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RESIDUAL ν-METRIC SPACE AND BANACH CONTRACTION PRINCIPLE

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Abstract. The purpose of this article is to introduce the concept of residual ν-metric space as a synthesis of a type of generalization of metric space and its extensions namely, b-metric space, extended b-metric space, strong b-metric space, strong controlled b-metric space, double controlled metric type space, suprametric space, b-suprametric space, rectangular metric space, rectangular b-metric space, extended rectangular b-metric space, homothetic rectangular metric space, controlled rectangular b-metric space, ν-generalized metric space and $b_\nu(s)$-metric space. Moreover we give a general form of the notion of contraction in a residual ν-metric space and we prove the analogue of the Banach Contraction Principle in this new framework.

Keywords: fixed points; metric space; residual ν-metric space; Banach contraction principle.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The most pivotal result in the development of nonlinear analysis is the Banach Contraction Principle (in short BCP). It states that on a complete metric space (X,d), every contraction $T : X \to X$, that is, there exists a constant $\lambda \in [0,1)$ such that $d(Tx,Ty) \le \lambda d(x,y)$ for all x,y in X, has a unique fixed point,[5]. Moreover, the sequence $(T^n x)$ of n-iterate of T converges to this

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fixed point for any initial point x in X. Due to its importance for several fields of mathematics, physics and applied sciences, many autors extended the BCP into more general forms either by working on a more generalized metric space, [1, 2, 3, 6, 7, 8, 10, 12, 13, 18, 20, 22, 30, 31], or by modifying the contractive type condition, [14, 15, 16, 17, 19, 24, 25, 28]. In this paper, we introduce the concept of residual ν-metric space as a synthesis of a type of generalization of metric space and its extensions namely, b-metric space, extended b-metric space, strong b-metric space, strong controlled b-metric space, double controled metric type space, suprametric space, b-suprametric space, rectangular metric space, rectangular b-metric space, extended rectangular b-metric space, homothetic rectangular metric space, controlled rectangular b-metric space, ν-generalized metric space and $b_\nu(s)$-metric space. Moreover we give a general form of the notion of contraction in a residual ν-metric space and we prove the analogue of the BCP in this new unification.

2. Preliminaries

In this section, we provide some definitions that will be helpful for readers to understand the main result. Let X be a nonempty set and the mapping d defined from $X \times X$ into $[0, +\infty)$ satisfies, for all $x, y \in X$,

- $d(x,y) = 0$, if and only if, $x = y$,
- $d(x,y) = d(y,x)$,

then, the ordered pair (X,d) is called a semimetric space (in short SMS). In the sequal (X,d) denote a SMS. Now, suppose that, for all $x,y,z \in X$,

$$d(x,y) \leq d(x,z) + d(z,y), (\text{the triangle inequality})$$

then, the ordered pair (X,d) is called a metric space (in short MS), [21]. If, we replace the triangle inequality by, for all $x,y,z \in X$,

$$d(x,y) \leq s[d(x,z) + d(z,y)],$$

for a given $s \geq 1$, then, the ordered pair (X,d) is called a b-metric space or quasi-metric space (in short bMS). For more details about b-metric spaces, we refer to *Bakhtin* [4] and *Czerwik* [10].

Kamran et al. introduced the concept of extended b-metric space [18], which generalizes MS and bMS, by replacing the constant s by a function θ depending on the parameters of the left-hand side of the triangle inequality. Indeed, let θ be a function from $X \times X$ to $[1, +\infty)$ such that, for all $x, y, z \in X$,

$$d(x,y) \leq \theta(x,y)[d(x,z) + d(z,y)],$$

then, (X,d) is called an extended b-metric space (in short EbMS).

A. Kirk and *Shahzad* introduced the notion of a strong b-metric space [32] as follows: assume that there exists $\kappa \geq 1$ such that for all $x, y, z \in X$,

$$d(x,y) \leq d(x,z) + \kappa d(z,y),$$

then, (X,d,κ) is called a strong b-metric space (in short SbMS). In 2022, *M. Berzig* introduced the notion of a suprametric space [6]. Thereafter the suprametric space was used by *Panda* et al., to analyze the stability and the existence of equilibrium points of some fractional-order complex-valued neural networks [29]. Indeed, if there exists ρ in $[0, +\infty)$ such that for all $x, y, z \in X$,

$$d(x,y) \leq d(x,z) + d(z,y) + \rho d(x,z) d(z,y),$$

then, (X,d) is called a suprametric space (in short SPMS). In 2024, The same author introduced the notion of a b-suprametric space [7], as a generalization of SPMS, indeed, suppose that, there exists $s \geq 1$ and ρ in $[0, +\infty)$ such that for all $x, y, z \in X$,

$$d(x,y) \leq s[d(x,z) + d(z,y)] + \rho d(x,z) d(z,y),$$

then, (X,d) is called a b-suprametric space (in short bSPMS). In 2023, *D. Santina* et al. introduced the notion of a strong controlled b-metric space [27], such that, there exists η a function defined from $X \times X$ to $[1, +\infty)$ where for all $x, y, z \in X$,

$$d(x,y) \leq d(x,z) + \eta(z,y) d(z,y),$$

therefore, (X,d) is called a strong controlled b-metric space (in short SCbMS). In 2018, *T. Abdeljawad* et al. introduced a different type extension for bMS by replacing s by a non-comparable functions α, μ to act separately on each term in the right-hand side of the triangle inequality, [1]. Indeed, let α, μ be a non-comparable functions defined from $X \times X$ into $[1, +\infty)$ such that, for all $x, y, z \in X$,

$$d(x,y) \leq \alpha(x,z)d(x,z) + \mu(z,y)d(z,y),$$

then, (X,d) is called a double controlled metric type space (in short DCMTS).

In 2000, *Branciari* introduced the concept of a ν-generalized metric space, [9]. Let ν be a positive integer, then, (X,d) is said to be a ν-generalized metric space (in short ν-GMS), if, for all $x, y \in X$ and for all distinct points $u_1, u_2, ..., u_\nu \in X$, each of them different from x and y,

$$d(x,y) \leq d(x,u_1) + d(u_1,u_2) + \ldots + d(u_\nu, y).$$

If, $\nu = 2$, then (X,d) is called a rectangular metric space (in short RMS) and the inequality, that is, for all x, y in X, and for all distinct points u, v in X each of them different from x, y,

$$d(x,y) \leq d(x,u) + d(u,v) + d(v,y),$$

is said to be the *quadrilateral inequality*. In 2015, *R.George* et al. introduced the concept of rectangular b-metric space [11], as a generalization of MS, bMS and RMS. Indeed, assume that, there exists a real number $s \geq 1$ and for all $x, y \in X$, for all distinct points $u, v \in X$ each of them different from x and y,

$$d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)],$$

then, (X,d) is called a rectangular b-metric space (in short RbMS). In 2019, *Mustapha, Z.* et al. introduced the concept of extended rectangular b-metric spaces [23], as a generalization of RbMS. Let θ be a function defined from $X \times X \to [1, +\infty)$, such that for all $x, y \in X$, for all distinct points $u, v \in X$ each of them different from x and y,

$$d(x,y) \leq \theta(x,y)[d(x,u) + d(u,v) + d(v,y)],$$

then, (X,d) is called an extended rectangular b-metric space (in short ERbMS). In 2022, *M. Rossafi* and *A. Kari* introduced the concept of controlled rectangular metric space [26], as a generalization of RMS, bRMS and ERbMS, indeed, let α be a function defined from $X \times X$ into $[1, +\infty)$ such that, for all distinct points $x, y, u, v \in X$,

$$d(x,y) \leq \alpha(x,u)d(x,u) + \alpha(u,v)d(u,v) + \alpha(v,y)d(v,y),$$

then, (X,d) is called a controled rectangular metric space (in short CRMS). In 2024, Harrafa C. and Mbarki A. introduced the concept of homothetic rectangular metric space [12], as an extension of rectangular metric space. Suppose that there exists a function ψ defined from $X \times X \to (-\infty, +\infty)$, where for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y,

$$d(x,y) \leq d(x,u) + \psi(x,y)d(u,v) + d(v,y),$$

then (X,d) is called a homothetic rectangular metric space (in short HRMS) and ψ is called the control function of X. The notable point of this extension is that the controle function ψ can take negative values and also values in $(0,1)$ and what makes this extension particularly interesting and intriguing is that a metric space does not necessarily satisfy the inequality associated with it. In 2017, Zoran D. and Stojan R. introduced the concept of $b_v(s)$-metric space [31] and proved the Banach contraction principle in this framework. Indeed, let v be a positive integer and let $s \geq 1$, then (X,d) is said to be a $b_v(s)$-metric space if for all $x, y \in X$ and for all distinct points $u_1, u_2, ..., u_v \in X$, each of them different from x and y such that,

$$d(x,y) \leq s[d(x,u_1) + d(u_1,u_2) + \ldots + d(u_v,y)].$$

Let T be a selfmap of X, then T is said to be a λ-contraction of (X,d), if there exists $\lambda \in [0,1)$, such that, for all x, y in X, $d(Tx,Ty) \leq \lambda d(x,y)$.

3. Main Results

In the sequal v denote a positive integer. We begin with the following definition,

Definition 3.1. *Let (X,d) be a SMS, such that there exists a symmetric function $\mathscr{R}$ defined from $\underbrace{X \times X \times \cdots \times X}_{v+2\, times}$ into $[0,+\infty)$, where for all x, y in X and for all distinct points $u_1, u_2, ..., u_v \in X$, each of them different from x and y,*

$$d(x,y) \leq d(x,u_1) + d(u_1,u_2) + \ldots + d(u_v,y) + \mathscr{R}(x,u_1,u_2,\cdots,u_v,y).$$

(The residual inequality)

Then d is called a *residual* ν-metric on X, $(X,d,\mathscr{R})$ is called a *residual* ν-metric space and $\mathscr{R}$ is said to be a residue of (X,d).

Remark 3.1. *Note that the residue $\mathscr{R}$ of a given residual ν-metric space $(X,d,\mathscr{R})$ can be independent of the metric d.*

Definition 3.2. *Let $(X,d,\mathscr{R})$ be a residual ν-metric space and let T be a selfmap of X. Then, T is said to be a (λ,k)-contraction of $(X,d,\mathscr{R})$, if there exists $\lambda,k \in [0,1)$, such that, for all $x,u_1,u_2,\cdots,u_\nu,y$ in X,*

$$\begin{cases} d(Tx,Ty) \leq \lambda d(x,y), \\ \mathscr{R}(Tx,Tu_1,\cdots,Tu_\nu,Ty) \leq k\mathscr{R}(x,u_1,\cdots,u_\nu,y). \end{cases}$$

Remark 3.2. *The definition* 3.2 *of the notion of contraction in a residual ν-metric space, is more general than the usual definition of contraction in MS, bMS, SbMS, SPMS, bSPMS, RMS, bRMS, ν-GMS and $b_\nu(s)$-metric space (See the remark* 3.3 *below).*

We define convergence and Cauchy sequence in a residual ν-metric space and completeness of residual ν-metric space as follows:

Definition 3.3. *Let $(X,d,\mathscr{R})$ be a residual ν-metric space, let (x_n) be a sequence in X and $x \in X$. Then,*

- *The sequence (x_n) is said to be convergent in $(X,d,\mathscr{R})$ and converges to x, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n,x) \leq \varepsilon$ for all $n \geq n_0$ and this fact is represented by $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$.*
- *The sequence (x_n) is said to be a Cauchy sequence in $(X,d,\mathscr{R})$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n,x_m) \leq \varepsilon$ for all $n,m \geq n_0$ or equivalently, if $\lim_{n,m\to+\infty} d(x_n,x_m) = 0$.*
- *$(X,d,\mathscr{R})$ is said to be a complete residual ν-metric space if every Cauchy sequence in X converges to some x in X.*

Let X be a nonempty set and n is an integer such that $n \geq 2$. Let h be a mapping defined from $X \times X$ into $[0,+\infty)$, then we define the mapping $\mathscr{P}_{(h,n)}$ from $\underbrace{X \times X \times \cdots \times X}_{n\ times}$ into $[0,+\infty)$ by, for all $x_1, x_2, \cdots, x_n$ in X,

$$\mathscr{P}_{(h,n)}(x_1, x_2, \cdots, x_n) = \sum_{1 \leq i < j \leq n} h(x_i, x_j).$$

Remark 3.3. *Let $\lambda \in [0,1)$, it follows that,*

1) *Let (X,d) be a MS, then $(X,d,\mathscr{R})$ is a residual 1-metric space where $\mathscr{R}$ is equal to the null function. Every T a λ-contraction of (X,d) is a (λ,k)-contraction of $(X,d,\mathscr{R})$ for all $k \in [0,1)$.*

2) *Let (X,d) be a bMS with $s \geq 1$, then $(X,d,\mathscr{R})$ is a residual 1-metric space, such that, for all x,y,z in X,*

$$\mathscr{R}(x,y,z) = (s-1)\mathscr{P}_{(d,3)}(x,y,z),$$

in this case, $\mathscr{P}_{(d,3)}(x,y,z)$ represents the perimeter of the triangle formed by the points x,y and z. Every T a λ-contraction of (X,d) is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all $k \in [\lambda,1)$.

3) *Let (X,d) be a EbMS with a function θ defined from $X \times X$ to $[1,\infty)$, then $(X,d,\mathscr{R})$ is a residual 1-metric space, such that for all x,y,z in X,*

$$\mathscr{R}(x,y,z) = [\mathscr{P}_{(\theta,3)}(x,y,z) - 3]\mathscr{P}_{(d,3)}(x,y,z).$$

Every T a λ-contraction of (X,d), such that for all x,y,z in X,

$$\mathscr{P}_{(\theta,3)}(Tx,Ty,Tz) \leq \mathscr{P}_{(\theta,3)}(x,y,z),$$

is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all k in $[\lambda,1)$.

4) *Let (X,d) be a DCMTS, with α,μ a non-comparable functions defined from $X \times X$ to $[1,\infty)$, then $(X,d,\mathscr{R})$ is a residual 1-metric space, such that, for all x,y,z in X,*

$$\mathscr{R}(x,y,z) = [\mathscr{P}_{(\alpha,3)}(x,y,z) + \mathscr{P}_{(\mu,3)}(x,y,z) - 6]\mathscr{P}_{(d,3)}(x,y,z).$$

Every T a λ-contraction of (X,d), such that, for all x,y,z in X,

$$\begin{cases} \mathscr{P}_{(\alpha,3)}(Tx,Ty,Tz) \leq \mathscr{P}_{(\alpha,3)}(x,y,z) \\ \mathscr{P}_{(\mu,3)}(Tx,Ty,Tz) \leq \mathscr{P}_{(\mu,3)}(x,y,z), \end{cases}$$

is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all $k \in [\lambda,1)$.

5) *Let (X,d) be a SPMS, with $\rho \geq 0$, then $(X,d,\mathscr{R})$ is a residual 1-metric space, such that for all x,y,z in X,*

$$\mathscr{R}(x,y,z) = \rho[d(x,z)d(y,z) + d(x,z)d(x,y) + d(y,z)d(x,y)],$$

and every T a λ-contraction of (X,d) is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all $k \in [\lambda,1)$.

6) *Let (X,d) be a bSPMS, with $s \geq 1$ and $\rho \geq 0$, then $(X,d,\mathscr{R})$ is a residual 1-metric space, such that for all x,y,z in X,*

$$\mathscr{R}(x,y,z) = (s-1)\mathscr{P}_{(d,3)}(x,y,z) + \rho[d(x,z)d(y,z) + d(x,z)d(x,y) + d(y,z)d(x,y)],$$

and every T a λ-contraction of (X,d) is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all $k \in [\lambda,1)$.

7) *Let (X,d) be a RMS, then $(X,d,\mathscr{R})$ is a residual 2-metric space with a residue $\mathscr{R}$ equal to the null function and every T a λ-contraction of (X,d) is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all $k \in [0,1)$.*

8) *Let (X,d) be a RbMS with $s \geq 1$, then $(X,d,\mathscr{R})$ is a residual 2-metric space, such that, for all x,u,v,y in X,*

$$\mathscr{R}(x,u,v,y) = (s-1)\mathscr{P}_{(d,4)}(x,u,v,y),$$

and every T a λ-contraction of (X,d) is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all $k \in [\lambda,1)$.

9) *Let (X,d) be a CRMS, with a function α defined from $X \times X$ to $[1,+\infty)$, then $(X,d,\mathscr{R})$ is a residual 2-metric space, such that for all x,u,v,y in X,*

$$\mathscr{R}(x,u,v,y) = [\mathscr{P}_{(\alpha,4)}(x,u,v,y) - 6]\mathscr{P}_{(d,4)}(x,u,v,y).$$

Every T a λ-contraction of (X,d), such that, for all x,u,v,y in X,

$$\mathscr{P}_{(\alpha,4)}(Tx,Tu,Tv,Ty) \leq \mathscr{P}_{(\alpha,4)}(x,u,v,y),$$

is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all k in $[\lambda,1)$.

10) *Let (X,d) be a HRMS, with a function ψ defined from $X \times X$ to $(-\infty,+\infty)$. If, for all x,y in X, $\psi(x,y) \leq 1$, then (X,d) is a RMS which is a residual 2-metric space. Otherwise, $(X,d,\mathscr{R})$ is a residual 2-metric space, such that for all x,u,v,y in X,*

$$\mathscr{R}(x,u,v,y) = [\mathscr{P}_{(\rho,4)}(x,u,v,y) - 6]\mathscr{P}_{(d,4)}(x,u,v,y),$$

where ρ is the function defined from $X\times X$ into $[1,+\infty)$, such that, for all x,y in X,

$$\rho(x,y)=\begin{cases}1, & \text{if, } \psi(x,y)\leq 1,\\ \psi(x,y), & \text{otherwise.}\end{cases}$$

Every T a λ-contraction of (X,d), such that, for all x,u,v,y in X,

$$\mathscr{P}_{(\rho,4)}(Tx,Tu,Tv,Ty)\leq \mathscr{P}_{(\rho,4)}(x,u,v,y),$$

is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all $k\in[\lambda,1)$.

11) *Let (X,d) be a ν-generalized metric space then $(X,d,\mathscr{R})$ is a residual ν-metric space where $\mathscr{R}$ is equal to the null function. Every T a λ-contraction of (X,d) is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all $k\in[0,1)$.*

12) *Let (X,d) be a $b_\nu(s)$-metric space then $(X,d,\mathscr{R})$ is a residual ν-metric space, such that, for all $x_1,x_2,\cdots,x_{\nu+2}$ in X,*

$$\mathscr{R}(x_1,x_2,\cdots,x_{\nu+2})=(s-1)\mathscr{P}_{(d,\nu+2)}(x_1,x_2,\cdots,x_{\nu+2}),$$

and every T a λ-contraction of (X,d) is a (λ,k)-contraction of $(X,d,\mathscr{R})$, for all $k\in[\lambda,1)$.

Example 3.1. *Let $X=\left\{(x,y)\in\mathbb{R}\times\mathbb{R}\mid y=x^2\right\}$, where $\mathbb{R}$ denote the set of real numbers and let d be the function defined from $X\times X$ to $[0,+\infty)$, such that, for all $A,B\in X$,*

$$d(A,B)=\frac{|x_A-x_B|^3}{6},$$

where $A=(x_A,y_A)$, and $B=(x_B,y_B)$. Note that, $d(A,B)$, is the area between the graph of the function $x\mapsto x^2$ and the the straight line AB. Therefore, for all A,B in X,

- *$d(A,B)=0$, if and only if, $A=B$,*
- *$d(B,A)=d(A,B)$.*

Then, (X,d) is a SMS. Now, let A,B in X and $C\in X$ different from A and B, therefore,

$$d(A,B)\leq d(A,C)+d(C,B)+\mathscr{R}(A,B,C),$$

where $\mathscr{R}$ is the symmetric function defined from $X\times X\times X$ into $[0,+\infty)$, by:

$$\mathscr{R}(A,B,C)=\frac{|(x_B-x_A)(x_C-x_B)(x_C-x_A)|}{2}.$$

Note that $\mathscr{R}(A,B,C)$, is the area of the triangle formed by A, B, and C. Indeed, if, $A=B$ the result is trivial. Otherwise, let $A\neq B$ in X and let $C\in X$ different from A and B, by symmetry, assume that $x_A<x_B$, then,

- *If, $x_C<x_A$, then, $d(A,B)\leq d(C,B)$,*
- *If, $x_B<x_C$, then $d(A,B)\leq d(A,C)$,*
- *Suppose that $x_A<x_C<x_B$, therefore,*

$$\begin{aligned} d(A,B) &= \frac{(x_B-x_A)^3}{6} \\ &= \frac{(x_B-x_C+x_C-x_A)^3}{6} \\ &= \frac{(x_B-x_C)^3}{6}+\frac{(x_C-x_A)^3}{6}+\frac{(x_B-x_A)(x_B-x_C)(x_C-x_A)}{2}, \\ &\leq d(A,C)+d(C,B)+\mathscr{R}(A,B,C). \end{aligned}$$

Therefore, $(X,d,\mathscr{R})$ is a residual 1-metric space. Note that, (X,d) is not a ν-GMS, for all positive integer ν. Indeed, suppose that there exists ν a positive integer such that (X,d) is a ν-GMS. Let $A_i\in X$, where $x_{A_i}=\frac{i}{\nu+1}$, for all $i\in\{0,1,\cdots,\nu+1\}$. Therefore,

$$\begin{aligned} d(A_0,A_{\nu+1}) &= \frac{1}{6} \\ &\leq d(A_0,A_1)+d(A_1,A_2)+\cdots+d(A_\nu,A_{\nu+1}) \\ &= \frac{1}{6(\nu+1)^3}+\frac{1}{6(\nu+1)^3}+\cdots+\frac{1}{6(\nu+1)^3} \\ &= \frac{1}{6(\nu+1)^2}, \end{aligned}$$

a contradiction. Then, (X,d) is not a ν-GMS, for all positive integer ν. Therefore, (X,d) is not a MS and (X,d) is not a RMS.

Example 3.2. *Let (X,d) be a MS and let x,y,z in X, then,*

$$d(x,y)\leq d(x,z)+d(z,y),$$

it follows that,

$$\begin{aligned} d^2(x,y) &\leq [d(x,z)+d(z,y)]^2 \\ &= d^2(x,z)+d^2(z,y)+2d(x,z)d(z,y) \end{aligned}$$

$$\begin{aligned} &\leq d^2(x,z)+d^2(z,y) \\ &+ 2[d(x,y)d(y,z)+d(y,z)d(x,z)+d(x,z)d(x,y)]. \end{aligned}$$

$(X,d^2,\mathscr{R})$ is a residual 1-metric space, where for all x,y,z in X,

$$\mathscr{R}(x,y,z)=2[d(x,y)d(y,z)+d(y,z)d(x,z)+d(x,z)d(x,y)].$$

Example 3.3. *Let $X=[0,+\infty)$. Define $d:X\times X\to[0,+\infty)$ such that $d(x,y)=d(y,x)$ for all $x,y\in X$ and*

$$d(x,y)=\begin{cases} 0, & \text{if, } x=y \\ y, & \text{if, } x=0,\ y\in(0,1) \\ y^2, & \text{if, } x=0,\ y\in[1,+\infty) \\ max(x,y), & \text{if } x,y\in[1,+\infty), \\ min(x,y), & \text{otherwise.} \end{cases}$$

Then, $(X,d,\mathscr{R})$ is a residual ν-metric space for all positive integer ν such that, for all $x_1,\cdots,x_{\nu+2}$ in X,

$$\mathscr{R}(x_1,\cdots,x_{\nu+2})=max(\sqrt{x_1}+x_1^2,\cdots,\sqrt{x_{\nu+2}}+x_{\nu+2}^2).$$

However we have the following:

1) *Suppose that there exists $s\geq 1$ and q a positive integer such that, for all $x,y\in X$ and for all distinct points $u_1,u_2,...,u_q\in X$, each of them different from x and y,*

$$d(x,y)\leq s[d(x,u_1)+d(u_1,u_2)+\ldots+d(u_q,y)].$$

Then for all $x>1$ and for all distinct points $u_1,u_2,...,u_q\in(0,1)$,

$$\begin{aligned} d(0,x)=x^2 &\leq s[d(0,u_1)+d(u_1,u_2)+\ldots+d(u_q,x)] \\ &= s[u_1+min(u_1,u_2)+\ldots+min(u_{q-1},u_q)+u_q] \\ &\leq (q+1)s, \end{aligned}$$

a contradiction. Therefore, (X,d) is not a $b_q(s)$-metric for all $s\geq 1$ and for all a positive integer q, it follows that,

- (X,d) *is not a MS,*
- (X,d) *is not a bMS, for all* $s>1$*,*
- (X,d) *is not a RMS,*
- (X,d) *is not a RbMS, for all* $s>1$*,*
- (X,d) *is not a* ν*-GMS, for all a positive integer* ν*.*

2) *Suppose that* (X,d) *is a SbMS with* $\kappa \geq 1$*. Therefore, for all* $x>1$ *and for all* $z\in(0,1)$*,*

$$d(0,x)=x^2 \leq d(0,z)+\kappa d(z,x)=z[1+\kappa]\leq 1+\kappa,$$

a contradiction. Then (X,d) *is not a SbMS, for all* $\kappa \geq 1$*.*

3) *Suppose* (X,d) *is a SPMS with* $\rho \in [0,+\infty)$*. Therefore, for all* $x>1$ *and for all* $z\in(0,1)$*,*

$$d(0,x)=x^2 \leq d(0,z)+d(z,x)+\rho d(0,z)d(0,z)=2z+\rho z^2 \leq 2+\rho,$$

a contradiction. Then (X,d) *is not a SPMS, for all* ρ *in* $[0,+\infty)$*.*

4) *Suppose* (X,d) *is a bSPMS with* $s\geq 1$ *and* $\rho \in [0,+\infty)$*. Therefore, for all* $x>1$ *and for all* $z\in(0,1)$*,*

$$d(0,x)=x^2 \leq s[d(0,z)+d(z,x)]+\rho d(0,z)d(0,z)=2zs+\rho z^2 \leq 2s+\rho,$$

a contradiction. Then (X,d) *is not a bSPMS, for all* $s\geq 1$ *and* ρ *in* $[0,+\infty)$*.*

5) *Suppose that* (X,d) *is an EbMS with the function* θ*. Let* $x>1$*, then, for* $z=\frac{1}{\theta(0,x)}$*,*

$$\begin{aligned} d(0,x)=x^2 &\leq \theta(0,x)[d(0,\frac{1}{\theta(0,x)})+d(\frac{1}{\theta(0,x)},x)] \\ &= \theta(0,x)[\frac{1}{\theta(0,x)}+\frac{1}{\theta(0,x)}] \\ &= 2, \end{aligned}$$

a contradiction. Then (X,d) *is not an EbMS for all* θ *defined from* $X\times X$ *to* $[1,+\infty)$ *.*

6) *Suppose that* (X,d) *is a SCbMS with a functions* η *defined from* $X\times X$ *to* $[1,+\infty)$*. Let* $z_0\in(0,1)$ *and* $y_0\in[1,+\infty)$*, for all* $x>y_0\geq 1$*,*

$$d(x,y_0)=max(x,y_0)=x\leq d(x,z_0)+\eta(z_0,y_0)d(z_0,y_0)=z_0[1+\eta(z_0,y_0)],$$

a contradiction. Then, (X,d) *is not an SCbMS for all* η *defined from* $X\times X$ *to* $[1,+\infty)$*.*

7) *Suppose that (X,d) is an ERbMS with the function θ and let $x>1$. Then, we encounter the followings cases:*

   i) *If, $\theta(0,x)\leq 2$, then for all $u\neq v\in(0,1)$,*

$$\begin{aligned} d(0,x)=x^2 &\leq \theta(0,x)[d(0,u)+d(u,v)+d(v,x)] \\ &\leq 2[d(0,u)+d(u,v)+d(v,x)] \\ &\leq 2[u+min(u,v)+v] \\ &\leq 6, \end{aligned}$$

   then $1<x\leq\sqrt{6}$.

   ii) *Otherwise, let $u=\frac{1}{\theta(0,x)}$ and $v=\frac{2}{\theta(0,x)}$ in $(0,1)$, then*

$$\begin{aligned} d(0,x)=x^2 &\leq \theta(0,x)[d(0,\frac{1}{\theta(0,x)})+d(\frac{1}{\theta(0,x)},\frac{2}{\theta(0,x)}) \\ &+ d(\frac{2}{\theta(0,x)},x)] \\ &= \theta(0,x)[\frac{1}{\theta(0,x)}+\frac{1}{\theta(0,x)}+\frac{2}{\theta(0,x)}] \\ &= 4, \end{aligned}$$

   then $1<x\leq 2$.

   Therefore, $\forall x>1$, $x\leq\sqrt{6}$, a contradiction. Then, (X,d) is not an ERbMS for all θ defined from $X\times X$ to $[1,+\infty)$.

8) *Suppose that (X,d) is a HRMS with a function ψ defined from $X\times X$ into $(-\infty,+\infty)$. Let n be a positive integer and $x>1$, therefore,*

$$\begin{aligned} d(0,x)=x^2 &\leq d(0,\frac{1}{n})+\psi(0,x)d(\frac{1}{n},\frac{1}{2n})+d(\frac{1}{2n},x) \\ &\leq \frac{3+\psi(0,x)}{2n}, \end{aligned}$$

   taking the limit as $n\to+\infty$, we obtain, $x^2\leq 0$, a contradiction. Then (X,d) is not a HRMS for all function ψ defined from $X\times X$ into $(-\infty,+\infty)$.

9) *Suppose that (X,d) is a DCMTS with a non comparable functions α and μ defined from $X\times X$ into $[1,+\infty)$, such that α or μ is bounded. Assume that α is bounded, then there*

exists $M \geq 1$, such that for all $x,y \in X$, $\alpha(x,y) \leq M$. Let $z_0 \in (0,\frac{1}{M})$ and $y_0 \in [1,+\infty)$, for all $x > y_0 \geq 1$,

$$\begin{aligned} d(x,y_0) = max(x,y_0) = x &\leq \alpha(x,z_0)d(x,z_0) + \mu(z_0,y_0)d(z_0,y_0) \\ &= z_0[\alpha(x,z_0) + \mu(z_0,y_0)], \\ &\leq 1 + \frac{\mu(z_0,y_0)}{M}, \end{aligned}$$

a contradiction. Then (X,d) is not a DCMTS for all non comparable functions α and μ defined from $X \times X$ into $[1,+\infty)$, such that α or μ is bounded.

Proposition 3.1. *Let $(X,d,\mathscr{R})$ be a residual ν-metric space. Then, for all positive integer n, $(X,d,\mathscr{R}_n)$ is a residual $n\nu$-metric space, where $\mathscr{R}_n$ is the symmetric function defined from $\underbrace{X \times X \times \cdots \times X}_{n\nu+2\ times}$ into $[0,+\infty)$, by,*

$$\mathscr{R}_n(x_1,x_2,\cdots,x_{n\nu+2}) = \sum_{1\leq i_1<i_2<\cdots<i_{\nu+2}\leq n\nu+2} \mathscr{R}(x_{i_1},x_{i_2},\cdots,x_{i_{\nu+2}}),$$

where $i_1,i_2,\cdots,i_{\nu+2} \in \{1,2,\cdots,n\nu+2\}$.

Proof. Let $(X,d,\mathscr{R})$ be a residual ν-metric space. We employ the induction methodology, indeed, for $n = 1$,

$$\begin{aligned} \mathscr{R}_1(x_1,x_2,\cdots,x_{\nu+2}) &= \sum_{1\leq i_1<i_2<\cdots<i_{\nu+2}\leq \nu+2} \mathscr{R}(x_{i_1},x_{i_2},\cdots,x_{i_{\nu+2}}) \\ &= \mathscr{R}(x_1,x_2,\cdots,x_{\nu+2}), \end{aligned}$$

then, $(X,d,\mathscr{R}_1)$ is a residual ν-metric space where the symmetric function $\mathscr{R}_1 = \mathscr{R}$. Assume that the said property holds for certain positive integer $n \geq 2$. Therefore, for all x,y in X, and for all distinct points $u_1,\cdots,u_{n\nu}$ each of them different from x and y in X,

$$d(x,y) \leq d(x,u_1) + d(u_1,u_2) + \ldots + d(u_{n\nu},y) + \mathscr{R}_n(x,u_1,u_2,\cdots,u_{n\nu},y),$$

such that,

$$\mathscr{R}_n(x_1,x_2,\cdots,x_{n\nu+2}) = \sum_{1\leq i_1<i_2<\cdots<i_{\nu+2}\leq n\nu+2} \mathscr{R}(x_{i_1},x_{i_2},\cdots,x_{i_{\nu+2}}),$$

since $(X,d,\mathscr{R})$ is a residual ν-metric space it follows that, for different $s_1,s_2,\cdots,s_\nu$ in X, each of them different from x, $u_1,\cdots,u_{n\nu}$ and y,

$$d(u_{n\nu},y) \leq d(u_{n\nu},s_1)+\ldots+d(s_\nu,y)+\mathscr{R}(u_{n\nu},s_1,\cdots,s_\nu,y),$$

then,

$$\begin{aligned} d(x,y) &\leq d(x,u_1)+d(u_1,u_2)+\ldots+d(u_{n\nu},y) \\ &+ \mathscr{R}_n(x,u_1,u_2,\cdots,u_{n\nu},y) \\ &\leq d(x,u_1)+d(u_1,u_2)\ldots+d(u_{(n-1)\nu},u_{n\nu}) \\ &+ d(u_{n\nu},s_1)+\cdots+d(s_\nu,y) \\ &+ \mathscr{R}(u_{n\nu},s_1,\cdots,s_\nu,y)+\mathscr{R}_n(x,u_1,u_2,\cdots,u_{n\nu},y). \end{aligned}$$

Let $u_{n\nu+i}=s_i$, for all $i\in\{1,2,\cdots,\nu\}$, then, for different $u_1,\cdots,u_{(n+1)\nu}$ in X, each of them different from x and y,

$$d(x,y) \leq d(x,u_1)+d(u_1,u_2)\ldots+d(u_{(n+1)\nu},y)+\mathscr{R}_{n+1}(x,u_1,u_2,\cdots,u_{(n+1)\nu},y).$$

Therefore, $(X,d,\mathscr{R}_{n+1})$ is a residual $(n+1)\nu$-metric space, where $\mathscr{R}_{n+1}$ is the symmetric function defined from $\underbrace{X\times X\times\cdots\times X}_{(n+1)\nu+2\ times}$ into $[0,+\infty)$, by,

$$\mathscr{R}_{n+1}(x_1,x_2,\cdots,x_{(n+1)\nu+2}) = \sum_{1\leq i_1<i_2<\cdots<i_{\nu+2}\leq(n+1)\nu+2} \mathscr{R}(x_{i_1},x_{i_2},\cdots,x_{i_{\nu+2}}).$$

□

Remark 3.4. *For any $x\in X$ we define the open ball with center x and radius $r>0$ by: $\mathscr{B}_r(x)=\{y\in X \mid d(x,y)<r\}$. Since a residual ν-metric space is a generalization of the previous spaces, then the open balls in $(X,d,\mathscr{R})$ are not necessarily open. Let $\mathscr{U}$ be the collection of all subsets $\mathscr{A}$ of X satisfying the condition that for each $x\in\mathscr{A}$ there exists $r>0$ such that $\mathscr{B}_r(x)\subseteq\mathscr{A}$. Then $\mathscr{U}$ defines a topology for the residual ν-metric space $(X,d,\mathscr{R})$, which is not necessarily Hausdorff. Not that, limit of sequence in residual ν-metric space is not necessarily unique and also every convergent sequence in a residual ν-metric space is not a Cauchy sequence.*

3.1. BCP in a residual ν-metric space. In this section we prove the analogue of BCP in residual ν-metric space. In the sequal X will denotes a nonempty set and $T : X \to X$ will be a self mapping. Let $x_0 \in X$, the *Picard sequence* of T based on x_0 is the sequence (x_n), given by $x_{n+1} = Tx_n$ for all $n \geq 0$. In particular, $x_n = T^n x_0$ for all $n \geq 0$, where T^n denotes the n th-iterates of T. In [31], Zoran D. Mitrovic and Stojan Radenovic proved the following lemma.

Lemma 3.1. *Let (X,d) be a $b_\nu(s)$-metric space, $T : X \to X$ and let (x_n) be a sequence in X defined by $x_0 \in X$ and $x_{n+1} = Tx_n$ such that $x_n \neq x_{n+1}$, for all nonnegative integer n. Suppose that $\lambda \in [0,1)$ such that, for all nonnegative integer n,*

$$d(x_{n+1},x_n) \leq \lambda\, d(x_n,x_{n-1}).$$

Therefore, $x_n \neq x_m$ whenever $n \neq m$.

Lemma 3.2. *Let $(X,d,\mathscr{R})$ be a residual ν-metric space and (x_n) is a sequence of X, such that, $x_n \neq x_m$ whenever $n \neq m$. Let n be a nonnegative integer. Then, for all $p \geq 1$ and for all $m > n + p\nu$,*

$$\begin{aligned} d(x_n,x_m) &\leq \sum_{i=n}^{n+p\nu-1} d(x_i,x_{i+1}) + d(x_{n+p\nu},x_m) \\ &+ \sum_{i=0}^{p-1} \mathscr{R}(x_{n+i\nu},x_{n+1+i\nu},\cdots,x_{n+(1+i)\nu},x_m). \end{aligned}$$

Proof. Let $(X,d,\mathscr{R})$ be a residual ν-metric space and (x_n) is a sequence of X. Let $n \geq 0$, we employ the induction methodology, indeed, for $p = 1$, let $m > n + \nu$, since $x_n \neq x_m$ whenever $n \neq m$, then the residual inequality lead to,

$$\begin{aligned} d(x_n,x_m) &\leq \sum_{i=n}^{n+\nu-1} d(x_i,x_{i+1}) + d(x_{n+\nu},x_m) + \mathscr{R}(x_n,x_{n+1},\cdots,x_{n+\nu},x_m) \\ &\leq \sum_{i=n}^{n+\nu-1} d(x_i,x_{i+1}) + d(x_{n+\nu},x_m) \\ &+ \sum_{i=0}^{p-1} \mathscr{R}(x_{n+i\nu},x_{n+1+i\nu},\cdots,x_{n+(1+i)\nu},x_m). \end{aligned}$$

Assume that the said property holds for certain positive integer $p \geq 2$. Let $m > n + (p+1)\nu$, therefore, $m > n + p\nu$, it follows that,

$$
\begin{aligned}
d(x_n,x_m) &\leq \sum_{i=n}^{n+p\nu-1} d(x_i,x_{i+1}) + d(x_{n+p\nu},x_m) \\
&+ \sum_{i=0}^{p-1} \mathscr{R}(x_{n+i\nu},x_{n+1+i\nu},\cdots,x_{n+(1+i)\nu},x_m).
\end{aligned}
$$

Since $m > n+(p+1)\nu$, then, the residual inequality lead to,

$$
\begin{aligned}
d(x_{n+p\nu},x_m) &\leq \sum_{i=n+p\nu}^{n+(p+1)\nu-1} d(x_i,x_{i+1}) + d(x_{n+(p+1)\nu},x_m) \\
&+ \mathscr{R}(x_{n+p\nu},x_{n+1+p\nu},\cdots,x_{n+(p+1)\nu},x_m),
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d(x_n,x_m) &\leq \sum_{i=n}^{n+(p+1)\nu-1} d(x_i,x_{i+1}) + d(x_{n+(p+1)\nu},x_m) \\
&+ \sum_{i=0}^{p} \mathscr{R}(x_{n+i\nu},x_{n+1+i\nu},\cdots,x_{n+(1+i)\nu},x_m).
\end{aligned}
$$

□

Theorem 3.1. *Let $(X,d,\mathscr{R})$ be a complete residual ν-metric space and let $T : X \to X$ be a (λ,k)-contraction of $(X,d,\mathscr{R})$. Suppose that,*

(i) *There exists $\delta > 0$ where for all x_0, y in X,*

$$\mathscr{R}(x_0,Tx_0,\cdots,T^{\nu}x_0,Ty) \leq \delta,$$

(ii) *For all a Cauchy sequence (x_n) of $(X,d,\mathscr{R})$, where $\lim_{n\to+\infty} x_n = u$, then,*

$$\liminf \mathscr{R}(u,x_{n+1},\cdots,x_{n+\nu},Tu) = 0.$$

Therefore, T has a unique fixed point.

Proof. Let (x_n) be a Picard sequence of the self mapping T of X based on a given $x_0 \in X$. We shall show that (x_n) is Cauchy sequence. If, $x_n = x_{n+1}$ then x_n is fixed point of T. Suppose that for all $n \geq 0$, $x_n \neq x_{n+1}$. Setting $d_n(r_0) = d(x_n,x_{n+r_0})$ for a given positive integer r_0, it follows that,

$$d_n(r_0) \ = \ d(x_n,x_{n+r_0})$$

$$\begin{aligned}
&= d(Tx_{n-1}, Tx_{n+r_0-1}) \\
&\leq \lambda\, d(x_{n-1}, x_{n+r_0-1}) \\
&= \lambda\, d_{n-1}(r_0).
\end{aligned}$$

Repeating this process we obtain, $d_n(r_0) \leq \lambda^n d_0(r_0)$. Since every $b_\nu(s)$-metric space is a residual ν-metric space, then according to the lemma 3.1, $x_n \neq x_m$ whenever $n \neq m$. Then, $d_0(r_0) > 0$ and $d_n(r_0) \to 0$ as $n \to +\infty$. Let $m > n \geq 0$ and let q, r be the quotient and the remainder of the Euclid's Division Lemma of $m-n$ by ν respectively, i.e $m-n = q\nu + r$, where $0 \leq r < \nu$. Then, we encounter the following cases:

(i) if, $q = 0$, therefore,

$$d(x_n, x_m) = d(x_n, x_{n+r}) \leq \lambda^n d_0(r) \leq \lambda^n \sum_{i=1}^{\nu} d_0(i).$$

(ii) Otherwise, i.e $q \geq 1$, then,

(a) if, $r = 0$, therefore,

a-1) if, $q = 1$, i.e $m = n + \nu$, it follows that,

$$d(x_n, x_m) = d_n(\nu) \leq \lambda^n d_0(\nu).$$

a-2) Otherwise, i.e $q > 1$ and $m = n + q\nu$. Since $(X, d, \mathscr{R})$ is a residual ν-metric space, then, according to the Lemma 3.2, it follows that,

$$\begin{aligned}
d(x_n, x_m) &\leq \sum_{i=n}^{n+(q-1)\nu-1} d(x_i, x_{i+1}) + d(x_{n+(q-1)\nu}, x_m) \\
&+ \sum_{i=0}^{q-2} \mathscr{R}(x_{n+i\nu}, x_{n+1+i\nu}, \cdots, x_{n+(1+i)\nu}, x_m) \\
&\leq \sum_{i=n}^{n+(q-1)\nu-1} d_i(1) + d_{n+(q-1)\nu}(\nu) \\
&+ \sum_{i=0}^{q-2} k^{n+i\nu} \mathscr{R}(x_0, x_1, \cdots, x_\nu, x_{m-(n+i\nu)}) \\
&\leq \sum_{i=n}^{n+(q-1)\nu-1} \lambda^i d_0(1) + \lambda^{n+(q-1)\nu} d_0(\nu) \\
&+ k^n \sum_{i=0}^{q-2} k^{i\nu} \mathscr{R}(x_0, x_1, \cdots, x_\nu, x_{m-(n+i\nu)})
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda^n d_0(1)}{1-\lambda} + \lambda^n d_0(\nu) + \delta k^n \sum_{i=0}^{q-2} k^{i\nu} \\
&\leq \lambda^n [\frac{d_0(1)}{1-\lambda} + d_0(\nu)] + \frac{k^n \delta}{1-k^\nu},
\end{aligned}$$

(b) If, $r \neq 0$. According to the Lemma 3.2, it follows that,

$$\begin{aligned}
d(x_n, x_m) &\leq \sum_{i=n}^{n+q\nu-1} d_i(1) + d(x_{n+q\nu}, x_m) \\
&+ \sum_{i=0}^{q-1} \mathscr{R}(x_{n+i\nu}, x_{n+1+i\nu}, \cdots, x_{n+(1+i)\nu}, x_m) \\
&\leq \sum_{i=n}^{n+q\nu-1} \lambda^i d_0(1) + \lambda^{n+q\nu} d_0(r) \\
&+ \sum_{i=0}^{q-1} k^{n+i\nu} \mathscr{R}(x_0, x_1, \cdots, x_\nu, x_{m-(n+i\nu)}) \\
&\leq \lambda^n d_0(1) \sum_{i=0}^{q\nu-1} \lambda^i + \lambda^n \sum_{i=1}^{\nu} d_0(i) \\
&+ k^n \sum_{i=0}^{q-1} k^{i\nu} \mathscr{R}(x_0, x_1, \cdots, x_\nu, x_{m-(n+i\nu)}) \\
&\leq \frac{\lambda^n \sum_{i=1}^{\nu} d_0(i)}{1-\lambda} + \frac{k^n \delta}{1-k^\nu}.
\end{aligned}$$

Therefore, for all $m > n \geq 0$, $d(x_n, x_m) \to 0$ as $n \to +\infty$, thus (x_n) is a Cauchy sequence in X. By completness of $(X, d, \mathscr{R})$ there exists $u \in X$ such that $\lim_{n \to +\infty} x_n = u$. we shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have,

$$\begin{aligned}
d(u, Tu) &\leq d(u, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+\nu}, Tu) \\
&+ \mathscr{R}(u, x_{n+1}, \cdots, x_{n+\nu}, Tu) \\
&\leq d(u, x_{n+1}) + d_{n+1}(1) + \cdots + \lambda d(x_{n+\nu-1}, u) \\
&+ \mathscr{R}(u, x_{n+1}, \cdots, x_{n+\nu}, Tu).
\end{aligned}$$

According to (ii) and taking the limit as $n \to +\infty$, we obtain $d(u, Tu) = 0$, i.e., $Tu = u$. Thus u is a fixed point of T. For uniqueness, let v be another fixed point of T. Then, $d(u, v) = d(Tu, Tv) \leq \lambda d(u, v) < d(u, v)$, a contradiction. Therefore, we must have $d(u, v) = 0$, i.e., $u = v$. Thus fixed point is unique. □

Remark 3.5. *Note that we can replace the condition* (ii) *in the Theorem* 3.1 *by,*

(a) *for all* $u_1, \cdots, u_{\nu+2}$ *in* X*,*

$$\mathscr{R}(u_1, \cdots, u_{\nu+2}) = 0, \text{ whenever } u_i = u_j,$$

where $i \neq j \in \{1, \cdots, \nu+2\}$*,*

(b) $\mathscr{R}$ *is continuous in each variable.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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