eta(\Gamma\mathfrak{s}(\varsigma),\Gamma\mathfrak{s}(\varsigma),\Gamma\mathfrak{e}(\varsigma))\succ 0_{\mathscr{A}}$;

(i_6) for each $\varsigma\in\mathscr{E}$ and $\{\mathfrak{s}_{\mathfrak{p}}\}\subseteq\pounds$ be a sequence such that $\mathfrak{s}_{\mathfrak{p}}\to\mathfrak{s}$ in $\pounds$ and $\eta(\mathfrak{s}_{\mathfrak{p}}(\varsigma),\mathfrak{s}_{\mathfrak{p}}(\varsigma),\mathfrak{s}_{\mathfrak{p}+1}(\varsigma))\succ 0_{\mathscr{A}}$ for all $p\in\mathbb{N}$ then $\eta(\mathfrak{s}_{\mathfrak{p}}(\varsigma),\mathfrak{s}_{\mathfrak{p}}(\varsigma),\mathfrak{s}(\varsigma))\succ 0_{\mathscr{A}}$ for all $p\in\mathbb{N}$;

(i_7) $\int_{\mathscr{E}}|\mathfrak{K}(\iota,\varsigma)|d\varsigma\le 1$ for every $\iota\in\mathscr{E}$ and $\varsigma\in\mathbb{R}$.

Theorem 3.1.1: Under the assumption (i_0)-(i_7), the equation (11) has a solution in $L^\infty(\mathscr{E})$.

Proof Consider the mapping Γ defined by Eq.12. By (i_3), we deduce

$$\rho_\varsigma(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e})=\mathbb{M}_{(2|\Gamma\mathfrak{s}-\Gamma\mathfrak{e}|)^{\mathfrak{p}}}$$

We obtain that

$$\begin{aligned}
||\rho_\varsigma(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e})|| &= \sup_{||h||=1}\left\langle\mathbb{M}_{(2|\Gamma\mathfrak{s}(\iota)-\Gamma\mathfrak{e}(\iota)|)^{\mathfrak{p}}}h,h\right\rangle\\
&= \sup_{||h||=1}\left\langle 2^{\mathfrak{p}}\mathbb{M}_{|\Gamma\mathfrak{s}(\iota)-\Gamma\mathfrak{e}(\iota)|^{\mathfrak{p}}}h,h\right\rangle\\
&= \sup_{||h||=1}\int_{\mathscr{E}}(2^{\mathfrak{p}}|\Gamma\mathfrak{s}(\iota)-\Gamma\mathfrak{e}(\iota)|^{\mathfrak{p}})h(\varsigma)\overline{h(\varsigma)}d\varsigma
\end{aligned}$$

$$\leq \quad 2^{\mathfrak{v}} \sup_{||h||=1} \int_{\mathscr{E}} \left[\int_{\mathscr{E}} |\mathfrak{K}(\iota,\varsigma)||\mathfrak{g}(\varsigma,\mathfrak{s}(\varsigma)) - \mathfrak{g}(\varsigma,\mathfrak{e}(\varsigma))| \right]^{\mathfrak{v}} |h(\varsigma)|^2 d\varsigma$$

$$\leq \quad 2^{\mathfrak{v}} \sup_{||h||=1} \int_{\mathscr{E}} \left[\int_{\mathscr{E}} |\mathfrak{K}(\iota,\varsigma)| d\varsigma \right]^{\mathfrak{v}} |h(\varsigma)|^2 (\frac{\lambda}{4\sqrt{2}})^{\mathfrak{v}} ||\mathfrak{s}-\mathfrak{e}||_{\infty}^{\mathfrak{v}}$$

$$\leq \quad \frac{\lambda}{(4\sqrt{2})^{\mathfrak{v}}} \sup_{\iota\in\mathscr{E}} \left[\int_{\mathscr{E}} |\mathfrak{K}(\iota,\varsigma)| d\varsigma \right]^{\mathfrak{v}} \sup_{||h||=1} \int_{\mathscr{E}} |h(\varsigma)|^2 ||2(\mathfrak{s}-\mathfrak{e})||_{\infty}^{\mathfrak{v}}$$

$$\leq \quad \frac{\lambda}{(4\sqrt{2})^{\mathfrak{v}}} \sup_{\iota\in\mathscr{E}} \left[\int_{\mathscr{E}} |\mathscr{K}(\iota,\varsigma)| d\varsigma \right]^{\mathfrak{v}} ||\mathbb{M}_{(|\mathfrak{s}-\mathfrak{e}|+|\mathfrak{s}-\mathfrak{e}|)^{\mathfrak{v}}}||.$$

By setting $p=2$ and $a=\lambda 1_{B(L^2(\mathscr{E}))}$, then $a\in B(L^2(\mathscr{E}))$ so that $||a||=\lambda<1$, since $\int_{\mathscr{E}} |\mathscr{K}(\iota,\varsigma)|d\varsigma < 1$, then it follows that

$$||\rho_{\varsigma}(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e})|| \quad \leq \tfrac{1}{2}\tfrac{1}{||\varsigma||^2}||a||^2||\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e})|| \tag{13}$$

Let $\varphi:\mathscr{A}_+\to\mathscr{A}$ as $\varphi(x)=\frac{x}{2}\ \forall\, x\in\mathscr{A}_+$. For $\varsigma\in[0,1]$ the following is defined:
$\alpha,\beta : \pounds\times\pounds\times\pounds\to\mathscr{A}_+$ as $\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})=\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})=\begin{cases} 1_{\mathscr{A}} & \text{if } \eta(\mathfrak{s}(\varsigma),\mathfrak{s}(\varsigma),\mathfrak{e}(\varsigma))\succ 0_{\mathscr{A}} \\ 0_{\mathscr{A}} & \text{Otherwise}\end{cases}$
From (i_4) and (i_5) we have $||\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})||\geq 1$ and $||\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})||\geq 1$.
From (13), we have

$$||\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})||||\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})||||\rho_{\varsigma}(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e})|| \leq \frac{1}{||\varsigma||^2}||a||^2||\varphi\left(\rho_{\varsigma}(\mathfrak{s},\mathfrak{s},\mathfrak{e})\right)|| \tag{14}$$

Hence

$$\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})\rho_{\varsigma}(\Gamma\mathfrak{s},\Gamma\mathfrak{s},\Gamma\mathfrak{e}) \preceq \frac{1}{\varsigma^2}\varphi\left(a^{\star}\mathbb{M}_b^i(\mathfrak{s},\mathfrak{e})a\right).$$

Since Theorem 3.6's hypotheses are all satisfied, the mapping Γ has a FP, and this FP is a solution in $L^{\infty}(\mathscr{E})$ for the integral equation (12).

3.2. Applications to Homotopy.

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 3.2.1: If $(\pounds,\mathscr{A},\rho_{\varsigma})$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$ with $||\varsigma||\geq 1$, then $\mathscr{D}$ and $\overline{\mathscr{D}}$ are open and closed subsets of $\pounds$ such that $\mathscr{D}\subseteq\overline{\mathscr{D}}$. Let $\mathscr{L}_{\varsigma}:\overline{\mathscr{D}}\times[0,1]\to\pounds$ be an homotopy operator meeting the requirements listed below.

(τ_0) $\mathfrak{s} \neq \mathscr{L}_\varsigma(\mathfrak{s},s)$, for each $\mathfrak{s} \in \partial\mathscr{D}$ and $s \in [0,1]$ (here $\partial\mathscr{D}$ is boundary of $\mathscr{D}$ in £)

(τ_1) there exist $\mathfrak{s},\mathfrak{e} \in \overline{\mathscr{D}}$ and $a \in \mathscr{A}$ with $||a|| < 1$ such that
$\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e})\beta(\mathfrak{s},\mathfrak{s},\mathfrak{e})\rho_\varsigma(\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{e},s)) \preceq \frac{1}{4\varsigma^2}\varphi(a^*\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{e})a)$
where, $\varphi \in \mathfrak{D}$ and $s \in [0,1]$;

(τ_2) $\exists\, M_b \succeq 0_{\mathscr{A}} \ni \rho_\varsigma(\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{s},t)) \leq ||M_b|||s-t|$ for every $\mathfrak{s} \in \overline{\mathscr{D}}$ and $s,t \in [0,1]$;

(τ_3) $\mathscr{L}_\varsigma$ is a twisted (α,β)-admissible mapping;

(τ_4) $\exists\, \mathfrak{s}_0 \in$ £ such that $\alpha(\mathfrak{s}_0,\mathfrak{s}_0,\mathscr{L}_\varsigma(\mathfrak{s}_0,s)) \succeq 1_{\mathscr{A}}$, $\beta(\mathfrak{s}_0,\mathfrak{s}_0,\mathscr{L}_\varsigma(\mathfrak{s}_0,s)) \succeq 1_{\mathscr{A}}$.

Then $\mathscr{L}_\varsigma(.,0)$ has a FP $\iff$ $\mathscr{L}_\varsigma(.,1)$ has a FP.

Proof Consider the following set: $\mathscr{G} = \{s \in [0,1] : \mathfrak{s} = \mathscr{L}_\varsigma(\mathfrak{s},s)$ for some $\mathfrak{s} \in \mathfrak{s}\}$. We have that $0 \in \mathscr{G}$ since $\mathscr{L}_\varsigma(\cdot,0)$ has an FP in $\mathscr{D}$. This means that the set $\mathscr{G}$ is not empty. By proving that $\mathscr{G}$ is both open and closed in $[0,1]$, we will be able to demonstrate that $\mathscr{G} = [0,1]$. As a result, $\mathscr{L}_\varsigma(\cdot,1)$ has an FP in $\mathscr{D}$. We first demonstrate the closure of $\mathscr{G}$ in $[0,1]$. In order to observe this, let $\{s_\mathfrak{v}\}_{\mathfrak{v}=1}^\infty \subseteq \mathscr{G}$ and assign $s_\mathfrak{v} \to s \in [0,1]$ as $\mathfrak{v} \to \infty$. s must be proven to be in $\mathscr{G}$. There is $\mathfrak{s}_\mathfrak{v}$ in $\mathscr{G}$ given that $s_\mathfrak{v}$ for $\mathfrak{v} = 0,1,2,...$, we have $\mathfrak{s}_\mathfrak{v} = \mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v})$.
Since $\mathscr{L}_\varsigma$ is a twisted (α,β)-adm, then $\alpha(\mathfrak{s}_0,\mathfrak{s}_0,\mathfrak{s}_1) = \alpha(\mathfrak{s}_0,\mathfrak{s}_0,\mathscr{L}_\varsigma(\mathfrak{s}_1,s_1)) \succeq 1_{\mathscr{A}}$ then
$\alpha(\mathfrak{s}_2,\mathfrak{s}_2,\mathfrak{s}_1) = \alpha(\mathscr{L}_\varsigma(\mathfrak{s}_2,s_2),\mathscr{L}_\varsigma(\mathfrak{s}_2,s_2),\mathscr{L}_\varsigma(\mathfrak{s}_1,s_1)) \succeq 1_{\mathscr{A}}$ implies
$\alpha(\mathfrak{s}_2,\mathfrak{s}_2,\mathfrak{s}_3) = \alpha(\mathscr{L}_\varsigma(\mathfrak{s}_2,s_2),\mathscr{L}_\varsigma(\mathfrak{s}_2,s_2),\mathscr{L}_\varsigma(\mathfrak{s}_3,s_3)) \succeq 1_{\mathscr{A}}$. By repeating similar process, we obtain $\alpha(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1}) \succeq 1_{\mathscr{A}}$ and $\alpha(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}-1}) \succeq 1_{\mathscr{A}}$.
Similarly, we have $\beta(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1}) \succeq 1_{\mathscr{A}}$ and $\beta(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}-1}) \succeq 1_{\mathscr{A}}$ for every $\mathfrak{v} \in N$.
Now consider,

$$\begin{aligned}
\rho_\varsigma(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1}) &= \rho_\varsigma(\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{v}+1},s_{\mathfrak{v}+1}))\\
&\preceq 2\varsigma\rho_\varsigma(\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{v}+1},s_\mathfrak{v}))\\
&\quad +\varsigma^2\rho_\varsigma(\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{v}+1},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{v}+1},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{v}+1},s_{\mathfrak{v}+1}))\\
&\preceq 2\varsigma\rho_\varsigma(\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{v}+1},s_\mathfrak{v}))+\varsigma^2||M_b|||s_\mathfrak{v}-s_{\mathfrak{v}+1}|.
\end{aligned}$$

Letting $\mathfrak{v} \to \infty$, we obtain

$$\frac{1}{2\varsigma}\lim_{\mathfrak{v}\to\infty}\rho_\varsigma(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1}) \preceq \lim_{\mathfrak{v}\to\infty}\rho_\varsigma(\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{v}+1},s_\mathfrak{v}))+0_{\mathscr{A}}$$

$$
\begin{aligned}
&\preceq \lim_{\mathfrak{v}\to\infty} \alpha(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1})\beta(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1})\rho_\varsigma(\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s_\mathfrak{v}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{v}+1},s_\mathfrak{v}))\\
&\preceq \lim_{\mathfrak{v}\to\infty} \frac{1}{4\varsigma^2}\varphi\left(a^*\rho_\varsigma(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1})a\right).
\end{aligned}
$$

Since φ is continuous and non-decreasing, we obtain

$$
(2||\varsigma||-||a||^2)\lim_{\mathfrak{v}\to\infty}||\rho_\varsigma(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1})||<0.
$$

So that

$$
\lim_{\mathfrak{v}\to\infty}||\rho_\varsigma(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1})||=0.
$$

It is now time to demonstrate the $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-CS $\{\mathfrak{s}_\mathfrak{v}\}$ in $(£,\mathscr{A},\rho_\varsigma)$. On the other hand, suppose $\{\mathfrak{s}_\mathfrak{v}\}$ is not a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-CS. Natural numbers $\{\mathfrak{z}_k\}$ and $\{\mathfrak{v}_k\}$ can be arranged in a monotone increasing sequence with $\varepsilon>0$ such that $\mathfrak{v}_k>\mathfrak{z}_k$,

$$
\rho_\varsigma\left(\mathfrak{s}_{\mathfrak{z}_k},\mathfrak{s}_{\mathfrak{z}_k},\mathfrak{s}_{\mathfrak{v}_k}\right)\succeq\varepsilon \tag{15}
$$

and

$$
\rho_\varsigma\left(\mathfrak{s}_{\mathfrak{z}_k},\mathfrak{s}_{\mathfrak{z}_k},\mathfrak{s}_{\mathfrak{v}_{k-1}}\right)\prec\varepsilon. \tag{16}
$$

From (15) and (16), we have

$$
\begin{aligned}
\varepsilon &\preceq \rho_\varsigma\left(\mathfrak{s}_{\mathfrak{z}_k},\mathfrak{s}_{\mathfrak{z}_k},\mathfrak{s}_{\mathfrak{v}_k}\right)\\
&\preceq 2\varsigma\rho_\varsigma\left(\mathfrak{s}_{\mathfrak{z}_k},\mathfrak{s}_{\mathfrak{z}_k},\mathfrak{s}_{\mathfrak{z}_{k+1}}\right)+\varsigma^2\rho_\varsigma\left(\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{v}_k}\right).
\end{aligned}
$$

Letting $p\to\infty$, we obtain

$$
\frac{\varepsilon}{\varsigma^2} \preceq \lim_{\mathfrak{v}\to\infty}\rho_\varsigma\left(\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{v}_k}\right). \tag{17}
$$

But

$$
\begin{aligned}
&\lim_{\mathfrak{v}\to\infty}\rho_\varsigma\left(\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{v}_k}\right)=\lim_{\mathfrak{v}\to\infty}\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{z}_{k+1}},s_{\mathfrak{z}_{k+1}}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{z}_{k+1}},s_{\mathfrak{z}_{k+1}}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{v}_k},s_{\mathfrak{v}_k})\right)\\
\preceq\ &\lim_{\mathfrak{v}\to\infty}\alpha(\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{v}_k})\beta(\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{v}_k})\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{z}_{k+1}},s_{\mathfrak{z}_{k+1}}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{z}_{k+1}},s_{\mathfrak{z}_{k+1}}),\mathscr{L}_\varsigma(\mathfrak{s}_{\mathfrak{v}_k},s_{\mathfrak{v}_k})\right)\\
\preceq\ &\lim_{\mathfrak{v}\to\infty}\frac{1}{4\varsigma^2}\varphi\left(a^\star\rho_\varsigma(\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{v}_k})a\right)\\
\prec\ &\lim_{\mathfrak{v}\to\infty}\frac{1}{4\varsigma^2}a^\star\rho_\varsigma(\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{v}_k})a.
\end{aligned}
$$

It follows that

$$\left(4||\varsigma||^2-||a||^2\right)\lim_{\mathfrak{v}\to\infty}||\rho_\varsigma\left(\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{z}_{k+1}},\mathfrak{s}_{\mathfrak{v}_k}\right)||<0.$$

From Eq.(17), we get that $\varepsilon<0$, which is a contradiction.

In the C^*-AV-ρ_ςMS $(\pounds,\mathscr{A},\rho_\varsigma)$, the sequence $\{\mathfrak{s}_\mathfrak{v}\}$ is a C^*-AV-ρ_ς-CS. The sequence $\{\mathfrak{s}_\mathfrak{v}\}\to\mathfrak{e}\in\left(\pounds,\mathscr{A},\rho_\varsigma\right)$ comes from the completeness of $(\pounds,\mathscr{A},\rho_\varsigma)$.

$$\lim_{\mathfrak{v}\to\infty}\mathfrak{s}_{\mathfrak{v}+1}=\mathfrak{e}=\lim_{\mathfrak{v}\to\infty}\mathfrak{s}_\mathfrak{v}.$$

We can prove $\mathfrak{e}=\mathscr{L}_\varsigma(\mathfrak{e},s)$.

From (τ_4), we have $\alpha\left(\mathscr{L}_\varsigma(\mathfrak{e},s),\mathscr{L}_\varsigma(\mathfrak{e},s),\mathfrak{e}\right)\succeq 1_\mathscr{A}$, $\beta\left(\mathscr{L}_\varsigma(\mathfrak{e},s),\mathscr{L}_\varsigma(\mathfrak{e},s),\mathfrak{e}\right)\succeq 1_\mathscr{A}$ $\forall$ $\mathfrak{e}\in\pounds$.

Now consider,

$$\begin{aligned}
&\frac{1}{2\varsigma}\rho_\varsigma(\mathscr{L}_\varsigma(\mathfrak{e},s),\mathscr{L}_\varsigma(\mathfrak{e},s),\mathfrak{e})\preceq\lim_{\mathfrak{v}\to\infty}\inf\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{e},s),\mathscr{L}_\varsigma(\mathfrak{e},s),\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s)\right)\\
\preceq\;&\lim_{\mathfrak{v}\to\infty}\inf\alpha(\mathscr{L}_\varsigma(\mathfrak{e},s),\mathscr{L}_\varsigma(\mathfrak{e},s),\mathfrak{s}_\mathfrak{v})\beta(\mathscr{L}_\varsigma(\mathfrak{e},s),\mathscr{L}_\varsigma(\mathfrak{e},s),\mathfrak{s}_\mathfrak{v})\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{e},s),\mathscr{L}_\varsigma(\mathfrak{e},s),\mathscr{L}_\varsigma(\mathfrak{s}_\mathfrak{v},s)\right)\\
\preceq\;&\lim_{\mathfrak{v}\to\infty}\inf\frac{1}{4\varsigma^2}\varphi\left(a^\star\rho_\varsigma\left(\mathfrak{e},\mathfrak{e},\mathfrak{s}_\mathfrak{v}\right)a\right)=0_\mathscr{A}.
\end{aligned}$$

Accordingly, $\mathscr{L}_\varsigma(\mathfrak{e},s)=\mathfrak{e}$ indicates that $\rho_\varsigma(\mathscr{L}_\varsigma(\mathfrak{e},s),\mathscr{L}_\varsigma(\mathfrak{e},s),\mathfrak{e})=0_\mathscr{A}$. So, s in $\mathscr{G}$. It is obvious that $\mathscr{G}$ is closed in $[0,1]$. Let $\mathscr{G}$ be s_0. Then, $\mathfrak{s}_0$ exists in $\mathscr{D}$ such that $\mathfrak{s}_0=\mathscr{L}_\varsigma(\mathfrak{s}_0,s_0)$. Because $\mathscr{D}$ is open, $\delta>0$ must exist for $B_{\rho_\varsigma}(\mathfrak{s}_0,\delta)\subseteq\mathscr{D}$. Select the value of $s\in(s_0-\varepsilon,s_0+\varepsilon)$ such that $|s-s_0|\leq\frac{1}{||M_b||^\mathfrak{v}}<\varepsilon$.

Consequently, for $\overline{B_b(\mathfrak{s}_0,\delta)}=\{\mathfrak{s}\in\pounds:\frac{1}{4}||\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{s}_0)||\leq\delta+\varsigma^2||\rho_\varsigma(\mathfrak{s}_0,\mathfrak{s}_0,\mathfrak{s}_0)||\}$.

$$\begin{aligned}
\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{s},s),\mathfrak{s}_0\right)\;&=\;\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{s}_0,s_0)\right)\\
&\preceq\;2\varsigma\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{s},s_0)\right)\\
&\qquad+\varsigma^2\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s}_0,s_0)\right)\\
&\preceq\;2\varsigma||M_b|||s-s_0|+\varsigma^2\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s}_0,s_0)\right)\\
&\preceq\;\frac{2\varsigma}{||M_b||^{\mathfrak{v}-1}}+\varsigma^2\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s}_0,s_0)\right).
\end{aligned}$$

Letting $\mathfrak{v}\to\infty$, we get that

$$\frac{1}{\varsigma^2}\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{s},s),\mathfrak{s}_0\right)\preceq\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s}_0,s_0)\right)$$

$$\preceq \alpha\left(\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathfrak{s}_0\right)\beta\left(\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathfrak{s}_0\right)\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s},s_0),\mathscr{L}_\varsigma(\mathfrak{s}_0,s_0)\right)$$
$$\preceq \frac{1}{4\varsigma^2}\varphi(a^\star\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{s}_0)a)$$

Since $||a||<1$ and φ is continuous and non-decreasing, we obtain implies that

$$||\rho_\varsigma\left(\mathscr{L}_\varsigma(\mathfrak{s},s),\mathscr{L}_\varsigma(\mathfrak{s},s),\mathfrak{s}_0\right)||<\frac{1}{4}||\rho_\varsigma(\mathfrak{s},\mathfrak{s},\mathfrak{s}_0)||\leq\delta+\varsigma^2||\rho_\varsigma(\mathfrak{s}_0,\mathfrak{s}_0,\mathfrak{s}_0)||.$$

Consequently, for every fixed $s\in(s_0-\varepsilon,s_0+\varepsilon)$ and the function $\mathscr{L}_\varsigma(.,0):\overline{B_b(\mathfrak{s}_0,\delta)}\to\overline{B_b(\mathfrak{s}_0,\delta)}$. At that point, Theorem (3.2.1) holds in all cases.Consequently, we deduce that there is an FP for $\mathscr{L}_\varsigma(.,0)$ in $\overline{\mathscr{D}}$. However, this needs to be in $\mathscr{D}$. Consequently, for $s\in(s_0-\varepsilon,s_0+\varepsilon)$, $s\in\mathscr{G}$. Thus, $(s_0-\varepsilon,s_0+\varepsilon)\subseteq\mathscr{G}$. $\mathscr{G}$ is obviously open in $[0,1]$.
The contrary can be shown using a similar process.

4. Conclusions

In complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-S_b-$\mathbb{MS}$, this study proposed a framework to establish FP results by using (α,β)-φ-type contraction via twisted (α,β)-adm. The framework is supported by relevant examples that highlight the main findings. There are also applications to homotopy and integral equations provided.

Author Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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