Available online at http://scik.org
Adv. Fixed Point Theory, 2025, 15:41
https://doi.org/10.28919/afpt/9489
ISSN: 1927-6303

ISHIKAWA ITERATION FOR DOUBLE SEQUENCES VIA PREDECESSOR PATHS AND RH-REGULAR MATRIX TRANSFORMS

SAMI M. HAMID*, RICHARD F. PATTERSON

Department of Mathematics & Statistics, University of North Florida, Jacksonville, FL 32224, USA

Abstract. This paper extends the Ishikawa fixed-point iteration scheme to double sequences through a diagonal processing framework with unique predecessor paths. We establish monotonicity along these paths and impose parameter conditions on boundary sequences to prove P-convergence of the iterative double sequences to fixed points of compact nonexpansive mappings in Hilbert space. Our results show that convergence extends from boundaries to interior points via diagonal ordering that resolves the two-step dependency structure. We prove that four-dimensional RH-regular matrix transforms preserve both P-convergence and asymptotic regularity, providing a unified treatment that integrates predecessor paths with RH-regular matrix transforms for extending Ishikawa iteration to double sequences.

Keywords: Ishikawa iteration; double sequences; four-dimensional matrices; predecessor paths; Pringsheim convergence; RH-regular matrices; nonexpansive mappings; asymptotic regularity; fixed point theorem.

2020 AMS Subject Classification: 40B05, 40C05, 47H09, 47H10.

1. Introduction

In 1974, S. Ishikawa [6] proposed a new iteration scheme for constructing fixed points of a nonlinear mapping as follows

$$y_n = (1-\beta_n)x_n + \beta_n T(x_n),$$

*Corresponding author
E-mail address: shamid@unf.edu
Received July 13, 2025

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(y_n),$$

where C is a compact convex subset of a Hilbert space $\mathscr{H}$, $T : C \to C$ is a Lipschitzian pseudo-contraction mapping, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that $0 < \alpha_n, \beta_n < 1$ for all integers $n \geq 0$. Ishikawa proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T if, in addition, $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions $\alpha_n \leq \beta_n$ for all n, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_n \alpha_n \beta_n = \infty$. This method generalizes the Mann iteration [8], which is obtained when $\beta_n = 0$ for all n.

Several authors have studied the Ishikawa scheme and its generalizations. Rhoades [15] and Naimpally and Singh [10] asked whether the Ishikawa iteration could be extended to quasi-contraction mappings. This question was resolved by Liu [12, 13], Ding [4], and Zhao [18]. Moreover, Kalinde and Rhoades [7] established fundamental convergence conditions for Ishikawa iteration on the unit interval.

Rhoades [14] established connections between iteration schemes and summability theory by using infinite matrix transformations to analyze fixed point iterations. This matrix-theoretic approach, as presented in Chapter 12 of Mursaleen [9], shows how summability methods can be applied to study convergence of iterative schemes.

Extending Ishikawa iteration to double sequences presents unique challenges due to its two-step structure. Unlike Mann iteration, where the single sequence $x_n = T^n(x_0)$ naturally extends to double sequences as $z_{k,l} = T^{k+l}(x_0)$, Ishikawa's intermediate step y_n creates dependencies that must be carefully managed in the two-dimensional setting. The theory of double sequences, initiated by Pringsheim [11] and developed by Hamilton [5] and Robison [16], provides the framework for this extension. For a comprehensive treatment of double sequence spaces and four-dimensional matrices, the reader can refer to the recent monograph [1].

In this paper, we extend Ishikawa iteration to double sequences and establish convergence results for the iterative double sequences when applied to compact nonexpansive mappings in Hilbert space. Our approach introduces parameter double sequences $(\alpha_{k,l})$ and $(\beta_{k,l})$ that satisfy conditions adapted for the double sequence setting. We show that convergence along the boundary sequences ensures convergence of the entire double sequence through monotonicity properties.

The paper is organized as follows: Section 2 presents preliminaries on double sequences and four-dimensional matrices. Section 3 begins with a strong convergence lemma for the single sequence case, then develops our extension of Ishikawa iteration to double sequences using the predecessor path framework. Section 4 establishes our main convergence results and the preservation of asymptotic regularity under four-dimensional matrix transforms.

2. Preliminaries and Definitions

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ denotes the set of non-negative integers. We begin with the necessary background on double sequences and matrix transformations.

Definition 1 (Pringsheim [11]). *A double sequence $x = (x_{k,l})$ of complex (or real) numbers is called convergent to a scalar L in Pringsheim's sense (denoted by $P\text{-}\lim x = L$) if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\left|x_{k,l} - L\right| < \varepsilon$ whenever $k, l > N$. Such a sequence is described as "P-convergent."*

Definition 2. *A double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $\left|x_{k,l}\right| \leq M$ for all k and l, that is, if $\sup_{k,l} \left|x_{k,l}\right| < \infty$.*

The study of double sequences naturally leads to four-dimensional matrices, which transform one double sequence into another. These matrices play a crucial role in our analysis.

Definition 3. *Let $A = (a_{m,n,k,l})$ denote a four-dimensional matrix that maps complex double sequences $x = (x_{k,l})$ into the double sequence Ax where the (m,n)-th term of Ax is as follows*

$$(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

Definition 4. *A four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.*

The class of RH-regular matrices was completely characterized by Hamilton [5] and Robison [16] as follows.

Theorem 5. *A four-dimensional matrix $A = (a_{m,n,k,l})$ is RH-regular if and only if*

$RH_1 : P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*

$RH_2 : P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$*;*

$RH_3 : P\text{-}\lim_{m,n} \sum_{k=0}^{\infty} \left|a_{m,n,k,l}\right| = 0$ *for each l;*

$RH_4 : P\text{-}\lim_{m,n} \sum_{l=0}^{\infty} \left|a_{m,n,k,l}\right| = 0$ *for each k;*

$RH_5 : \sum_{k,l=0,0}^{\infty,\infty} \left|a_{m,n,k,l}\right|$ *is P-convergent; and*

$RH_6 :$ *there exist finite positive integers K_1 and K_2 such that* $\sum_{k,l>K_2} \left|a_{m,n,k,l}\right| \leq K_1$.

For our purposes, we require additional matrix properties beyond RH-regularity.

Definition 6. *A four-dimensional matrix $A = (a_{m,n,k,l})$ is called uniformly bounded if*

$$\text{(UB):} \quad \sup_{m,n\in\mathbb{N}_0} \sum_{k,l=0,0}^{\infty,\infty} \left|a_{m,n,k,l}\right| < \infty.$$

Definition 7. *A four-dimensional matrix $A = (a_{m,n,k,l})$ satisfies the normalization condition if*

$$\text{(C1):} \quad \sum_{k,l=0,0}^{\infty,\infty} \left|a_{m,n,k,l}\right| = 1, \text{ for all } m,n \in \mathbb{N}_0.$$

3. Ishikawa Iteration: From Single to Double Sequences

We now turn to extending the Ishikawa iteration method to double sequences. We begin with a convergence result for the single sequence case that will be instrumental in proving convergence along the boundary sequences in Theorem 16. The following lemma extends Ishikawa's convergence theorem [6] to compact nonexpansive mappings under relaxed parameter conditions, adapting Fejér monotonicity (see [2, Chapter 5]) to utilize Hilbert space structure and compactness.

Lemma 8. *Let $\mathscr{H}$ be a Hilbert space, $C \subseteq \mathscr{H}$ a nonempty bounded closed convex subset, and $T : C \to C$ a compact nonexpansive mapping. Let $\{x_n\}$ be the Ishikawa sequence defined by*

$$y_n = (1-\beta_n)x_n + \beta_n T(x_n),$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(y_n),$$

with parameters $\{\alpha_n\}$, $\{\beta_n\} \subseteq [0,1]$ *satisfying:*

(1) $0 < \liminf_{n\to\infty} \alpha_n$ *and* $\limsup_{n\to\infty} \alpha_n < 1$*;*
(2) $\lim_{n\to\infty} \beta_n = 0$.

Then $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. Let $F(T) = \{x \in C : T(x) = x\}$ denote the set of fixed points of T. Since $T : C \to C$ is a compact nonexpansive mapping on a nonempty bounded closed convex subset C of the Hilbert space $\mathscr{H}$, Schauder's fixed point theorem [17] guarantees that $F(T) \neq \emptyset$.

Fix $p \in F(T)$ and define the sequence $d_n := \|x_n - p\|$ for all $n \geq 0$. We shall establish that $\{d_n\}$ is nonincreasing. First, consider the intermediate iterate $y_n = (1-\beta_n)x_n + \beta_n T(x_n)$. Since T is nonexpansive and p is a fixed point of T, we have

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)(x_n - p) + \beta_n(T(x_n) - p)\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n \|T(x_n) - p\| \\
&= (1-\beta_n)\|x_n - p\| + \beta_n \|T(x_n) - T(p)\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n \|x_n - p\| \\
&= \|x_n - p\| = d_n.
\end{aligned}$$

For the main iterate $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(y_n)$, we employ the Hilbert space identity [2, Corollary 2.15]

$$\|(1-t)a + tb\|^2 = (1-t)\|a\|^2 + t\|b\|^2 - t(1-t)\|a-b\|^2,$$

for $t \in [0,1]$ and $a, b \in \mathscr{H}$. Applying this identity with $a = x_n - p$, $b = T(y_n) - p$ and $t = \alpha_n$ together with the fact that T is nonexpansive with fixed point p, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T(y_n) - p)\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|T(y_n) - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T(y_n)\|^2
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|T(y_n) - T(p)\|^2 - \alpha_n(1-\alpha_n)\|x_n - T(y_n)\|^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T(y_n)\|^2 \\
&\leq (1-\alpha_n)d_n^2 + \alpha_n d_n^2 - \alpha_n(1-\alpha_n)\|x_n - T(y_n)\|^2 \\
&= d_n^2 - \alpha_n(1-\alpha_n)\|x_n - T(y_n)\|^2 \leq d_n^2.
\end{aligned}
$$

Therefore, $d_{n+1} = \|x_{n+1} - p\| \leq d_n$, establishing that $\{d_n\}$ is nonincreasing. Since $d_n \geq 0$ for all n, the sequence converges to some limit $d^* \geq 0$. From the inequality $\|x_{n+1} - p\|^2 \leq d_n^2 - \alpha_n(1-\alpha_n)\|x_n - T(y_n)\|^2$ established above, we obtain

$$\alpha_n(1-\alpha_n)\|x_n - T(y_n)\|^2 \leq d_n^2 - d_{n+1}^2.$$

By condition (1), we have $\alpha_* := \liminf_{n\to\infty} \alpha_n > 0$ and $\alpha^* := \limsup_{n\to\infty} \alpha_n < 1$. This ensures that the sequence $\{\alpha_n\}$ stays bounded away from both 0 and 1. Specifically, we can find $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$,

$$\frac{\alpha_*}{2} \leq \alpha_n \leq \frac{1+\alpha^*}{2}.$$

Since the function $f(t) = t(1-t)$ is continuous and positive on the open interval $(0,1)$, and our sequence $\{\alpha_n\}_{n\geq N_0}$ lies in the compact interval $\left[\frac{\alpha_*}{2}, \frac{1+\alpha^*}{2}\right] \subset (0,1)$, we have

$$\alpha_n(1-\alpha_n) \geq \frac{\alpha_*}{2}\left(1 - \frac{1+\alpha^*}{2}\right) = \frac{\alpha_*(1-\alpha^*)}{4} =: c > 0$$

for all $n \geq N_0$. Note that $c > 0$ since $0 < \alpha_* \leq \alpha^* < 1$. Thus, for all $n \geq N_0$, we have

$$c\|x_n - T(y_n)\|^2 \leq d_n^2 - d_{n+1}^2.$$

Summing this inequality from $n = N_0$ to $n = M$ for any $M > N_0$, we obtain

$$c \sum_{n=N_0}^{M} \|x_n - T(y_n)\|^2 \leq \sum_{n=N_0}^{M} (d_n^2 - d_{n+1}^2) = d_{N_0}^2 - d_{M+1}^2.$$

Since $d_{M+1}^2 \geq 0$, we obtain

$$\sum_{n=N_0}^{M} \|x_n - T(y_n)\|^2 \leq \frac{d_{N_0}^2}{c}$$

for all $M > N_0$. Taking the limit as M tends to infinity, we conclude that

$$\sum_{n=N_0}^{\infty} \|x_n - T(y_n)\|^2 \leq \frac{d_{N_0}^2}{c} < \infty.$$

The convergence of this series implies that $\lim_{n\to\infty} \|x_n - T(y_n)\|^2 = 0$, and hence

$$\lim_{n\to\infty} \|x_n - T(y_n)\| = 0.$$

We now establish the asymptotic regularity of the sequence $\{x_n\}$. By the nonexpansiveness of T, we get

$$\begin{aligned}\|x_n - T(x_n)\| &\leq \|x_n - T(y_n)\| + \|T(y_n) - T(x_n)\| \\ &\leq \|x_n - T(y_n)\| + \|y_n - x_n\|.\end{aligned}$$

Since $y_n = (1-\beta_n)x_n + \beta_n T(x_n)$, we have

$$y_n - x_n = \beta_n(T(x_n) - x_n),$$

and therefore $\|y_n - x_n\| = \beta_n \|T(x_n) - x_n\|$. Thus,

$$\|x_n - T(x_n)\| \leq \|x_n - T(y_n)\| + \beta_n \|T(x_n) - x_n\|.$$

Rearranging terms, we obtain

$$(1-\beta_n)\|x_n - T(x_n)\| \leq \|x_n - T(y_n)\|.$$

By condition (2), we can find $N_1 \in \mathbb{N}$ such that $\beta_n < \frac{1}{2}$ for all $n \geq N_1$. This implies that for all $n \geq N_1$,

$$1 - \beta_n > \frac{1}{2}, \quad \text{and hence} \quad \frac{1}{1-\beta_n} < 2.$$

Consequently, for all $n \geq \max\{N_0, N_1\}$, we have

$$\|x_n - T(x_n)\| \leq \frac{\|x_n - T(y_n)\|}{1-\beta_n} < 2\|x_n - T(y_n)\|.$$

Since we established that $\lim_{n\to\infty} \|x_n - T(y_n)\| = 0$, it follows that $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$, confirming the asymptotic regularity of the sequence $\{x_n\}$.

Since T is compact and $\{x_n\}$ is bounded, there exists a subsequence $\{n_k\}$ such that $T(x_{n_k})$ converges to some $w \in C$. From the asymptotic regularity, we have $\lim_{k\to\infty} \|x_{n_k} - T(x_{n_k})\| = 0$. Since

$$\|x_{n_k} - w\| \leq \|x_{n_k} - T(x_{n_k})\| + \|T(x_{n_k}) - w\|,$$

and both terms on the right tend to zero as k tends to infinity, we obtain $\lim_{k\to\infty} \|x_{n_k} - w\| = 0$, so the subsequence $\{x_{n_k}\}$ converges to w. By continuity of T, we have $\lim_{k\to\infty} T(x_{n_k}) = T(w)$. But $T(x_{n_k})$ also converges to w by our choice of subsequence, so by uniqueness of limits, $T(w) = w$. From our earlier analysis, $\{\|x_n - w\|\}$ is nonincreasing for any fixed point. Since this monotone decreasing sequence is bounded below by zero, it must converge to some limit $L \geq 0$. But the subsequence $\{\|x_{n_k} - w\|\}$ converges to zero forcing $L = 0$. Therefore, $\lim_{n\to\infty} \|x_n - w\| = 0$. □

The central challenge in extending Ishikawa iteration to double sequences lies in preserving the two-step structure while ensuring mathematical consistency. In the single-sequence case, the iteration proceeds linearly: compute y_n from x_n, then x_{n+1} from y_n. For double sequences, we must determine an appropriate processing order that maintains this dependency structure. We resolve this through a *predecessor path* model where each point has a unique predecessor, combined with *diagonal processing order*.

Definition 9 (Predecessor Path)**.** *For each point $(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0$ with $k+l > 0$, define its unique predecessor as*

$$pred(k,l) = \begin{cases} (k, l-1) & \text{if } l > 0, \\ (k-1, 0) & \text{if } l = 0 \text{ and } k > 0. \end{cases}$$

The predecessor path from $(0,0)$ to (k,l) is the unique sequence $(P_0, P_1, \ldots, P_{k+l})$ where $P_0 = (0,0)$, $P_{k+l} = (k,l)$, and $pred(P_t) = P_{t-1}$ for each $t > 0$.

The predecessor path consists of exactly $k+l+1$ points and can be explicitly written as

$$(0,0) \to (1,0) \to \cdots \to (k,0) \to (k,1) \to \cdots \to (k,l).$$

This path first traverses horizontally along the first row to reach column k, then vertically along column k to reach row l. Each point P_t with $t > 0$ satisfies $\text{pred}(P_t) = P_{t-1}$.

Definition 10 (Diagonal Processing Order)**.** *The diagonal s consists of all points $(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0$ satisfying $k+l = s$. Points are processed in diagonal order by:*

(1) Processing diagonals in increasing order: $s = 0, 1, 2, \ldots$

(2) Within diagonal s, processing points as: $(0,s)$, $(1,s-1)$, $(2,s-2)$, …, $(s,0)$.

For example, the first few diagonals are

$$
\begin{aligned}
&\text{Diagonal } 0: \quad (0,0)\\
&\text{Diagonal } 1: \quad (0,1) \to (1,0)\\
&\text{Diagonal } 2: \quad (0,2) \to (1,1) \to (2,0)\\
&\text{Diagonal } 3: \quad (0,3) \to (1,2) \to (2,1) \to (3,0)\\
&\qquad\qquad \dots
\end{aligned}
$$

This diagonal ordering ensures the iteration is well-defined: for any point (k,l) with $k+l>0$, its predecessor lies on diagonal $(k+l)-1$, which is processed before diagonal $k+l$. Specifically, if $l>0$, then $\text{pred}(k,l)=(k,l-1)$ lies on diagonal $k+(l-1)=(k+l)-1$; if $l=0$ and $k>0$, then $\text{pred}(k,l)=(k-1,0)$ lies on diagonal $(k-1)+0=(k+l)-1$. Thus when computing $x_{k,l}$, the required values $x_{i,j}$ and $y_{i,j}$ from its predecessor $(i,j)=\text{pred}(k,l)$ are already available, ensuring the iteration is well-defined. For example, the first few diagonals are processed as: $(0,0)$; then $(0,1)\to(1,0)$; then $(0,2)\to(1,1)\to(2,0)$; and so on.

With this framework in place, we can now formally define the Ishikawa iteration for double sequences.

Definition 11 (Ishikawa Iteration for Double Sequences)**.** *Let $T: C \to C$ be a mapping on a convex subset C of a Hilbert space $\mathscr{H}$. Let $(\alpha_{k,l})$, $(\beta_{k,l}) \subseteq [0,1]$ be parameter sequences. The double sequences $(x_{k,l})$ and $(y_{k,l})$ are defined by*

Initial condition: $x_{0,0} = x_0 \in C$.

Recurrence relations:

$$
\begin{aligned}
y_{k,l} &= (1-\beta_{k,l})x_{k,l} + \beta_{k,l}T(x_{k,l}) \quad \text{for all } (k,l) \in \mathbb{N}_0 \times \mathbb{N}_0\\
x_{k,l} &= \begin{cases} x_0 & \text{if } (k,l) = (0,0),\\ (1-\alpha_{k-1,0})x_{k-1,0} + \alpha_{k-1,0}T(y_{k-1,0}) & \text{if } l = 0 \text{ and } k > 0,\\ (1-\alpha_{k,l-1})x_{k,l-1} + \alpha_{k,l-1}T(y_{k,l-1}) & \text{if } l > 0. \end{cases}
\end{aligned}
$$

The following definition specifies the parameter conditions for the double-sequence Ishikawa iteration.

Definition 12 (Double Sequence Parameter Conditions)**.** *The parameter sequences* $(\alpha_{k,l})$, $(\beta_{k,l})$ *are said to satisfy the double sequence Ishikawa conditions if:*

(IC$_1$) Parameter boundedness: $0 \le \alpha_{k,l}, \beta_{k,l} \le 1$ *for all* $k,l \ge 0$*;*

(IC$_2$) Boundary summability: $\sum_{k=0}^{\infty} \beta_{k,0} < \infty$ *and* $\sum_{l=0}^{\infty} \beta_{0,l} < \infty$*;*

(IC$_3$) Uniform β*-boundedness:* $\sup_{k,l} \beta_{k,l} < 1$*;*

(IC$_4$) Row boundary control: $0 < \liminf_{k\to\infty} \alpha_{k,0}$ *and* $\limsup_{k\to\infty} \alpha_{k,0} < 1$*;*

(IC$_5$) Column boundary control: $0 < \liminf_{l\to\infty} \alpha_{0,l}$ *and* $\limsup_{l\to\infty} \alpha_{0,l} < 1$.

Remark 13. *Conditions (IC$_4$) and (IC$_5$) keep* $\alpha_{k,0}$ *and* $\alpha_{0,l}$ *bounded away from* 0 *and* 1*, which ensures* $\sum \alpha_{k,0} = \infty$ *and* $\sum \alpha_{0,l} = \infty$*, necessary for reaching the fixed point. Condition (IC$_2$) implies* $\lim_{k\to\infty} \beta_{k,0} = 0$ *and* $\lim_{l\to\infty} \beta_{0,l} = 0$*, which ensures boundary convergence.*

To connect our iteration scheme with the theory of four-dimensional matrices established in Section 2, we introduce the following matrix transform.

Definition 14 (Ishikawa Matrix Transform)**.** *Given the double sequence* $(x_{k,l})$ *from Definition 11 and an RH-regular four-dimensional matrix* $A = (a_{m,n,k,l})$ *satisfying the uniform boundedness condition (UB), then the Ishikawa matrix transform is defined as*

$$w_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

This transform generalizes single sequence matrix methods to double sequences, allowing classical summability techniques to be applied to Ishikawa iterates while preserving convergence properties under uniformly bounded RH-regular matrices.

4. Convergence Theory

We begin our convergence analysis with a fundamental monotonicity property that holds along predecessor paths. This property is crucial for establishing that boundary convergence extends throughout the double sequence.

Lemma 15. *Under the setting of Definition 11 with mapping* $T : C \to C$*, let* p *be a fixed point of* T*. For any point* $(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0$ *and any point* (i, j) *on the predecessor path from* $(0,0)$ *to*

(k,l), *we have*

$$\|x_{k,l}-p\| \le \|x_{i,j}-p\|.$$

Proof. We first establish the result for immediate predecessors. Let $(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0$ with $k+l > 0$, and let $(i,j) = \mathrm{pred}(k,l)$. From Definition 11,

$$x_{k,l} = (1-\alpha_{i,j})x_{i,j} + \alpha_{i,j}T(y_{i,j}).$$

Since T is nonexpansive and p is a fixed point,

$$\begin{aligned}\|x_{k,l}-p\| &= \|(1-\alpha_{i,j})(x_{i,j}-p)+\alpha_{i,j}(T(y_{i,j})-p)\| \\ &\le (1-\alpha_{i,j})\|x_{i,j}-p\| + \alpha_{i,j}\|T(y_{i,j})-T(p)\| \\ &\le (1-\alpha_{i,j})\|x_{i,j}-p\| + \alpha_{i,j}\|y_{i,j}-p\|.\end{aligned}$$

For the intermediate value $y_{i,j} = (1-\beta_{i,j})x_{i,j} + \beta_{i,j}T(x_{i,j})$, we similarly have

$$\begin{aligned}\|y_{i,j}-p\| &\le (1-\beta_{i,j})\|x_{i,j}-p\| + \beta_{i,j}\|T(x_{i,j})-T(p)\| \\ &\le (1-\beta_{i,j})\|x_{i,j}-p\| + \beta_{i,j}\|x_{i,j}-p\| \\ &= \|x_{i,j}-p\|.\end{aligned}$$

Combining these inequalities yields $\|x_{k,l}-p\| \le \|x_{i,j}-p\|$.

For the general case, consider any point (i',j') on the predecessor path from $(0,0)$ to (k,l). This path can be written as

$$(0,0) = P_0 \to P_1 \to \cdots \to P_s = (i',j') \to P_{s+1} \to \cdots \to P_m = (k,l),$$

where each P_{t+1} has P_t as its immediate predecessor. By repeatedly applying the result for immediate predecessors, we obtain

$$\begin{aligned}\|x_{k,l}-p\| &= \|x_{P_m}-p\| \le \|x_{P_{m-1}}-p\| \\ &\le \cdots \le \|x_{P_1}-p\| \le \|x_{P_0}-p\| = \|x_{i',j'}-p\|.\end{aligned}$$

This completes the proof. $\square$

This monotonicity property enables us to establish our main convergence theorem. The key insight is that convergence along the boundary sequences, combined with the monotonicity along predecessor paths, ensures convergence of the entire double sequence.

Theorem 16. *Let $(\mathscr{H}, \|\cdot\|)$ be a Hilbert space, $C \subseteq \mathscr{H}$ a nonempty bounded closed convex subset, and $T : C \to C$ a compact nonexpansive mapping. Let $F(T) = \{p \in C : T(p) = p\}$ denote the set of fixed points of T, and let $A = (a_{m,n,k,l})$ be an RH-regular matrix satisfying the uniform boundedness condition (UB). If the parameter sequences $(\alpha_{k,l})$, $(\beta_{k,l})$ satisfy the double sequence Ishikawa conditions (IC_1)-(IC_5), then:*

(1) The double sequence $(x_{k,l})$ defined by Definition 11 is bounded and P-converges to a fixed point $p^* \in F(T)$.

(2) The matrix transform $w_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$ P-converges to the same fixed point p^*.

Proof. Since $T : C \to C$ is a compact nonexpansive mapping on a nonempty bounded closed convex subset C of the Hilbert space $\mathscr{H}$, Schauder's fixed point theorem [17] guarantees that $F(T) \neq \emptyset$. The boundedness of C implies the existence of $M \geq 0$ such that $\|x\| \leq M$ for all $x \in C$. If $M = 0$, then $C = \{0\}$, and the theorem holds trivially with $x_{k,l} = 0$ for all (k,l) and $p^* = 0$. For $M > 0$, we first establish the boundedness of the double sequence $(x_{k,l})$. By Definition 11, each iterate is a convex combination of points in C. Since C is convex and $T : C \to C$, all iterates remain in C. Therefore, $(x_{k,l})$ is bounded. We now analyze convergence of the boundary sequences. Consider first the row boundary sequence $\{x_{k,0}\}_{k=0}^{\infty}$. By the predecessor path construction, consecutive terms satisfy

$$y_{k,0} = (1-\beta_{k,0})x_{k,0} + \beta_{k,0}T(x_{k,0}) \text{ and}$$

$$x_{k+1,0} = (1-\alpha_{k,0})x_{k,0} + \alpha_{k,0}T(y_{k,0}).$$

From condition (IC_2), we have $\sum_{k=0}^{\infty} \beta_{k,0} < \infty$, which yields $\lim_{k\to\infty} \beta_{k,0} = 0$ by the convergence test. Combined with condition (IC_4), which ensures

$$0 < \liminf_{k\to\infty} \alpha_{k,0} \text{ and } \limsup_{k\to\infty} \alpha_{k,0} < 1,$$

these are precisely the hypotheses required by Lemma 8 for the single sequence Ishikawa iteration. Therefore, the row boundary sequence $\{x_{k,0}\}_{k=0}^{\infty}$ converges strongly to some fixed point

$p_1 \in F(T)$. The column boundary sequence $\{x_{0,l}\}_{l=0}^{\infty}$ exhibits analogous behavior. The iteration follows

$$y_{0,l} = (1-\beta_{0,l})x_{0,l} + \beta_{0,l}T(x_{0,l}) \text{ and}$$
$$x_{0,l+1} = (1-\alpha_{0,l})x_{0,l} + \alpha_{0,l}T(y_{0,l}).$$

By the same reasoning, from condition (IC$_2$), we have $\sum_{l=0}^{\infty} \beta_{0,l} < \infty$, which yields $\lim_{l\to\infty} \beta_{0,l} = 0$, while condition (IC$_5$) provides

$$0 < \liminf_{l\to\infty} \alpha_{0,l} \text{ and } \limsup_{l\to\infty} \alpha_{0,l} < 1.$$

Applying Lemma 8 again, we conclude that $\{x_{0,l}\}_{l=0}^{\infty}$ converges strongly to some fixed point $p_2 \in F(T)$. Next, we show that the two boundary fixed points coincide. Since $T : C \to C$ is nonexpansive on a closed convex subset of a Hilbert space, the set of fixed points $F(T)$ is closed and convex (cf. [3], Proposition 2.1.11). We now apply the predecessor path inequality from Lemma 15, which states that for any $q \in F(T)$ and any (i, j) on a predecessor path from $(0,0)$ to (k,l), we have $\|x_{k,l} - q\| \leq \|x_{i,j} - q\|$. By Definition 11, the predecessor path from $(0,0)$ to any boundary point $(k,0)$ passes through $(0,0)$, and similarly for any $(0,l)$. Therefore, we obtain

$$\|x_{k,0} - q\| \leq \|x_{0,0} - q\| \quad \text{for all } k \geq 0 \text{ and all } q \in F(T), \text{ and} \tag{1}$$

$$\|x_{0,l} - q\| \leq \|x_{0,0} - q\| \quad \text{for all } l \geq 0 \text{ and all } q \in F(T). \tag{2}$$

Since the boundary sequences converge strongly to p_1 and p_2 respectively, and the norm is continuous, taking limits in (1) and (2) yields

$$\|p_1 - q\| \leq \|x_{0,0} - q\| \quad \text{for all } q \in F(T), \text{ and} \tag{3}$$

$$\|p_2 - q\| \leq \|x_{0,0} - q\| \quad \text{for all } q \in F(T). \tag{4}$$

These inequalities show that both p_1 and p_2 minimize the distance from $x_{0,0}$ over all points in $F(T)$. Let $p^* := P_{F(T)}(x_{0,0})$ be the metric projection of $x_{0,0}$ onto $F(T)$, that is, the unique point in $F(T)$ that minimizes the distance to $x_{0,0}$:

$$\|x_{0,0} - p^*\| = \inf_{q\in F(T)} \|x_{0,0} - q\|. \tag{5}$$

Since $F(T)$ is nonempty, closed, and convex, such a projection exists and is unique (cf. [3], Theorem 1.2.3). As both p_1 and p_2 achieve the minimum distance from $x_{0,0}$ by (3) and (4), we conclude that

$$p_1 = p_2 = p^* = P_{F(T)}(x_{0,0}).$$

We next establish the P-convergence of the entire double sequence to p^*. From the convergence of both boundary sequences to p^*, given any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $\left\|x_{k,0} - p^*\right\| < \varepsilon$ for all $k > N_1$, and $N_2 \in \mathbb{N}$ such that $\left\|x_{0,l} - p^*\right\| < \varepsilon$ for all $l > N_2$. Set $N = \max\{N_1, N_2\}$. We claim that $\left\|x_{k,l} - p^*\right\| < \varepsilon$ for all (k,l) satisfying $k > N$ and $l > N$. To verify this claim, consider any point (k,l) with $k > N$ and $l > N$. By Definition 9, the unique predecessor path from $(0,0)$ to (k,l) is

$$(0,0) \to (1,0) \to \cdots \to (k,0) \to (k,1) \to \cdots \to (k,l),$$

which first traverses horizontally along the first row to $(k,0)$, then vertically along column k to (k,l). Every predecessor path from $(0,0)$ to (k,l) with $k > 0$, $l > 0$ must include $(k,0)$ by the structure of our predecessor function. Since $k > N \geq N_1$, we have $\left\|x_{k,0} - p^*\right\| < \varepsilon$. As $(k,0)$ lies on the predecessor path to (k,l), Lemma 15 yields

$$\left\|x_{k,l} - p^*\right\| \leq \left\|x_{k,0} - p^*\right\| < \varepsilon.$$

Similarly, for any point $(0,l)$ with $l > N$, since $l > N \geq N_2$, we have $\left\|x_{0,l} - p^*\right\| < \varepsilon$ directly from the convergence of the column boundary sequence. Thus, the claim holds for all points (k,l) with $k > N$ and $l > N$. This establishes that $(x_{k,l})$ is P-convergent to p^*. Finally, since $(x_{k,l})$ is bounded and P-converges to p^*, and the matrix $A = (a_{m,n,k,l})$ is RH-regular, we can apply the matrix transform. By the uniform boundedness condition (UB), we have $\sup_{m,n\in\mathbb{N}_0} \sum_{k,l=0,0}^{\infty,\infty} \left|a_{m,n,k,l}\right| < \infty$, which ensures that the series $w_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$ converges absolutely for the bounded sequence $(x_{k,l})$. The definition of RH-regularity then ensures that the transform preserves the P-limit. Therefore, $P\text{-}\lim_{m,n\to\infty} w_{m,n} = p^*$. This completes the proof that both the original double sequence and its matrix transform converge to the same fixed point of T. $\square$

Having established convergence of the double-sequence Ishikawa iteration, we now turn to the asymptotic behavior of these iterates. The following theorem shows that the four-dimensional matrix transform preserves asymptotic regularity.

Theorem 17. *Let $(\mathscr{H}, \|\cdot\|)$ be a Hilbert space, $C \subseteq \mathscr{H}$ a nonempty bounded closed convex subset, and $T : C \to C$ a compact nonexpansive mapping. Let $A = (a_{m,n,k,l})$ be a nonnegative RH-regular matrix satisfying uniform boundedness (UB) and normalization (C1) conditions. If the parameter sequences $(\alpha_{k,l})$, $(\beta_{k,l})$ satisfy conditions (IC$_1$)-(IC$_5$), then the Ishikawa matrix transform*

$$w_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$$

satisfies

$$P\text{-}\lim_{m,n\to\infty} \|w_{m,n} - T(w_{m,n})\| = 0.$$

Proof. By Theorem 16, the double sequence $(x_{k,l})$ P-converges to the fixed point $p \in F(T)$, this implies that

$$P\text{-}\lim_{k,l\to\infty} \left\|x_{k,l} - p\right\| = 0.$$

We first establish the asymptotic regularity of the base sequence. For any $(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0$, since T is nonexpansive and p is a fixed point of T, we have

$$\begin{aligned} \left\|x_{k,l} - T(x_{k,l})\right\| &\leq \left\|x_{k,l} - p\right\| + \left\|p - T(x_{k,l})\right\| \\ &= \left\|x_{k,l} - p\right\| + \left\|T(p) - T(x_{k,l})\right\| \\ &\leq 2\left\|x_{k,l} - p\right\|. \end{aligned}$$

Since the right-hand side P-converges to zero, we conclude that

$$P - \lim_{k,l\to\infty} \left\|x_{k,l} - T(x_{k,l})\right\| = 0.$$

Define the double sequence $z_{k,l} = \left\|x_{k,l} - T(x_{k,l})\right\|$ for all $(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then $P\text{-}\lim_{k,l\to\infty} z_{k,l} = 0$, and since $(x_{k,l})$ is bounded and T maps bounded sets to bounded sets, the sequence $(z_{k,l})$ is bounded. To analyze the asymptotic regularity of the matrix transform $w_{m,n}$, we first note that by nonnegativity of A and the normalization condition (C1), we have $a_{m,n,k,l} \geq 0$ and

$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$. Since each $x_{k,l} \in C$ and C is convex, $w_{m,n}$ is a convex combination of points in C, hence $w_{m,n} \in C$. This ensures $T(w_{m,n})$ is well-defined. Now we write

$$\|w_{m,n} - T(w_{m,n})\| \leq \left\| w_{m,n} - \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} T(x_{k,l}) \right\| + \left\| \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} T(x_{k,l}) - T(w_{m,n}) \right\|.$$

Note that by the uniform boundedness condition (UB), both series converge absolutely since $(x_{k,l})$ and $(T(x_{k,l}))$ are bounded sequences. For the first term in the above inequality, by the definition of $w_{m,n}$ and nonnegativity of A, we have

$$\begin{aligned}\left\| w_{m,n} - \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} T(x_{k,l}) \right\| &= \left\| \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l} - \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} T(x_{k,l}) \right\| \\ &= \left\| \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} (x_{k,l} - T(x_{k,l})) \right\| \\ &\leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left\| x_{k,l} - T(x_{k,l}) \right\| = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} z_{k,l}.\end{aligned}$$

Since $\left(z_{k,l}\right)$ is a bounded double sequence that P-converges to zero, and the matrix A is RH-regular, the definition of RH-regularity ensures that $P\text{-}\lim_{m,n\to\infty} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} z_{k,l} = 0$, and therefore,

$P\text{-}\lim_{m,n\to\infty} \left\| w_{m,n} - \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} T(x_{k,l}) \right\| = 0.$

For the second term, since T is nonexpansive and p is a fixed point of T, we have

$$\begin{aligned}\left\| \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} T(x_{k,l}) - T(w_{m,n}) \right\| &\leq \left\| \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} T(x_{k,l}) - p \right\| + \|p - T(w_{m,n})\| \\ &= \left\| \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} T(x_{k,l}) - p \right\| + \|T(p) - T(w_{m,n})\| \\ &\leq \left\| \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} T(x_{k,l}) - p \right\| + \|p - w_{m,n}\|.\end{aligned}$$

For the first summand in this last inequality, observe that since $\left(x_{k,l}\right)$ P-converges to p and T is continuous, the sequence $\left(T(x_{k,l})\right)$ P-converges to $T(p) = p$. Moreover, $\left(T(x_{k,l})\right)$ is bounded since T maps the bounded set $\left(x_{k,l}\right)$ to a bounded set. Since $\left(T(x_{k,l})\right)$ is a bounded P-convergent sequence converging to p, and A is RH-regular, the definition of RH-regularity grants us that $P\text{-}\lim_{m,n\to\infty} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} T(x_{k,l}) = p$. For the second summand, Theorem 16 directly gives us

$P\text{-}\lim_{m,n\to\infty} w_{m,n} = p$. Since both summands P-converge to zero, the second term P-converges to zero. Combining our estimates for both terms, we have $P\text{-}\lim_{m,n\to\infty} \|w_{m,n} - T(w_{m,n})\| = 0$, establishing the asymptotic regularity of the matrix transform. □

This completes our analysis of double-sequence Ishikawa iteration, establishing both convergence and asymptotic regularity under the specified parameter conditions.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] F. Başar, M.Y. Savaşcı, Double Sequence Spaces and Four-Dimensional Matrices, Chapman and Hall/CRC, Boca Raton, 2022. https://doi.org/10.1201/9781003285786.

[2] H.H. Bauschke, P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, Cham, 2017. https://doi.org/10.1007/978-3-319-48311-5.

[3] A. Cegielski, Iterative Methods for Fixed Point Problems in Hilbert Spaces, Springer, Berlin, 2012. https://doi.org/10.1007/978-3-642-30901-4.

[4] X.P. Ding, Iteration Method to Construct Fixed Point of Nonlinear Mappings, Math. Numer. Sin. 3 (1981), 285–295.

[5] H.J. Hamilton, Transformations of Multiple Sequences, Duke Math. J. 2 (1936), 29–60.

[6] S. Ishikawa, Fixed Points by a New Iteration Method, Proc. Am. Math. Soc. 44 (1974), 147–150. https://doi.org/10.2307/2039245.

[7] A.K. Kalinde, B. Rhoades, Fixed Point Ishikawa Iterations, J. Math. Anal. Appl. 170 (1992), 600–606. https://doi.org/10.1016/0022-247x(92)90040-k.

[8] W.R. Mann, Mean Value Methods in Iteration, Proc. Am. Math. Soc. 4 (1953), 506–510. https://doi.org/10.2307/2032162.

[9] M. Mursaleen, Applied Summability Methods, Springer, Cham, 2014. https://doi.org/10.1007/978-3-319-04609-9.

[10] S. Naimpally, K. Singh, Extensions of Some Fixed Point Theorems of Rhoades, J. Math. Anal. Appl. 96 (1983), 437–446. https://doi.org/10.1016/0022-247x(83)90052-5.

[11] A. Pringsheim, Zur Theorie der Zweifach Unendlichen Zahlenfolgen, Math. Ann. 53 (1900), 289–321. https://doi.org/10.1007/bf01448977.

[12] L. Qihou, On Naimpally and Singh's Open Questions, J. Math. Anal. Appl. 124 (1987), 157–164. https://doi.org/10.1016/0022-247x(87)90031-x.

[13] L. Qihou, A Convergence Theorem of the Sequence of Ishikawa Iterates for Quasi-Contractive Mappings, J. Math. Anal. Appl. 146 (1990), 301–305. https://doi.org/10.1016/0022-247x(90)90303-w.

[14] B.E. Rhoades, Fixed Point Iterations Using Infinite Matrices, Trans. Am. Math. Soc. 196 (1974), 161–176. https://doi.org/10.1090/s0002-9947-1974-0348565-1.

[15] B. Rhoades, Comments on Two Fixed Point Iteration Methods, J. Math. Anal. Appl. 56 (1976), 741–750. https://doi.org/10.1016/0022-247x(76)90038-x.

[16] G.M. Robison, Divergent Double Sequences and Series, Trans. Am. Math. Soc. 28 (1926), 50–73. https://doi.org/10.2307/1989172.

[17] J. Schauder, Der Fixpunktsatz in Funktionalraümen, Studia Math. 2 (1930), 171–180. https://eudml.org/doc/217247.

[18] H.B. Zhao, Successive Approximations of Fixed Points for Some Nonlinear Mappings, Math. Numer. Sin. 7 (1985), 131–137.