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BETA-IDEAL LOCAL FUNCTIONS IN NANO ANTI-HAUSDORFF SPACES

P. GAYATHRI[1], G. SELVI[1,*], K. RUTH ISABELS[2], A. THIRUPPATHI[3]

[1]Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, SIMATS, Chennai-602105, Tamil Nadu, India

[2]Department of Mathematics, Saveetha Engineering College - Autonomous, Chennai -602105, Tamil Nadu, India

[3]Department of Mathematics, Panimalar Engineering College,Varadharajapuram, Poonamallee , Chennai-602123, Tamil Nadu, India

Abstract. Recent research has expanded upon established generalized set classes within the system of nano anti-Hausdorff β- local functions, specifically under conditions of ideals. In this research, a number of new set classes are established which underscore an important place of beta-ideals in the topological structure of the nano anti-Hausdorff, $\mathscr{N}\mathscr{A}\mathscr{H}^X_\star$. These novel collections are examined in order to investigate the delicate structural features as well as the interconnections of beta-ideals, within nano topological spaces, as gauged through the protracted view of $\mathscr{N}\mathscr{A}\mathscr{H}^X_\star$. The results lead to a more theoretical insight into the area of nano topology and a superior insight into the applications thereof concerning the use of beta-ideals.

Keywords: $_\beta\mathbb{I}^{nc}_{t}$-set; $_\beta\mathbb{I}^{nc}_{t_\alpha}$-set; $_\beta\mathbb{I}^{nc}_{\mathscr{R}}$-set; $_\beta\mathbb{I}^{nc}_{\mathscr{R}_\alpha}$-set; $_\beta\mathbb{I}^{nah}_{t^\#}$-set; $_\beta\mathbb{I}^{nah}_{t^\#_\alpha}$-set; $_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#}$-set; $_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#_\alpha}$-set.

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1. Introduction

The concept of ideals in topological spaces was invented independently by R. Vaidyanathaswamy [17, 18] and K. Kuratowski [3] simplifying much modern activity in nano

*Corresponding author
E-mail address: mslalima11@gmail.com
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topology and generalized open sets. During their work on early set topology, they focused on the importance of algebraic and order theoretic structures, e.g. ideals, in investigating more refined classifications of open and closed sets. The lower approximation, upper approximation, and boundary regions are the concepts that are introduced in the exploration of rough set theory by Z. Pawlak [7]. These tools provided a new perspective in terms of which uncertainty in topological constructions might be represented in a formal way and tackled therewith. The relationship between nano topology and rough set theory subsequently allowed a re-realisation of nano sets as approximation spaces.

M. Lellis Thivagar and Carmel Richard [4] advanced this idea by investigating weaker forms of nano open sets. Their work brought clarity to the understanding of nano semi-open, pre-open, and generalized weakly open sets, providing vital groundwork for analyzing continuity and separation axioms in nano topological contexts. In particular, the study of contra continuity and weak continuity by M. Lellis Thivagar, Saeid Jafari, and V. Sutha Devi [5] further enriched the framework by defining novel functions and exploring their preservation under nano structures.

K. Bhuvaneswari and K. Mythili Gnanapriya [1] contributed by characterizing nano generalized locally closed sets and introducing NGLC-continuous functions. Their work helps define finer classes of continuous mappings, where local closedness is preserved under nano constraints. Similarly, the contributions of O. Nethaji et al. [6] in developing ideal nano topological spaces through generalized classes have bridged the gap between ideal theory and nano structures.

Recent works by Parimala and Jafari [8, 9] continue this thread by formulating nano ideal generalized closed sets and exploring new notions of nano closure. These investigations complement foundational studies by I. Rajasekaran and collaborators [10, 11, 12], who introduced various semi-local and simple forms of nano open sets. Their ongoing research highlights how these generalizations influence both local and global topological behavior.

More contemporary efforts by Gayathri, Selvi, and Rajasekaran [13] propose weak forms of ideal nano topologies tailored to semi-local functions. This is a natural extension of the foundational notion of localization introduced by Vaidyanathaswamy, reinforcing the relevance of these early ideas. Furthermore, Sekar et al. [14] examined regular closed sets in these environments, refining understanding of closure operations in nano ideal topologies.

The work of Selvi and Rajasekaran [15] on nano Mr-sets and M*r-sets offered new classifications that help identify regularity and separation at finer levels. Meanwhile, the study on nano semi pre-neighbourhoods by Sathishmohan et al. [16] addressed how neighbourhood structures behave under nano frameworks, thus supporting topological function analysis.

Finally, the introduction of nano semi-local functions by P. Gayathri (2024) [13, 19] presents a new frontier for examining continuity, compactness, and connectedness in ideal nano topological spaces. These functions serve as a unifying thread among various forms of generalized continuity, giving rise to new applications in nano rough set theory and generalized topology.

Altogether, this lineage of research showcases the evolution from classical ideals and approximation spaces to intricate nano topological structures, revealing rich interdependencies and opening the door for deeper studies in nano generalized spaces, ideal continuity, and semi-local analysis.

The following symbol is used by us throughout this paper: nano connected-open, nano connected-closed (resp. *nc*-$\mathscr{OS}$, *nc*-$\mathscr{CS}$) and $(U,\mathscr{NC})$ Let us define by $\mathscr{NC}^X$, for ideal nano topological spaces $(U,\mathbb{I},\mathscr{NC})$ Let us define by $\mathscr{NC}^X_\star$. The following symbol is used by us throughout this paper: nano anti-haussdorff -open, nano anti-haussdorff -closed (resp. *nah*-$\mathscr{OS}$, *nah*-$\mathscr{CS}$) and $(U,\mathscr{NAH})$ Let us define by $\mathscr{NAH}^X$, for ideal nano topological spaces $(U,\mathbb{I},\mathscr{NAH})$ Let us define by $\mathscr{NAH}^X_\star$.

In the current work, Further research is being conducted using the existing generalised classes in nano anti-haussdorff beta-local functions in ideal, and we have constructed and introduced the notions of some new sets that look at and deal with beta-ideal nano anti-haussdorff topological spaces. Further research is being carried out in beta-ideal nano anti-haussdorff topological spaces using the created generalised classes of $\mathscr{NAH}^X_\star$

2. Preliminaries

Definition 2.1. *Consider a non-empty finite set U of elements, referred to as the* universe*, and let $\mathfrak{R}$ be an equivalence relation on U, known as the* indiscernibility relation*. This relation partitions U into mutually disjoint equivalence classes . That is* $L_{\mathfrak{R}}(X), U_{\mathfrak{R}}(X), B_{\mathfrak{R}}(X)\}$

where $L_{\mathfrak{R}}(X) = \bigcup_{x \in U} \{\mathfrak{R}(x) : \mathfrak{R}(x) \subseteq X\}$ *and* $U_{\mathfrak{R}}(X) = \bigcup_{x \in U} \{\mathfrak{R}(x) : \mathfrak{R}(x) \cap X \neq \phi\}$ *and* $B_{\mathfrak{R}}(X) = U_{\mathfrak{R}}(X) - L_{\mathfrak{R}}(X)$.

Definition 2.2. *Let U be a universal set and* $\mathfrak{R}$ *an equivalence relation over the set U. Then define the set as:* $\Omega_{\mathfrak{R}(X)} = [U, emptyset, L_{\mathfrak{R}}(X), U_{\mathfrak{R}}(X), B_{\mathfrak{R}}(X)]$, *where X is a subset of U. The set* $\Omega_{\mathfrak{R}(X)}$ *satisfies the following conditions:*

(1) *The universal set U, and the empty set* $\emptyset$ *are elements of* $\Omega_{\mathfrak{R}(X)}$;
(2) *The union of any number of sets from* $\Omega_{\mathfrak{R}(X)}$ *is also the element of* $\Omega_{\mathfrak{R}(X)}$;
(3) *The intersection of any finite set of sets in* $\Omega_{\mathfrak{R}(X)}$, *is in* $\Omega_{\mathfrak{R}(X)}$. *So,* $\Omega_{\mathfrak{R}(X)}$*fulfills the axioms of a System of open sets on U, and the couple (U,* $\Omega_{\mathfrak{R}(X)}$ *) is called a Nano-structurated topological spaces.*

3. SOME NEW NANO CONNECTED SETS USING β-IDEAL LOCAL FUNCTIONS

Definition 3.1. *A triple is a Nano Connected* β*-ideal Local Function given by the definition.* $(X, \mathscr{I}, \mathscr{N}\mathscr{C})$ *and where* $\mathscr{I}$ *is not empty and where X is a not empty set.* $\mathscr{I}$ *is ideal on X whereas* $\mathscr{N}\mathscr{C}$ *is nano topology were produced through lower approximations of an indiscernibility relation.*

A space is called nano connected provided that it cannot be decomposed into two disjoint non-empty nano open sets their union having equality to X. An element, $A \subseteq X$ *is a nano connected beta-ideal local function in the following situation:*

$$A \subseteq (C^{\star}_{nc}(I_{nc}(C^{\star}_{nc}(A))),$$

in which $C^{\star}_{nc}$ *is nano connected closure ideal based. A subset G in* $\mathscr{N}\mathscr{C}^{X}_{\star}$ *is referred to as nano connected*

(1) $_{\beta}\mathbb{I}\mathbb{C}^{nc}_{t}$*-set if* $I_{nc}(G) = I_{nc}(C^{\star}_{nc}(G))$,
(2) $_{\beta}\mathbb{I}\mathbb{C}^{nc}_{t\alpha}$*-set if* $I_{nc}(G) = I_{nc}(C^{\star}_{nc}(I_{nc}(G)))$,
(3) $_{\beta}\mathbb{I}^{n}_{\mathscr{R}}$*-set if* $G = S_1 \cap S_2$, *where* S_1 *is nc-*$\mathscr{O}\mathscr{S}$ *and* S_2 *is* $_{\beta}\mathbb{I}^{nc}_{t}$*-set,*
(4) $_{\beta}\mathbb{I}^{nc}_{\mathscr{R}\alpha}$*-set if* $G = S_1 \cap S_2$, *where* S_1 *is nc-*$\mathscr{O}\mathscr{S}$ *and* S_2 *is* $_{\beta}\mathbb{I}^{nc}_{t\alpha}$*-set.*

Example 3.2. *Let* $U = \{a, e, i, o\}$ *with* $U/R = \{\{e\}, \{o\}, \{a, i\}\}$ *and* $X = \{a, i, o\}$. *Then* $\mathscr{N}\mathscr{C} = \{\phi, \{o\}, \{a, i\}, \{a, i, o\}, U\}$ *and* $\mathbb{I} = \{\phi, \{i\}\}$.

(1) ${}_{\beta}\mathbb{I}^{nc}_{t}$*-set* = $\{\phi, \{a\}, \{e\}, \{i\}, \{o\}, \{a,e\}, U\}$.

(2) ${}_{\beta}\mathbb{I}^{nc}_{t_{\alpha}}$*-set* = ${}_{\beta}\mathbb{I}^{nc}_{t}$*-set* = $\{\phi, \{a\}, \{e\}, \{i\}, \{o\}, \{a,e\}, U\}$.

(3) ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$*-set* =$\{\phi, \{e\}, \{i\}, \{o\}, \{a,e\}, \{a,i\}, U\}$.

(4) ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}_{\alpha}}$*-set* = $\{\phi, \{a\}, \{e\}, \{i\}, \{o\}, \{a,e\}, \{a,i\}, \{a,o\}, \{e,i\}, \{e,o\}, \{i,o\},$
$\{a,e,i\}, \{a,e,o\}, \{a,i,o\}\{e,i,o\}, U\}$.

Remark 3.3. *In space* $\mathscr{N}\mathscr{C}^{X}_{\star}$*,*

(1) if L is nc-$\mathscr{O}\mathscr{S}$ $\Longrightarrow$ *L is* ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$*-set.*

(2) if L is ${}_{\beta}\mathbb{I}^{nc}_{t}$*-set* $\Longrightarrow$ *L is* ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$*-set.*

Remark 3.4. *The reverse implications of Remark 3.3 do not hold, as illustrated by the following examples.*

Example 3.5. *Let* $U = \{11,22,33,44\}$ *with* $U/R = \{\{22\},\{44\},\{11,33\}\}$ *and* $X = \{11,33,44\}$*. Then* $\mathscr{N}\mathscr{C} = \{\phi,\{44\},\{11,33\},\{11,33,44\},U\}$ *and* $\mathbb{I} = \{\phi,\{33\}\}$*.*

(1) $\{22\}$ *is not nc- Open Set but* ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$*-set.*

(2) $\{11,33\}$ *is not* ${}_{\beta}\mathbb{I}^{nc}_{t}$*-set but* ${}_{\mathbb{S}}\mathbb{I}^{nc}_{\mathscr{R}}$*-set.*

Proposition 3.6. *Let G and* G_1 *be subsets of* $\mathscr{N}\mathscr{C}^{X}_{\star}$*. If G and* G_1 *are* ${}_{\beta}\mathbb{I}^{nc}_{t}$*-sets, then* $G \cap G_1$ *is* ${}_{\beta}\mathbb{I}^{nc}_{t}$*-set.*

Proof.

Let G and G_1 be ${}_{\beta}\mathbb{I}^{nc}_{t}$-sets. Then there is $I_{nc}(G \cap G_1) \subseteq I_{nc}(C^{\star}_{nc}(G \cap G_1)) \subseteq I_{nc}(C^{\star}_{nc}(G) \cap C^{\star}_{nc}(G_1)) = I_{nc}(C^{\star}_{nc}(G)) \cap I_{nc}(C^{\star}_{nc}(G_1)) = I_{nc}(G) \cap I_{nc}(G_1) = I_{nc}(G \cap G_1)$. Then $I_{nc}(G \cap G_1) = I_{nc}(C^{\star}_{nc}(G \cap G_1))$ and hence $G \cap G_1$ is a ${}_{\beta}\mathbb{I}^{nc}_{t}$-set. □

Example 3.7. *In the above Example 3.5,* $\{22\}$ *and* $\{11,22\}$ *is* ${}_{\beta}\mathbb{I}^{nc}_{t}$*-set. But* $\{22\} \cap \{11,22\} = \{22\}$ *is* ${}_{\beta}\mathbb{I}^{nc}_{t}$*-set.*

Proposition 3.8. *The next characteristics are identical for a G subset of a* $\mathscr{N}\mathscr{C}^{X}_{\star}$*:*

(1) £ is nc-$\mathscr{O}\mathscr{S}$*,*

(2) £ is ${}_{\beta}\mathbb{I}^{nc}_{p}$*-*$\mathscr{O}\mathscr{S}$ & ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$*-set.*

Proof.

(1) $\Longrightarrow$ (2): Let £ be nc-$\mathscr{OS}$. Then $£ = I_{nc}(£) \subseteq I_{nc}(C^{\star}_{nc}(£))$ and £ is ${}_{\beta}\mathbb{I}^{nc}_{p}$-$\mathscr{OS}$. Moreover by Remark 3.3, L is ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$-set.

(2) $\Longrightarrow$ (1): Given £ is ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$-set. So $£ = C_1 \cap C_2$ such that C_1 is nc-$\mathscr{OS}$ and $I_{nc}(C_2) = I_{nc}(C_{nc}(C_2))$. Then $£ \subseteq C_1 = I_{nc}(C_1)$. Also, £ is ${}_{\beta}\mathbb{I}^{nc}_{p}$-$\mathscr{OS} \Longrightarrow £ \subseteq I_{nc}(C_{nc}(£)) \subseteq I_{nc}(C^{\star}_{nc}(C_2)) = I_{nc}(C_2)$ by assuming. Thus $£ \subseteq I_{nc}(C_1) \cap I_{nc}(C_2) = I_{nc}(C_1 \cap C_2) = I_{nc}(£)$ as well as L is nc-$\mathscr{OS}$. □

Remark 3.9. *In space $\mathscr{NC}^{X}_{\star}$, the families of ${}_{\beta}\mathbb{I}^{nc}_{p}$-$\mathscr{OS}$ and ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$-set are independent.*

Example 3.10. *In the above Example 3.5,*

(1) $\{11,44\}$ *is not ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$-set but ${}_{\beta}\mathbb{I}^{nc}_{p}$-$\mathscr{OS}$.*

(2) $\{22\}$ *is not ${}_{\beta}\mathbb{I}^{nc}_{p}$-$\mathscr{OS}$ but ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$-set.*

Remark 3.11. *In space $\mathscr{NC}^{X}_{\star}$,*

(1) if £ is nc-$\mathscr{OS} \Longrightarrow$ £ is ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}_{\alpha}}$-set.

(2) if £ is ${}_{\beta}\mathbb{I}^{nc}_{t_{\alpha}}$-set $\Longrightarrow$ £ is ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}_{\alpha}}$-set.

The diagram depicts these connections..

$$\begin{array}{ccccc} {}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}\textbf{-set} & \longrightarrow & {}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}_{\alpha}}\textbf{-set} & \longleftarrow & {}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}\textbf{-set} \\ \uparrow & & & & \uparrow \\ {}_{\beta}\mathbb{I}^{nc}_{t}\textbf{-set} & \longleftarrow & nc\text{-}\mathscr{OS} & \longrightarrow & {}_{\beta}\mathbb{I}^{nc}_{t_{\alpha}}\textbf{-set} \end{array}$$

The reverse of the figure does not hold, as evidenced by the following example.

Example 3.12. *In the above Ex 3.5,*

(1) $\{22\}$ *is not nc-$\mathscr{OS}$ but ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$-set.*

(2) $\{11,33\}$ *is not ${}_{\beta}\mathbb{I}^{nc}_{t}$-set but ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}}$-set.*

(3) $\{11,22\}$ *is not nc-$\mathscr{OS}$ but ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}_{\alpha}}$-set.*

(4) $\{22,33,44\}$ *is not ${}_{\beta}\mathbb{I}^{n}_{t_{\alpha}}$-set but ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}_{\alpha}}$-set.*

Proposition 3.13. *If G_1 and G_2 are ${}_{\beta}\mathbb{I}^{nc}_{t_{\alpha}}$-sets in $\mathscr{NC}^{X}_{\star}$, then $G_1 \cap G_2$ is ${}_{\beta}\mathbb{I}^{nc}_{t_{\alpha}}$-set.*

Proof.

Let G_1 and G_2 be ${}_{\beta}\mathbb{I}^{nc}_{t\alpha}$-sets. Then there is $I_{nc}(G_1 \cap G_2) \subseteq I_{nc}(C^{\star}_{nc}(I_{nc}(G_1 \cap G_2))) \subseteq I_{nc}[C^{\star}_{nc}(G_{nc}(G_1)) \cap C^{\star}_{nc}(I_{nc}(G_2))] = I_{nc}(C^{\star}_{nc}(I_{nc}(G_1))) \cap I_{nc}(C^{\star}_{nc}(I_{nc}(G_2))) = I_{nc}(G_1) \cap I_{nc}(G_2) = I_{nc}(G_1 \cap G_2)$. Then $I_{nc}(G_1 \cap G_2) = I_{nc}(C^{\star}_{nc}(I_{nc}(G_1 \cap G_2)))$ and hence $G_1 \cap G_2$ is a ${}_{\beta}\mathbb{I}^{nc}_{t\alpha}$-set. □

Example 3.14. *In the above Example 3.5,* $\{22\}$ *and* $\{11,22\}$ *is* ${}_{\beta}\mathbb{I}^{nc}_{t\alpha}$*-set. But* $\{22\} \cap \{11,22\} = \{22\}$ *is* ${}_{\beta}\mathbb{I}^{nc}_{t\alpha}$*-set.*

Proposition 3.15. *The next characteristics are identical for a £ subset of a* $\mathscr{N}\mathscr{C}^{X}_{\star}$*:*

(1) £ is nc-$\mathscr{O}\mathscr{S}$*.*

(2) £ is ${}_{\beta}\mathbb{I}^{nc}_{\alpha}$*-*$\mathscr{O}\mathscr{S}$ *and* ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}_{\alpha}}$*-set.*

Proof.

(1) $\Longrightarrow$ (2): Let £ be *nc-*$\mathscr{O}\mathscr{S}$. Then $£ = I_{nc}(£) \subseteq C^{\star}_{nc}(I_{nc}(£))$ and $£ = I_{nc}(£) \subseteq I_{nc}(C^{\star}_{nc}(I_{nc}(£)))$. Therefore £ is ${}_{\beta}\mathbb{I}^{nc}_{\alpha}$-$\mathscr{O}\mathscr{S}$. Likewise by (1) of Remark 3.11, £ is a ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}_{\alpha}}$-set.

(2) $\Longrightarrow$ (1): Given £ is a ${}_{\beta}\mathbb{I}^{nc}_{\mathscr{R}_{\alpha}}$-set. So $£ = C_1 \cap C_2$ where C_1 is *nc-*$\mathscr{O}\mathscr{S}$ and $I_{nc}(C_2) = I_{nc}(C^{\star}_{nc}(I_{nc}(C_2)))$. Then $£ \subseteq C_1 = I_{nc}(C_1)$. Also £ is ${}_{\beta}\mathbb{I}^{nc}_{\alpha}$-$\mathscr{O}\mathscr{S}$ implies $£ \subseteq I_{nc}(C^{\star}_{nc}(I_{nc}(£))) \subseteq I_{nc}(C^{\star}_{nc}(I_{nc}(C_2))) = I_{nc}(C_2)$ by assumption. Thus $£ \subseteq I_{nc}(C_1) \cap I_{nc}(C_2) = I_{nc}(C_1 \cap C_2) = I_{nc}(£)$ and £ is *nc-*$\mathscr{O}\mathscr{S}$. □

4. On a Few Novel Kinds of β-Ideal Nano Anti-Haussdorff sets

Definition 4.1. *A triple gives a Nano Anti-Haussdorff* β*-ideal Topological structure of the space* $(\mathbb{X}, \mathscr{I}, \mathscr{N}\mathscr{A}\mathscr{H})$*, with* $\mathbb{X}$ *a non empty set. The ideal* $\mathscr{I}$ *is on* $\mathbb{X}$ *and the nano anti- haussdorff topology is* $\mathscr{N}\mathscr{A}\mathscr{H}$ *are produced based on a lower approximation of an indiscernibility relation.*

The space is nano haussdorff, in case it decomposes into disjoint nano open sets in such a way that their union becomes $\mathbb{X}$*. Otherwise it is called as nano anti-haussdorff topological space*

A subset A of is β*-ideal nano anti-haussdorff when it is* β*-ideal anti-haussdorff in the previous sense.* $A \subseteq (C^{\star}_{nah}(I_{nah}(C^{\star}_{nah}(A))))$*. where* $C^{\star}_{nah}$ *denotes the ideal-based nano anti-haussdorff closure.*

A subset S in $\mathscr{N}\mathscr{A}\mathscr{H}^{X}_{\star}$ *is referred to as nano anti-haussdorff*

(1) ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$*-set if* $I_{nah}(S) = C^{\star}_{nah}(I_{nah}(S))$.

(2) ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$*-set if* $I_{nah}(S) = C^{\star}_{nah}(I_{nah}(C^{\star}_{nah}(S)))$.

(3) ${}_{\beta}\mathbb{I}^{nah}_{\mathscr{R}^{\#}}$*-set if* $S = J \cap K$*, where* J *is n-*$\mathscr{OAHS}$ *and* K *is* ${}_{\mathbb{S}}\mathbb{I}^{nah}_{t^{\#}}$*-set.*

(4) ${}_{\beta}\mathbb{I}^{nah}_{\mathscr{R}^{\#}_{\alpha}}$*-set if* $S = J \cap K$*, where* J *is n-*$\mathscr{OAHS}$ *and* K *is* ${}_{\mathbb{S}}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$*-set.*

(5) ${}_{\beta}\mathbb{I}^{nah}_{\mathscr{SR}}$*-set if* $S = J \cap K$*, where* L *is n-*$\mathscr{OAHS}$ *and* J *is* ${}_{\mathbb{S}}\mathbb{I}^{nah}_{t}$*-set and* $I_{nah}(C^{\star}_{nah}(J)) = C^{\star}_{nah}(I_{nah}(J))$.

Example 4.2. *Let* $U = \{1,8,27,64\}$ *with* $U/R = \{\{8\},\{64\},\{1,27\}\}$ *and* $X = \{27,64\}$*. Then* $\mathscr{NAH} = \{\phi,\{64\},\{1,27\},\{1,27,64\},U\}$ *and* $\mathbb{I} = \{\phi,\{27\}\}$.

(1) ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$*-set* = $\{\phi,\{1\},\{8\},\{27\},\{64\},\{1,8\},U\}$.

(2) ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$*-set* = $\{\phi,\{8\},\{64\},U\}$.

(3) ${}_{\beta}\mathbb{I}^{n}_{\mathscr{R}^{\#}}$*-set* = $\{\phi,\{1\},\{8\},\{27\},\{64\},\{1,8\},\{1,27\},U\}$.

(4) ${}_{\beta}\mathbb{I}^{nah}_{\mathscr{R}^{\#}_{\alpha}}$*-set* = $\{\phi,\{8\},\{64\},\{1,27\},\{1,27,64\},U\}$.

(5) ${}_{\beta}\mathbb{I}^{n}_{\mathscr{SR}}$*-set* = $\{\phi,\{64\},\{1,27\},\{1,64\},\{8,27\},\{8,64\},\{27,64\},\{1,8,27\},$ $\{1,8,64\},\{1,27,64\},\{8,27,64\},U\}$.

Remark 4.3. *In* $\mathscr{NAH}^{X}_{\star}$ *space,*

(1) *if* S *is n-*$\mathscr{OAHS} \Longrightarrow S$ *is* ${}_{\beta}\mathbb{I}^{nah}_{\mathscr{R}^{\#}}$*-set.*

(2) *if* S *is* ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$*-set* $\Longrightarrow S$ *is* ${}_{\beta}\mathbb{I}^{nah}_{\mathscr{R}^{\#}}$*-set.*

Remark 4.4. *As illustrated in the next two examples, the opposite in every portion of the Remark 4.3 is not necessarily satisfied.*

Example 4.5. *From the above Ex 3.2,*

(1) $\{8\}$ *is not nah-*$\mathscr{OS}$ *but* ${}_{\beta}\mathbb{I}^{nah}_{\mathscr{R}^{\#}}$*-set.*

(2) $\{1,8\},\{1,27\}$ *are not* ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$*-set but* ${}_{\beta}\mathbb{I}^{nah}_{\mathscr{R}^{\#}}$*-set.*

Proposition 4.6. *If* $\Im$ *and* $\aleph$ *are* ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$*-sets in* $\mathbb{NAH}^{X}_{\star}$*, then* $\Im \cap \aleph$ *is* ${}_{\mathbb{S}}\mathbb{I}^{nah}_{t^{\#}}$*-set.*

Proof.

Let $\Im$ and $\aleph$ be ${}_{\beta}\mathbb{I}^{nha}_{t^{\#}}$-sets. $I_{nah}(\Im \cap \aleph) \subseteq I_{nah}(\Im \cap \aleph) \subseteq C^{\star}_{nah}(I_{nah}(\Im \cap \aleph)) = C^{\star}_{nah}(I_{nah}(\Im) \cap I_{nah}(\aleph)) \subseteq C^{\star}_{nah}(I_{nah}(\Im)) \cap C^{\star}_{nah}(I_{nah}(\aleph)) = I_{nah}(\Im) \cap I_{nah}(\aleph)$ (by guess) $= I_{nah}(\Im \cap \aleph)$. Thus $I_{nah}(\Im \cap \aleph) = C^{\star}_{nah}(I_{nah}(\Im \cap \aleph))$ and hence $\Im \cap \aleph$ is ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$-set. □

Theorem 4.7. *The next characteristics are identical for a $\Im$ subset of a $\mathscr{N}\mathscr{A}\mathscr{H}^X_\star$:*

(1) *$\Im$ is nah-$\mathscr{O}\mathscr{S}$,*

(2) *$\Im$ is ${}_\beta\mathbb{I}^{nah}_p$-$\mathscr{O}\mathscr{S}$ & ${}_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#}$-set.*

Proof.

(2)$\Longleftarrow$(1): (2) is followed by Remark 3.3 of [11] and (1) of 4.3.

(1)$\Longleftarrow$(2): Given that $\Im$ is ${}_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#}$-set. So $\Im = \Im_1 \cap \Im_2$ where $\Im_1$ is *nah-$\mathscr{O}\mathscr{S}$* and $I_{nah}(T_2) = C^\star_{nah}(I_{nah}(\Im_2))$. Then $\Im \subseteq \Im_1 = I_{nah}(\Im_1)$. Also $\Im$ is ${}_\beta\mathbb{I}^{nah}_p$-$\mathscr{O}\mathscr{S}$ implies $\Im \subseteq C^\star_{nah}(I_{nah}(\Im)) \subseteq C^\star_{nah}(I_{nah}(\Im_2)) = I_{nah}(\Im_2)$ by guess. Thus $\Im \subseteq I_{nah}(\Im_1) \cap I_{nah}(\Im_2) = I_{nah}(\Im_1 \cap \Im_2) = I_{nah}(\Im_1)$ and so $\Im$ is *nah-$\mathscr{O}\mathscr{S}$*. □

Remark 4.8. *Regarding $\mathscr{N}\mathscr{A}\mathscr{H}^X_\star$ Space, the collections of ${}_\beta\mathbb{I}^{nah}_p$ Open set and ${}_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#}$-set are independent.*

Example 4.9. *In the above Ex 3. 2,*

(1) *$\{1,64\}$ is not ${}_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#}$-set but ${}_\beta\mathbb{I}^{nah}_p$- Open Set.*

(2) *$\{8\}$ is not ${}_\beta\mathbb{I}^{nah}_p$- Open Set, however ${}_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#}$-set.*

Remark 4.10. *In space $\mathscr{N}\mathscr{A}\mathscr{H}^X_\star$,*

(1) *if $\Im$ is nah-$\mathscr{O}\mathscr{S} \Longrightarrow \Im$ is ${}_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#_\alpha}$-set.*

(2) *if $\Im$ is ${}_\beta\mathbb{I}^{nah}_{t^\#_\alpha}$-set $\Longrightarrow \Im$ is ${}_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#_\alpha}$-set.*

Remark 4.11. *As illustrated in the next two examples, the opposite in every portion of the Remark 4.10 is not need to be true.*

Example 4.12. *From the above Ex 3.2,*

(1) *$\{8\}$ is not nah-$\mathscr{O}\mathscr{S}$ but ${}_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#_\alpha}$-set.*

(2) *$\{1,27\}$ is not ${}_\beta\mathbb{I}^{nah}_{t^\#_\alpha}$-set but ${}_\beta\mathbb{I}^{nah}_{\mathscr{R}^\#_\alpha}$-set.*

Proposition 4.13. *If $\Im$ and $\aleph$ are ${}_\beta\mathbb{I}^{nah}_{t^\#_\alpha}$-sets in $\mathscr{N}\mathscr{A}\mathscr{H}^X_\star$, then $\Im \cap \aleph$ is ${}_\beta\mathbb{I}^{nah}_{t^\#_\alpha}$-set.*

Proof.

Let $\Im$ and $\aleph$ be ${}_\beta\mathbb{I}^{nah}_{t^\#_\alpha}$-sets. $I_{nah}(\Im \cap \aleph) \subseteq I_{nah}(\Im \cap \aleph) \subseteq I_{nah}(C^\star_{nah}(\Im \cap \aleph)) \subseteq C^\star_{nah}(I_{nah}(C^\star_{nah}(\Im \cap \aleph))) \subseteq C^\star_{nah}(I_{nah}(C^\star_{nah}(\Im))) \cap C^\star_{nah}(I_{nah}(C^\star_{nah}(\aleph))) = I_{nah}(\Im) \cap I_{nah}(\aleph)$ (by guess) $= I_{nah}(\Im \cap \aleph)$. Then $I_{nah}(\Im \cap \aleph) = C^{\star}_{nah}(I_{nah}(C^{\star}_{nah}(\Im \cap \aleph))$ and hence $\Im \cap \aleph$ is ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$-set. □

Theorem 4.14. *In $\mathscr{N}\mathscr{A}\mathscr{H}^{X}_{\star}$ space, every ${}_{\beta}\mathbb{I}^{nah}_{r}$-$\mathscr{O}\mathscr{S}$ are ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$-set but the converse need not be true.*

Proof. For ${}_{\beta}\mathbb{I}^{nah}_{r}$-$\mathscr{O}\mathscr{S}$ $\Im = I_{nah}(C^{\star}_{nah}(\Im))$ which implies $I_{nah}(\Im) = I_{nah}(I_{nah}(C^{\star}_{nah}(\Im)))$
$I_{nah}(\Im) = I_{nah}(C^{\star}_{nah}(\Im)) \Rightarrow$ which is ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$-set. □

Example 4.15. *In the above Example 3.2,*

(1) *$\{64\}$ is ${}_{\beta}\mathbb{I}^{nah}_{r}$-$\mathscr{O}\mathscr{S}$ which is also ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$-set.*
(2) *$\{\{1\},\{8\},\{27\},\{1,8\}\}$ are in ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$-set but not ${}_{\beta}\mathbb{I}^{nah}_{r}$-$\mathscr{O}\mathscr{S}$.*

Theorem 4.16. *In $\mathscr{N}\mathscr{A}\mathscr{H}^{X}_{\star}$ space every ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$-set are ${}_{\beta}\mathbb{I}^{nah}_{\alpha}$-$\mathscr{O}\mathscr{S}$ but the converse need not be true.*

Proof. Let $\Im$ is a subset of X be a ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$-set.
By the definition of ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$-set
$I_{nah}(\Im) = I_{nah}(C^{\star}_{nah}(I_{nah}(\Im)))$, we know $I_{nah}(\Im) \subseteq \Im \subseteq (C^{\star}_{nah}(\Im)$ we have $I_{nah}(\Im) = I_{nah}(C^{\star}_{nah}(I_{nah}(\Im)))$ which implies $I_{nah}(\Im) \subseteq I_{nah}(C^{\star}_{nah}(I_{nah}(\Im)))$. Since $I_{nah}(\Im) \subseteq \Im$, transitive. Therefore, $\Im \subseteq I_{nah}(C^{\star}_{nah}(I_{nah}(\Im)))$. Which is ${}_{\beta}\mathbb{I}^{nah}_{\alpha}$ - Open Set. □

Example 4.17. *By using Ex 3.2,*

(1) *$\{64\}$ is ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$ which is also in ${}_{\beta}\mathbb{I}^{nah}_{\alpha}$-$\mathscr{O}\mathscr{S}$.*
(2) *$\{1,27\},\{1,64\},\{8,27\},\{8,64\},\{27,64\},\{1,8,27\},\{1,8,64\},\{1,27,64\},\{8,27,64\}$ are in ${}_{\beta}\mathbb{I}^{nah}_{\alpha}$-$\mathscr{O}\mathscr{S}$ but not ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$.*

Theorem 4.18. *Every space ${}_{\beta}\mathbb{I}^{nah}_{r}$-$\mathscr{O}\mathscr{S}$ are ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$ and ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$.*

Proof. Let $\Im \subseteq$ X be a ${}_{\beta}\mathbb{I}^{nah}_{r}$-$\mathscr{O}\mathscr{S}$. Which implies $\Im = I_{nah}(C^{\star}_{nah}(\Im)$ We know that $I_{nah}(\Im) \subseteq \Im \subseteq (C^{\star}_{nah}(\Im) \Longrightarrow I_{nah}(\Im) \subseteq I_{nah}(C^{\star}_{nah}(\Im)$. Also, $I_{nah}(\Im) \subseteq I_{nah}(C^{\star}_{nah}(\Im) \subseteq C^{\star}_{nah}(\Im)$. Since $\Im$= $I_{nah}(C^{\star}_{nah}(\Im)$ we conclude, $I_{nah}(\Im) = I_{nah}(C^{\star}_{nah}(\Im)$ which implies ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$.
For ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$, let us apply interior to both sides in the ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$ we get $I_{nah}(\Im) = I_{nah}(I_{nah}(C^{\star}_{nah}(\Im))$ = $I_{nah}(C^{\star}_{nah}(\Im))$. Now take closure on $I_{nah}(\Im)$, $C^{\star}_{nah}(I_{nah}(\Im)) \subseteq C^{\star}_{nah}(\Im) \subseteq I_{nah}(C^{\star}_{nah}(I_{nah}(\Im)))$

$\subseteq I_{nah}(C^{\star}_{nah}(\Im) = I_{nah}(\Im)$. But also $I_{nah}(\Im) \subseteq C^{\star}_{nah}(I_{nah}(\Im)) \Longrightarrow I_{nah}(\Im) \subseteq I_{nah}(C^{\star}_{nah} I_{nah}(\Im))$. So we have both $I_{nah}(\Im) \subseteq I_{nah}(C^{\star}_{nah}(I_{nah}(\Im)))$ and $I_{nah}(C^{\star}_{nah}(I_{nah}(\Im))) \subseteq I_{nah}(\Im)$. Therefore, $I_{nah}(\Im) = I_{nah}(C^{\star}_{nah}(I_{nah}(\Im))) \Longrightarrow {}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$. □

Example 4.19. *In the above Example 3.2,* $\{64\}$ ${}_{\beta}\mathbb{I}^{nah}_{r}$-$\mathscr{O}\mathscr{S}$ *which is also* ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$ *and* ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$ *but* $\{8\}$ *is in* ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}}$ *and* ${}_{\beta}\mathbb{I}^{nah}_{t^{\#}_{\alpha}}$ *which is not a* ${}_{\beta}\mathbb{I}^{nah}_{r}$-$\mathscr{O}\mathscr{S}$*.*

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] K. Bhuvaneswari, K.M. Gnanapriya, Nano Generalized Locally Closed Sets and NGLC-Continuous Functions in Nano Topological Spaces, Int. J. Math. Appl. 4 (2016), 101–106.

[2] D. Jankovic, T.R. Hamlett, Compatible Extensions of Ideals, Boll. Un. Mat. Ital. 7 (1992), 453–465.

[3] K. Kuratowski, Topology, Academic Press, 1966. https://doi.org/10.1016/C2013-0-11022-7.

[4] M.L. Thivagar, C. Richard, On Nano Forms of Weakly Open Sets, Int. J. Math. Stat. Invent. 1 (2013), 31–37.

[5] M.L. Thivagar, S. Jafari, V.S. Devi, On New Class of Contra Continuity in Nano Topology, Ital. J. Pure Appl. Math. 41 (2017), 25–36.

[6] O. Nethaji, R. Asokan, I. Rajasekaran, New Generalized Classes of an Ideal Nano Topological Spaces, Bull. Int. Math. Virtual Inst. 9 (2019), 543–552.

[7] Z. Pawlak, Rough Sets, Int. J. Comput. Inf. Sci. 11 (1982), 341–356. https://doi.org/10.1007/bf01001956.

[8] M. Parimala, S. Jafari, S. Murali, Nano Ideal Generalized Closed Sets in Nano Ideal Topological Spaces, Ann. Univ. Sci. Budapest. 61 (2018), 111–119.

[9] M. Parimala, S. Jafari, On Some New Notions in Nano Ideal Topological Spaces, Eurasian Bull. Math. 1 (2018), 85–93.

[10] I. Rajasekaran, G. Selvi, On Semi-Local Functions in Ideal Nano Topological Spaces, Communicated.

[11] I. Rajasekaran, O. Nethaji, Simple Forms of Nano Open Sets in an Ideal Nano Topological Spaces, J. New Theory 24 (2018), 35–43.

[12] I. Rajasekaran, O. Nethaji, R. Premkumar, Perceptions of Several Sets in Ideal Nano Topological Spaces, J. New Theory 23 (2018), 78–84.

[13] P. Gayathri, G. Selvi, I. Rajasekaran, Weak Form of Ideal Nano Topological Spaces for Semi-Local Functions, Adv. Fixed Point Theory 14 (2024), 58. https://doi.org/10.28919/afpt/8878.

[14] N. Sekar, R. Asokan, G. Selvi, I. Rajasekaran, On Regular Closed Sets in Ideal Nano Topological Spaces, Bull. Int. Math. Virtual Inst. 13 (2023), 135–141.

[15] G. Selvi, I. Rajasekaran, On Nano $\mathscr{M}_r$-Set and Nano $\mathscr{M}_r^\star$-Set in Nanotopological Spaces, Adv. Math.: Sci. J. 9 (2020), 9345–9351.

[16] P. Sathishmohan, V. Rajendran, C. Vignesh Kumar, P. Dhanasekaran, On Nano Semi Pre Neighbourhoods in Nano Topological Spaces, Malaya J. Mat. 06 (2018), 294–298. https://doi.org/10.26637/mjm0601/0034.

[17] R. Vaidyanathaswamy, The Localization Theory in Set Topology, Proc. Indian Acad. Sci. 20 (1945), 51–61.

[18] R. Vaidyanathaswamy, Set Topology, Chelsea Publishing Company, New York, 1946.

[19] G. Selvi, I. Rajasekaran, Some New Sets in Nano Semi-Local Functions, South East Asian J. Math. Math. Sci. 20 (2024), 359–368. https://doi.org/10.56827/seajmms.2024.2003.27.