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A FIXED POINT THEOREM FOR G_f-KANNAN CONTRACTIVE MAPPINGS IN GENERALIZED SUPRAMETRIC SPACES WITH A DIGRAPH

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Abstract. In this paper, we introduce a fixed point theorem for a new class of G_f-Kannan contractive mappings defined on generalized suprametric spaces with a reflexive digraph. Although various extensions of Kannan's fixed point theorem have been studied in many generalized metric spaces, few results address the interaction between contractive conditions and additional relational structures, such as digraphs. Our approach incorporates the structure of a digraph into the analysis considering the connectivity between points, allowing us to generalize existing theorems and unify different frameworks. We prove the uniqueness of fixed points under suitable conditions and provide a nontrivial example to illustrate the applicability of our result. This work contributes to the ongoing development of fixed point theory in generalized suprametric spaces and opens new directions for research involving relational structures defined by graphs.

Keywords: fixed point theorem; digraph; generalized suprametric space; property (P); G monotone preserving; D-convergent; D-Cauchy; complete subgraph; transitive; Kannan contraction.

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1. Introduction

Fixed-point theory remains a cornerstone of mathematical analysis and provides essential results for non-linear equations and optimization problems. Among the pivotal contributions

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to this field is Kannan's fixed-point theorem, introduced in 1968, which characterizes metric completeness via Kannan contractions. This theorem asserts that a metric space is complete if and only if every Kannan contraction has a fixed point. Despite its broad applications, there has been growing interest in extending Kannan's results to more general spaces and structures.

Recent studies have aimed to generalize Kannan's fixed point theorem in various settings. For example, Kamouss, Chaira, Eladraoui and Kabil [1] extended the theorem to generalized metric spaces equipped with a graph structure, building on the foundational ideas introduced by Jleli and Samet [4] in their study of generalized metric spaces and related fixed-point theorems. Similarly, Doan [2] developed a new version of Kannan's fixed point theorem in the framework of strong b-metric spaces, paving the way for further exploration of fixed points in more specialized environments. Berzig [3] also contributed significantly by establishing fixed-point results in generalized suprametric spaces, thereby broadening the class of spaces in which Kannan-type theorems are applicable.In addition, Chaira, Eladraoui, Kabil and Lazaiz [5] investigated extensions of the Kirk–Saliga fixed point theorem in metric spaces with reflexive digraphs, while Eladraoui, Kabil, and Lazaiz [6] studied generalized metric spaces and related fixed point theorems, further enriching the theory and its applications.

This paper builds on these contributions by studying fixed-point theorems for Kannan contractions in generalized suprametric spaces with a digraph. By combining the flexibility of metric spaces with the directed relationships inherent in digraphs, we provide new insights into the conditions under which Kannan contractions have fixed points, along with their uniqueness and convergence properties. The results presented here not only generalize classical fixed point theory but also offer a deeper understanding of the interaction between the space, the digraph, and the contraction conditions.

2. PRELIMINARIES

Definition 2.1. *A graph G is determined by a nonempty set $V(G)$ of its vertices and the set $E(G) \subset V(G) \times V(G)$ of its arcs. A directed graph, also called a digraph, is a type of graph in which the edges have a direction.*

If $\forall x \in V(G)$, $(x,x) \in E(G)$ then $G = (V,E)$ is reflexive.

G is said to be transitive whenever for any $x,y,z \in V(G)$

$$[(x,y) \in E(G) \quad and \quad (y,z) \in E(G) \implies (x,z) \in E(G)].$$

The converse of a digraph $G = (V(G),E(G))$, denoted by G^{-1}, is the digraph $(V(G),E(G^{-1}))$ where $(u,v) \in E(G^{-1})$ if and only if $(v,u) \in E(G)$
Also, $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring direction of the edges. Thus we have

$$E(\widetilde{G}) = E(G)\bigcup E(G^{-1})$$

Definition 2.2. *A sequence $(x_n)_{n\in\mathbb{N}} \in V(G)$ is said to be $G-$increasing if $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, $G-$decreasing if $(x_{n+1},x_n) \in E(G)$ for all $n \in \mathbb{N}$ and $G-$monotone if it is either $G-$increasing or $G-$decreasing.*

Definition 2.3. *Let $\mathscr{X}$ be a nonempty set, and let $\mathscr{D} : \mathscr{X} \times \mathscr{X} \to \overline{\mathbb{R}}^+ = [0,+\infty]$ be a given mapping. Let $x \in \mathscr{X}$ and $(x_n)_{n\geqslant 0}$ a sequence in $\mathscr{X}$. We say that $(x_n)_{n\geqslant 0}$ is $\mathscr{D}$-convergent to x, and we write $\lim_{n\to+\infty} \mathscr{D}(x_n,x) = 0$, if and only if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all intger $n \in \mathbb{N}$,*

$$n \geqslant n_0 \Rightarrow \mathscr{D}(x,x_n) < \varepsilon.$$

Definition 2.4. *Let $\mathscr{X}$ be a nonempty set. Suppose that the mapping $\mathscr{D} : \mathscr{X} \times \mathscr{X} \to \overline{\mathbb{R}}^+$ is defined, and satisfies following properties:*

- (D_1) $0 \leqslant \mathscr{D}(x,y)$, *for all x and y in $\mathscr{X}$; **(Non-negativity)**.*
- (D_2) $\mathscr{D}(x,y) = 0$ *implies that $x = y$; **(Identity of indiscernibles)**.*
- (D_3) $\mathscr{D}(x,y) = \mathscr{D}(y,x)$, *for all x and y in $\mathscr{X}$; **(Symmetry)**.*
- (D_4) *there exists $\theta : \mathscr{X} \times \mathscr{X} \to [0,+\infty[$ such that, for all $(x,y) \in \mathscr{X} \times \mathscr{X}$ and every sequence $(x_n)_{n\in\mathbb{N}} \in \mathscr{C}(\mathscr{D},\mathscr{X},x)$, the following conditions hold:*

$$\limsup_{n\to+\infty} \mathscr{D}(x_n,y)) < \infty$$

and

$$\mathscr{D}(x,y) \leqslant \theta(x,y) \limsup_{n\to+\infty} \mathscr{D}(x_n,y) \quad \textit{\textbf{(Generalized lim sup inequality).}}$$

In this case, $\mathscr{D}$ is called a generalized suprametric on $\mathscr{X}$, and the pair $(\mathscr{X},\mathscr{D})$ is referred to as a generalized suprametric space.

Example 2.1. *Let $\mathscr{X}=\mathbb{R}_+\bigcup\{-1\}$, such that $\theta(x,y)=5e^{xy}$, and consider the mapping $\mathscr{D}:\mathscr{X}\times\mathscr{X}\to\mathbb{R}_+\bigcup\{+\infty\}$ defined by:*

$$\mathscr{D}(x,y)=\begin{cases} x+y, & \text{if} \quad (x=0 \text{ or } y=0) \quad \text{and} \quad x,y\notin\{-1\},\\ 1+\dfrac{x+y}{5}, & \text{if} \quad x,y\notin\{0,-1\},\\ +\infty, & \text{if} \quad (x,y)\in(\mathbb{R}_+\times\{-1\}) \text{ or } (\{-1\}\times\mathbb{R}_+). \end{cases} \tag{2.1}$$

It is evident that the mapping $\mathscr{D}$ satisfies axioms (D_1), (D_2), and (D_3). Now, let us verify axiom (D_4).

Let $x\in\mathscr{X}$, we distinguish 3 cases :

Case 1: *If $x\notin\{0,-1\}$*

In this case, we have $C(\mathscr{D},\mathscr{X},x)=\emptyset$, because otherwise, there would exist a sequence $(x_n)_{n\geqslant 0}$ of elements of $\mathscr{X}$ such that

$$\lim_{n\to+\infty}\mathscr{D}(x_n,x)=0.$$

For $n\geqslant n_0$, we have $\mathscr{D}(x_n,x)=1+\dfrac{x_n+x}{5}$ or $\mathscr{D}(x_n,x)=x+x_n$, for any integer $n\geqslant n_0$. As a result

$$\lim_{n\to+\infty}\mathscr{D}(x_n,x)=\lim_{n\to+\infty}1+\frac{x_n+x}{5}\geqslant\frac{x}{5}$$

or

$$\lim_{n\to+\infty}\mathscr{D}(x_n,x)=\lim_{n\to+\infty}(x+x_n)\geqslant x$$

Implie that

$$0\geqslant x$$

Hence,

$$x=0 \quad \text{or} \quad x=-1$$

which contradicts our assumption that $x\notin\{0,-1\}$.

Case 2: *If $x=-1$, then $C(\mathscr{D},\mathscr{X},x)=\emptyset$, because for any sequence $(x_n)_{n\in\mathbb{N}}$, we have*

$$\mathscr{D}(x_n,-1)=+\infty$$

Case 3:*If $x = 0$. In this case, we have $C(\mathscr{D}, \mathscr{X}, x) \neq \emptyset$.*

Let $(x_n)_{n\in\mathbb{N}} \in C(\mathscr{D}, \mathscr{X}, 0)$, so from some rank $\mathscr{D}(x_n, 0) < +\infty$ and $x_n \neq -1$.

- *If the sequence $(x_n)_{n\in\mathbb{N}}$ admits an infinity of zero, there exists a subsequence $(x_{\phi(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{\phi(n)} = 0$, for all $n \in \mathbb{N}$.*

  Let $y \in X\backslash\{-1\}$, we have:

$$\mathscr{D}(0,y) = y = \lim_{n\to+\infty} \mathscr{D}(x_{\phi(n)}, y) \leqslant \limsup_{n\to+\infty} \mathscr{D}(x_n, y) \leqslant 5e^0 \limsup_{n\to+\infty} \mathscr{D}(x_n, y)$$

  For $y = -1$, we have $\mathscr{D}(0,y) = +\infty$.

  Then $\mathscr{D}(0,y) = 5e^{xy} \limsup_{n\to+\infty} \mathscr{D}(x_n, y)$

- *If the sequence $(x_n)_{n\in\mathbb{N}}$ doesn't admit an infinity of zero, then there exists $n_0 \in \mathbb{N}$ such that $x_n \neq 0$, for any integer $n \geqslant n_0$.*

  Let $y \in \mathscr{X}\backslash\{0,-1\}$. We have

$$D(x_n, y) = 1 + \frac{x_n + y}{5} = \mathscr{D}(x_n, 0) + \frac{y}{5},$$

  for any integer $n \geqslant n_0$. As a result

$$\mathscr{D}(0,y) = y = 5 \lim_{n\to+\infty} \mathscr{D}(x_n, y) \leqslant 5e^{xy} \limsup_{n\to+\infty} \mathscr{D}(x_n, y).$$

  For $y = 0$, we have

$$\mathscr{D}(0,y) = 0 = \lim_{n\to+\infty} \mathscr{D}(x_n, y).$$

  For $y = -1$, we have $\mathscr{D}(x_n, y) = +\infty$, and $\mathscr{D}(0,y) = +\infty$.

  Then

$$\mathscr{D}(0,y) = 5e^{xy} \limsup_{n\to+\infty} \mathscr{D}(x_n, y),$$

  Hence, for all $y \in \mathscr{X}$, there exists $\theta : \mathscr{X} \times \mathscr{X} \to [1, +\infty[$, such that, for all $(x,y) \in \mathscr{X} \times \mathscr{X}$ and $(x_n)_{n\in\mathbb{N}} \in \mathscr{C}(\mathscr{D}, \mathscr{X}, x)$, we have

$$\mathscr{D}(x,y) \leqslant \theta(x,y) \limsup_{n\to+\infty} \mathscr{D}(x_n, y)$$

  Therefore, $(\mathscr{X}, \mathscr{D})$ is a generalized suprametric space.

Remark 2.1. *Let $(\mathscr{X}, \mathscr{D})$ be a generalized suprametric space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{X}$. If $(x_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-convergent to some $x \in \mathscr{X}$, then this limit is unique.*

Proof. Suppose the sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-converges to x and y . By the property (D_4), we have.

$$\mathscr{D}(x,y) \leqslant \theta(x,y) \limsup_{n\to+\infty} max\{\mathscr{D}(x_n,x), \mathscr{D}(x_n,y)\}$$

Thus $\mathscr{D}(x,y) = 0$, we conclude that $x = y$. □

Definition 2.5. *Let $(\mathscr{X},\mathscr{D})$ be a generalized suprametric space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{X}$.*

(i) A sequence $(x_n)_{n\in\mathbb{N}}$ is called a $\mathscr{D}$-Cauchy sequence if, for every $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that

$$\mathscr{D}(x_n,x_m) < \varepsilon, \textit{ for all } n,m \geqslant N,$$

(ii) The space $(\mathscr{X},\mathscr{D})$ is said to be $\mathscr{D}$-Complete if every $\mathscr{D}$-Cauchy sequence is $\mathscr{D}$-convergent to some element in $\mathscr{X}$.

Example 2.2. *We return to Example 2.1 and show that $(\mathscr{X},\mathscr{D})$ is $\mathscr{D}$-Complete.*
It is easy to see that the only $\mathscr{D}-$Cauchy sequences those that eventually become zero from some rank.
Let $(x_n)_{n\in\mathbb{N}}$ be a $\mathscr{D}-$Cauchy, meaning that

$$\forall\varepsilon > 0\,, (\exists N \in \mathbb{N})\,, \forall n,m > N, \mathscr{D}(x_n,x_m) < \varepsilon$$

Since $(x_n)_{n\in\mathbb{N}}$ is a $\mathscr{D}-$Cauchy, we analyze different cases:
Case 1: $x_n, x_m \notin \{0,-1\}$
In this case, we have:

$$0 < \frac{x_n + x_m}{5} < \varepsilon - 1 < 0,$$

which is a contradiction.
Case 2: $x_n = 0$ *or* $x_m = 0$.
Since $(x_n)_{n\in\mathbb{N}}$ is $\mathscr{D}-$Cauchy, we obtain $x_n + x_m < \varepsilon$.
Letting $m\to\infty$, this implies that $x_n \to 0$.
Thus, every $\mathscr{D}$-Cauchy sequences in $(\mathscr{X},\mathscr{D})$ those converges to 0.
Hence $(\mathscr{X},\mathscr{D})$ is $\mathscr{D}$-Complete.

For every $x \in \mathscr{X}$, let us define the set

$$C(\mathscr{D}, \mathscr{X}, x) = \{(x_n)_{n\in\mathbb{N}} \subseteq \mathscr{X} : \lim_{n\to+\infty} \mathscr{D}(x_n, x) = 0\}.$$

Remark 2.2. *$(\mathscr{X}, \mathscr{D})$ is a generalized metric spaces implies that $(\mathscr{X}, \mathscr{D})$ is a generalized suprametric spaces.*

Definition 2.6. *Let G be a digraph.*
A mapping $T : V(G) \to V(G)$ is said to be a G_K-Kannan contraction if the following conditions are satisfied:

(i) T is $G-$monotone-preserving, meaning that

$$(x,y) \in E(G) \implies (T(x), T(y)) \in E(G)$$

(ii) For every $(x,y) \in E(G)$

$$\mathscr{D}(Tx, Ty) \leqslant K\Big[\mathscr{D}(Tx, x) + \mathscr{D}(y, Ty)\Big], \tag{2.2}$$

with $0 < K < \frac{1}{2}$.

Remark 2.3. *If T is $G-$monotone-preserving, then T is $G^{-1}-$monotone-preserving. Indeed, we have*

$$\begin{aligned}(x,y) \in E(G^{-1}) &\implies (y,x) \in E(G)\\ &\implies (T(y), T(x)) \in E(G)\\ &\implies (T(x), T(y)) \in E(G^{-1})\end{aligned}$$

Definition 2.7. *A digraph G is said to satisfy property $(\mathscr{P})$ if, for any $G-$monotone-preserving sequence $(x_n)_{n\in\mathbb{N}}$ that $\mathscr{D}-$converges to some $x \in V(G)$, we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

3. Main Results

Let $(x_n)_{n\geqslant 0}$ be the Picard sequence defined by $x_{n+1} = T^n x_0$, for all $n \in \mathbb{N}$.
Assume that G is reflexive.
Let $T : V(G) \to V(G)$ be a mapping. For any $x_0 \in V(G)$, denote by $G[\mathscr{O}_T(x_0)]$ the complete

subgraph induced by the orbit $\mathscr{O}_T(x_0) = \{T^n x_0 : n \in \mathbb{N}\}$.

The following notation will be used throughout:

$$\delta(\mathscr{D}, T, x_0) := \sup\left\{\mathscr{D}(T^i x_0, x_0) : i \in \mathbb{N} \setminus \{0\}\right\}, \text{ where } x_0 \in \mathscr{X}.$$

Lemma 3.1. *Let $T : V(G) \to V(G)$ be $G-$monotone-preserving. Suppose there exists $x_0 \in V(G)$ such that $(x_0, Tx_0) \in E(G)$ and the complete subgraph $G[\mathscr{O}_T(x_0)]$ is transitive, then the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is a $G-$increasing, and for any $m, n \in \mathbb{N}$ with $n < m$., we have $(T^n x_0, T^m x_0) \in E(G)$.*

Proof. We are given that T is $G-$monotone-preserving, and $(x_0, Tx_0) \in E(G)$.
Thus, for all $n \in \mathbb{N}$, we have $(T^n x_0, T^{n+1} x_0) \in E(G)$.
Therefore, the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is a $G-$increasing.
Since $(T^n x_0, T^{n+1} x_0), (T^{n+1} x_0, T^{n+2} x_0), \cdots, (T^{m-1} x_0, T^m x_0) \in E(G)$ and $G[\mathscr{O}_T(x_0)]$ is transitive , then for any $m, n \in \mathbb{N}$ with $n < m$, we have $(T^n x_0, T^m x_0) \in E(G)$. □

Lemma 3.2. *Suppose that $(x_0, Tx_0) \in E(G)$ for some $x_0 \in V(G)$.*
If T is a G_K-Kannan contraction , then

(1) For every $n \in \mathbb{N}^*$, we have

$$\mathscr{D}(T^n x_0, T^{n-1} x_0) \leqslant \delta_0 \beta^{n-1}$$

where $\beta = \dfrac{K}{1-K}$ and $\delta_0 = \delta(\mathscr{D}, T, x_0)$.

(2) For every $(n, m) \in \mathbb{N}^* \times \mathbb{N}^*$, we have

$$\mathscr{D}(T^n x_0, T^m x_0) \leqslant \frac{1}{2}\delta_0\left(\beta^{n-1} + \beta^{m-1}\right)$$

Proof. 1. Let $x_0 \in V(G)$ be arbitrary
Since $(x_0, Tx_0) \in E(G)$ and T is a $G-$monotone-preserving then $(T^{n-1} x_0, T^n x_0) \in E(G)$.
We have

$$\mathscr{D}(T^n x_0, T^{n-1} x_0) \leqslant K[\mathscr{D}(T^n x_0, T^{n-1} x_0) + \mathscr{D}(T^{n-1} x_0, T^{n-2} x_0)]$$

Hence

$$(1-K)\mathscr{D}(T^n x_0, T^{n-1} x_0) \leqslant K(\mathscr{D}(T^{n-1} x_0, T^{n-2} x_0))$$

implies that

$$\begin{aligned}\mathscr{D}(T^n x_0, T^{n-1}x_0) \leqslant &\frac{K}{K-1}(\mathscr{D}(T^{n-1}x_0, T^{n-2}x_0))\\ \leqslant &(\frac{K}{K-1})^2(\mathscr{D}(T^{n-2}x_0, T^{n-3}x_0))\\ &\vdots\\ \leqslant &(\frac{K}{K-1})^{n-1}\mathscr{D}(Tx_0, x_0)\end{aligned}$$

Let $\beta = \dfrac{K}{K-1}$
Since $K < \dfrac{1}{2}$,then $\beta < 1$, Then

$$\mathscr{D}(T^n x_0, T^{n-1}x_0) \leqslant \beta^{n-1}\mathscr{D}(Tx_0, x_0)$$

Finally

$$\mathscr{D}(T^n x_0, T^{n-1}x_0) \leqslant \delta_0\beta^{n-1}$$

2. Since $G[\mathscr{O}_T(x_0)]$ is transitive , then $(T^n x_0, T^m x_0) \in E(G)$ for any $n, m \in \mathbb{N}^*$.
Let $x_0 \in V(G)$, We have

$$\begin{aligned}\mathscr{D}(T^n x_0, T^m x_0) \leqslant &K[\mathscr{D}(T^n x_0, T^{n-1}x_0) + \mathscr{D}(T^m x_0, T^{m-1}x_0)]\\ \leqslant &K(\delta_0\beta^{n-1} + \delta_0\beta^{m-1})\\ \leqslant &\frac{1}{2}\delta_0(\beta^{n-1} + \beta^{m-1}).\end{aligned}$$

□

Remark 3.1. *Suppose that $(x_0, Tx_0) \in E(G)$ for some $x_0 \in V(G)$, the complete subgraph $G[\mathscr{O}_T(x_0)]$ is transitive and T is a G_K-Kannan mapping .*
If $(\omega, T\omega) \in E(G)$ such that $\mathscr{D}(\omega, T\omega) < \infty$, then $\mathscr{D}(T^n x_0, T\omega) < \infty$

Indeed, we have $(\omega, T\omega) \in E(G)$ and from Lemma 3.1 $(T^n x_0, T^m x_0) \in E(G)$ thus $(T^n x_0, T\omega) \in E(G)$ (because $G[\mathscr{O}_T(x_0)]$ is transitive).
We have also T is a G_K-Kannan mapping, then

$$\mathscr{D}(T^n x_0, T\omega) \leqslant K\big(\mathscr{D}(T^n x_0, T^{n-1}x_0) + \mathscr{D}(T\omega, \omega)\big)$$

$$\leqslant \frac{1}{2}\big(\delta_0\beta^{n-1} + \mathscr{D}(T\omega,\omega)\big)$$

Hence

$$\mathscr{D}(T^n x_0, T\omega) < \infty$$

Theorem 3.1. *Let $(V(G),\mathscr{D})$ be a $G-$complete generalized suprametric space with a reflexive digraph G and $T : V(G) \to V(G)$ be a G_K-Kannan contraction with $K < \inf\left\{\frac{1}{2}, \frac{1}{\|\theta\|}\right\}$. Suppose that there exists $x_0 \in V(G)$ such that $\delta(\mathscr{D},T,x_0) < \infty$, $(x_0,Tx_0) \in E(G)$ and $G[\mathscr{O}_T(x_0)]$ is transitive. Then the sequence $(T^n x_0)_{n\in\mathbb{N}}$ $\mathscr{D}-$converges to some point ω.*
furthermore, if G satisfies property $(\mathscr{P})$ and $\mathscr{D}(\omega,T\omega) < \infty$, then ω is the unique fixed point of T.

Proof. Select $(m,n) \in (\mathbb{N}^*)^2$ such that $n \leqslant m$.
Since $(x_0,Tx_0) \in E(G)$, by Lemma 3.1, we have $(T^n x_0, T^m x_0) \in E(G)$.

By Lemma 3.2, we have:

$$\mathscr{D}(T^n x_0, T^m x_0) \leqslant K\delta_0(\beta^{n-1} + \beta^{m-1}),$$

where $\beta = \dfrac{K}{K-1}$ and $\delta_0 = \delta(\mathscr{D},T,x_0)$.
Thus, $(T^n x_0)_{n\in\mathbb{N}}$ is a $\mathscr{D}$-Cauchy sequence.
Since $(V(G),\mathscr{D})$ is $G-$Complete, the sequence $(T^n x_0)_{n\in\mathbb{N}}$ $\mathscr{D}$-converges to some $\omega \in \mathscr{X}$.

Next, assume that G satisfies the property $(\mathscr{P})$ and that $\mathscr{D}(\omega,T\omega) < \infty$.
Since $(T^n x_0)_{n\in\mathbb{N}}$ is $G-$monotone-preserving and $\mathscr{D}-$converges to $\omega \in V(G)$, we have $(T^n x_0, \omega) \in E(G)$ for any $n \in \mathbb{N}$.
Let $n \in \mathbb{N}^*$
Since T is a G_K-Kannan contraction, by Remark 3.1, we obtain:

$$\mathscr{D}(T^n x_0, T\omega) \leqslant K\big(\delta_0\beta^{n-1} + \mathscr{D}(T\omega,\omega)\big) \tag{3.1}$$

Since $(T^p x_0)_{p\in\mathbb{N}}$ $\mathscr{D}-$converges to ω and $(T\omega,\omega) \in E(G)$, we have

$$\mathscr{D}(T\omega,\omega) \leqslant \theta(T\omega,\omega) \limsup_{p\to\infty} \mathscr{D}(T^p x_0, T\omega)$$

$$\leqslant \|\theta\| \limsup_{p\to\infty} \mathscr{D}(T^p x_0, T\omega)$$

From the previous inequality, we conclude:

$$\mathscr{D}(T^n x_0, T\omega) \leqslant K\big(\delta_0 \beta^{n-1} + \mathscr{D}(T\omega, \omega)\big)$$

Hence

$$\mathscr{D}(T^n x_0, T\omega) \leqslant K\big(\delta_0 \beta^{n-1} + \|\theta\| \limsup_{p\to\infty} \mathscr{D}(T^p x_0, T\omega)\big)$$

Now, taking the limit as $n \to \infty$, we obtain:

$$\begin{aligned}\limsup_{n\to\infty} \big(\mathscr{D}(T^n x_0, T\omega)\big) &\leqslant \limsup_{n\to\infty} \big[K\big(\delta_0 \beta^{n-1} + \|\theta\| \limsup_{p\to\infty} \mathscr{D}(T^p x_0, T\omega)\big)\big] \\ &\leqslant K\big(\delta_0 \limsup_{n\to\infty} \beta^{n-1} + \|\theta\| \limsup_{n\to\infty} \mathscr{D}(T^n x_0, T\omega)\big)\end{aligned}$$

Thus

$$(1 - K\|\theta\|) \limsup_{n\to\infty} \mathscr{D}(T^n x_0, T\omega) \leqslant K\delta_0 \limsup_{n\to\infty} \beta^{n-1}$$

Since $K < \inf\left\{\frac{1}{2}, \frac{1}{\|\theta\|}\right\}$ and $\beta = \frac{K}{K-1} < 1$, we conclude:

$$\limsup_{n\to\infty} \mathscr{D}(T^n x_0, T\omega) = 0$$

Thus $(T^n x_0)_{n\in\mathbb{N}}$ $\mathscr{D}$-converges to $T\omega$. By the uniqueness of the limit, we have $T\omega = \omega$.

Now, we show the uniqueness of the fixed point ω of T.

Assume that $(T^n x_0)_{n\in\mathbb{N}}$ $\mathscr{D}-$converges to two points ω_1 and ω_2, such that $T\omega_1 = \omega_1$ and $T\omega_2 = \omega_2$.

Since $(T^n x_0, \omega_1), (T^n x_0, \omega_2) \in E(G)$, and $(T^n x_0, T\omega_1), (T^n x_0, T\omega_2) \in E(G)$ we have $(\omega_1, \omega_2), (T\omega_1, T\omega_2) \in E(G)$. Thus, we get:

$$\mathscr{D}(T\omega_1, T\omega_2) \leqslant K\big(\mathscr{D}(T\omega_1, \omega_1) + \mathscr{D}(T\omega_2, \omega_2)\big)$$

By the condition (D_4), we have:

$$\begin{aligned}\mathscr{D}(T\omega_1, \omega_1) &\leqslant \theta(T\omega_1, \omega_1) \limsup_{n\to+\infty} \mathscr{D}(T^n x_0, \omega_1) \\ &\leqslant \|\theta\| \limsup_{n\to+\infty} \mathscr{D}(T^n x_0, \omega_1)\end{aligned}$$

Thus, we obtain:

$$\mathscr{D}(T\omega_1, \omega_1) = 0$$

Similarly, we find that

$$\mathscr{D}(T\omega_2, \omega_2) = 0.$$

Which implies

$$\omega_1 = \omega_2$$

Hence, ω is the unique fixed point of T. □

Let $\mathscr{S} = \{f :]0,\infty[\to [0, \frac{1}{2}[\quad ; \lim_{n\to\infty} f(t_n) = \frac{1}{2} \implies \lim_{n\to\infty} t_n = 0\}$

Definition 3.1. *Let G be a digraph .*

A mapping $T : V(G) \to V(G)$ is said to be a G_f-Kannan contraction if the following conditions are satisfied:

(i) T is $G-$monotone-preserving.

(ii) For every $(x,y) \in E(G)$

$$\mathscr{D}(Tx,Ty) \leqslant f(\mathscr{D}(x,y))(\mathscr{D}(Tx,x) + \mathscr{D}(Ty,y)), \tag{3.2}$$

with $f \in \mathscr{S}$.

Theorem 3.2. *Let $(X = V(G), \mathscr{D})$ be a G-Complete generalized suprametric space with a reflexive digraph G and let $\|\theta\| < 1$.*

Let $T : X \to X$ be a G_f-Kannan contraction.

Suppose there exists $x_0 \in V(G)$ such that $\delta(\mathscr{D}, T, x_0) < \infty$, $(x_0, Tx_0) \in E(G)$ and $G[\mathscr{O}_T(x_0)]$ is transitive.

Furthermore, assume that G satisfies property$(\mathscr{P})$ and $\mathscr{D}(\omega, T\omega) < \infty$.

Then T has a unique fixed point $\omega \in X$, and for any $x_0 \in X$, the sequence of iterates $(T^n x_0)_{n\in\mathbb{N}}$ converges to ω.

Proof. Fix $x_0 \in X$ and define a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ by $x_{n+1} = T^n x_0$ for all $n \in \mathbb{N}$.

The first step is to show that $(x_n)_{n\in\mathbb{N}}$ is a $\mathscr{D}-$Cauchy in X

Suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$

Set $D_n = \mathscr{D}(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

By hypothesis, we have:

$$\begin{aligned} D_{n+1} &= \mathscr{D}(x_{n+1}, x_{n+2}) = \mathscr{D}(T^n x_0, T^{n+1} x_0) \\ &\leqslant f(\mathscr{D}(x_n, x_{n+1}))\left[\mathscr{D}(x_n, x_{n+1}) + \mathscr{D}(x_{n+1}, x_{n+2})\right] \\ &< \frac{1}{2}(D_n + D_{n+1}) \quad for\, all \;\; n \in \mathbb{N}. \end{aligned}$$

Implies that $D_{n+1} < D_n$ for all $n \in \mathbb{N}$ Hence $(D_n)_{n\in\mathbb{N}}$ is decreasing and bounded below, so

$$(\exists \alpha \geqslant 0) : \lim_{n\to\infty} D_n = \alpha$$

Assume that $\alpha > 0$. Then by hypothesis, we have

$$\mathscr{D}(x_{n+1}, x_{n+2}) \leqslant f(\mathscr{D}(x_n, x_{n+1}))\left[\mathscr{D}(x_n, x_{n+1}) + \mathscr{D}(x_{n+1}, x_{n+2})\right] \quad for\, all \;\; n \in \mathbb{N}.$$

This gives the inequality:

$$\frac{D_{n+1}}{D_n + D_{n+1}} \leqslant f(D_n) \quad for\, all \;\; n \geqslant 0$$

This gives the inequality: $n \to \infty$, we obtain $\lim_{n\to\infty} f(D_n) \geqslant \dfrac{1}{2}$

Which is impossible because $f \in \mathscr{S}$.

Therefore

$$\lim_{n\to\infty} D_n = 0$$

On the other hand, for $m \neq n$, we have:

$$\begin{aligned} \mathscr{D}(x_{n+1}, x_{m+1}) &\leqslant f(\mathscr{D}(x_n, x_m)\left[\mathscr{D}(x_n, x_{n+1}) + \mathscr{D}(x_m, x_{m+1})\right] \\ &< \frac{1}{2}(D_n + D_m) \to 0 \quad as\; n, m \to \infty \end{aligned}$$

Then $(x_n)_{n\in\mathbb{N}}$ is a $\mathscr{D}-$Cauchy in X.

By the completeness of X, there is $\omega \in X$ such that $\lim_{n\to\infty} T^n x_0 = \omega$.

Since $\mathscr{D}(\omega, T\omega) \leqslant \infty$, we also have $\mathscr{D}(T^n x_0, T\omega) < \infty$ (by Remark 3.1)

We are given that $G[\mathscr{O}_T(x_0)]$ is transitive, so we can write:

$$\mathscr{D}(T\omega, \omega) \leqslant \theta(T\omega, \omega) \limsup_{n\to\infty} \mathscr{D}(T\omega, T^n x_0)$$

$$\leqslant \|\theta\| \limsup_{n\to\infty} \mathscr{D}(T\omega, T^n x_0).$$

Hence, we have

$$\mathscr{D}(T\omega,\omega) \leqslant \|\theta\| \mathscr{D}(T\omega,\omega)$$

Therefore

$$\mathscr{D}(T\omega,\omega)(1-\|\theta\|) \leqslant 0$$

Since $\|\theta\| < 1$, it follows that: $\mathscr{D}(T\omega,\omega) = 0$.

Thus, $T\omega = \omega$.

Now, suppose γ is another fixed point of T.

By hypothesis, we have

$$\mathscr{D}(\omega,\gamma) = \mathscr{D}(T\omega,T\gamma) \leqslant f(\mathscr{D}(\omega,\gamma))\{\mathscr{D}(\omega,T\gamma)+\mathscr{D}(T\omega,\gamma)\} = 0$$

Hence, $\omega = \beta$.

Finally, T has a unique fixed point $\omega \in X$. □

Example 3.1. *Let $X = [0,1]$, the mapping $\mathscr{D}$ be defined by: $\mathscr{D}(x,y) = (x-y)^2$ and a self mapping T on $\mathscr{X}$ defined by*

$$T(x) = \begin{cases} \frac{x}{4} & \text{if} \quad x \in \{0\} \bigcup \left\{ \frac{1}{4^n} : \quad n \in \mathbb{N} \right\} \\ 1 & \text{otherwise} \end{cases} \tag{3.3}$$

Let $f :]0,+\infty[\to [0,\frac{1}{2}[$ defined by :

$$f(x) = \frac{x^2-x+1}{2(x^2+1)}$$

We have $f \in \mathscr{S}$

Consider the graph G on $\mathscr{X}$ consisting of the transitive closure of the graph represented in FIGURE 1.

Note that

$$E(G) = \Delta \bigcup \left\{ \left(0, \frac{1}{4^n}\right) : n \in \mathbb{N} \right\} \bigcup \left\{ \left(\frac{1}{4^n}, \frac{1}{4^m}\right) : n,m \in \mathbb{N} \quad and \quad n \geqslant m \right\}$$

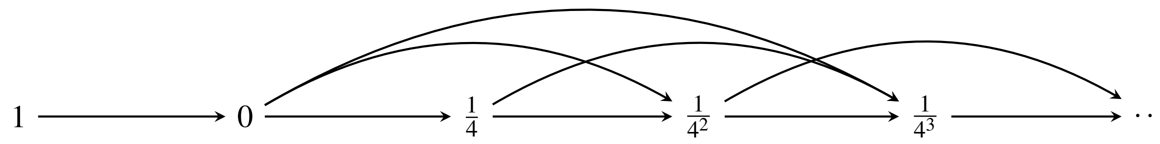

FIGURE 1. Transitive closure of the graph$G$.

We have that

$$\begin{cases} \left(T(0), T\left(\frac{1}{4^n}\right)\right) = \left(0, \frac{1}{4^{n+1}}\right) \in E(G) \quad \textit{for any } n \in \mathbb{N}, \\ \left(T\left(\frac{1}{4^n}\right), T\left(\frac{1}{4^m}\right)\right) = \left(\frac{1}{4^{n+1}}, \frac{1}{4^{m+1}}\right) \in E(G) \quad \textit{for any } n, m \in \mathbb{N} \textit{ such that } n \geqslant m, \end{cases}$$

Then T is $G-$monotone-preserving. For $x_0 = 1$, we have $(Tx_0, x_0) \in E(G)$, $G[\mathscr{O}_T(x_0)]$ is transitive and

$$\delta(\mathscr{D}, T, x_0) = \sup\left\{\mathscr{D}\left(\frac{1}{4^i}, 1\right) : \ i \in \mathbb{N}\right\} = 1 < \infty$$

Let $x, y \in \mathscr{X}$ such that $(x, y) \in E(G)$.
If $x = y \in \mathscr{X}$, then $\mathscr{D}(Tx, Tx) = 0 \leqslant f(\mathscr{D}(x, x))(\mathscr{D}(Tx, x) + \mathscr{D}(y, Ty)) = 0$.
If $(x, y) = \left(0, \frac{1}{4^n}\right)$, then

$$\begin{aligned} \mathscr{D}\left(T(0), T\left(\frac{1}{4^n}\right)\right) &= \frac{1}{4^{2(n+1)}} \leqslant \frac{9}{4^{2(n+1)}}\left(\frac{1 - 2^{2n} + 2^{4n}}{2(1 + 2^{4n}}\right) \\ &= f\left(\mathscr{D}\left(0, \frac{1}{4^n}\right)\right)\left[\mathscr{D}\left(T(0), 0\right) + \mathscr{D}\left(T\left(\frac{1}{4^n}\right), \frac{1}{4^n}\right)\right] \end{aligned}$$

If $(x, y) = \left(\frac{1}{4^n}, \frac{1}{4^m}\right)$, then

$$\begin{aligned} \mathscr{D}\left(T\left(\frac{1}{4^n}\right), T\left(\frac{1}{4^m}\right)\right) &= \frac{(4^n - 4^m)^2}{4^{2(n+m+1)}} \leqslant \frac{(4^n - 4^m)^2 + 4^{4(n+m)} + 4^{2m} - 4^{2n}}{2((4^n - 4^m)^2 + 4^{4(n+m)})} \times \frac{9(4^{n+1} + 4^{m+1})}{4^{2(n+1)(m+1)}} \\ &= f\left(\mathscr{D}\left(\frac{1}{4^n}, \frac{1}{4^m}\right)\right)\left[\mathscr{D}\left(T\left(\frac{1}{4^n}\right), \frac{1}{4^n}\right) + \mathscr{D}\left(T\left(\frac{1}{4^m}\right), \frac{1}{4^m}\right)\right] \end{aligned}$$

In all cases,

$$\mathscr{D}(Tx, Ty) \leqslant f(\mathscr{D}(x, y))\{\mathscr{D}(Tx, x) + \mathscr{D}(Ty, y)\},$$

implies that T is a G_f-Kannan contraction.
The sequence $(T^n x_0)_{n \in \mathbb{N}} = \left(\frac{1}{4^n}\right)_{n \in \mathbb{N}}$ is $\mathscr{D}-$ convergent to 0 and $\left(0, \frac{1}{4^n}\right) \in E(G)$, then G has the $(\mathscr{P})$ property from the previous theorem, T has a unique fixed point which is 0.

In the next theorem, we will consider the class of functions

$$\mathscr{F}_q = \{\psi :]0,\infty[\to [0,q[: \ \psi(t_n) \to q \implies t_n \to 0 \ \ as \ \ n \to \infty\}$$

where $q \in]0, \frac{1}{2}[$.

Theorem 3.3. *Let $(X = V(G), \mathscr{D})$ be a G-complete generalized suprametric space with a reflexive digraph G, and let $T : X \to X$ be a G_f−-Kannan mapping.*
Suppose that there exists $x_0 \in V(G)$ such that $\delta(\mathscr{D}, T, x_0) < \infty$, $(x_0, Tx_0) \in E(G)$, and $G[\mathscr{O}_T(x_0)]$ is transitive .
furthermore if $\psi \in \mathscr{F}_q$ satisfies

$$\frac{1}{\|\theta\|} D(Tx, x) \leqslant D(x, y)$$

Which implies

$$\mathscr{D}(Tx, Ty) \leqslant \psi(\mathscr{D}(x, y))(\mathscr{D}(Tx, x) + \mathscr{D}(Ty, y)), \quad for\ all\ \ x, y \in \mathscr{X}\ \ with\ x \neq y.$$

Then T has a unique fixed point $\omega \in X$ and for any $x \in X$ the sequence of iterates $(T^n x)_{n\in\mathbb{N}}$ converges to ω.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by $x_{n+1} = Tx_n$, for all $n \geqslant 0$.
Assume that there exists n such that $x_{n+1} = x_n$, then x_n is the fixed point of T.
Therefore, suppose that $x_{n+1} \neq x_n$ for all $n \geqslant 0$.
Set $D_n = D(x_n, x_{n+1})$ for all $n \geqslant 0$.
Since

$$\frac{1}{\|\theta\|} D(Tx_n, x_n) = \frac{1}{\|\theta\|} D(x_{n+1}, x_n) \leqslant D(x_{n+1}, x_n) = D(x_n, x_{n+1})$$

. By hypothesis, we have

$$\begin{aligned}
D_{n+1} &= D(x_{n+1}, x_{n+2}) \\
&= D(Tx_n, Tx_{n+1}) \\
&\leqslant \psi(\mathscr{D}(x_n, x_{n+1}))(\mathscr{D}(Tx_n, x_n) + \mathscr{D}(Tx_{n+1}, x_{n+1})) \\
&< q(\mathscr{D}(Tx_n, x_n) + \mathscr{D}(Tx_{n+1}, x_{n+1})) \\
&= q(D_n + D_{n+1})
\end{aligned}$$

So

$$D_{n+1} < \frac{q}{1-q} D_n = hD_n, \quad \text{where } \; h = \frac{q}{1-q} \in]0,1[$$

Thus

$$(D_n)_n \textit{ is decreasing} \quad \textit{and} \;\; D_n < h^n D_0, \quad \textit{for all } n \geqslant 1$$

Then

$$\lim_{n \to +\infty} D_n = 0$$

Thus

$$\forall \varepsilon > 0, (\exists N \in \mathbb{N}), (\forall n > N) : \mathscr{D}_n < \frac{\varepsilon}{2}$$

Let $m, n > N$.

We have

$$\begin{aligned}
\mathscr{D}(T^n x_0, T^m x_0) &\leqslant \psi(\mathscr{D}(x_n, x_m))(\mathscr{D}(x_{n+1}, x_n) + \mathscr{D}(x_{m+1}, x_m)) \\
&\leqslant \psi(\mathscr{D}(x_n, x_m))(\mathscr{D}_n + \mathscr{D}_m) \\
&\leqslant \psi(\mathscr{D}(x_n, x_m))(\varepsilon) \\
&\leqslant \varepsilon.
\end{aligned}$$

Then $(x_n)_n$ is a cauchy sequence in X.

Since X is complete, there exists $\omega \in X$ such that $\lim_{n \to +\infty} x_n = \omega \in X$.

According by (D_4) of generalized suprametric space, We have

$$\begin{aligned}
D(\omega, Tx_n) &\leqslant \|\theta\| \limsup_{n \to \infty} D(x_n, Tx_n) \\
&= \|\theta\| \limsup_{n \to \infty} D_n.
\end{aligned}$$

Then

$$D(\omega, Tx_n) = 0,$$

implies that $\lim_{n \to \infty} Tx_n = \omega$.

We know that

$$\frac{1}{\|\theta\|} D(x_n, Tx_n) \leqslant D(\omega, x_n)$$

to say that

$$D(x_{n+1}, T\omega) \leqslant \psi(D(x_n, \omega)\{D(x_n, Tx_n) + D(\omega, T\omega)\}$$

We can choose $(p(n))_n \subset (n)_n$ monotone strictly increasing sequence of natural numbers. Therefore, $(x_{p(n)})_n$ is a subsequence of $\{x_n\}$ and

$$\begin{aligned} D(x_{p(n)+1}, T\omega) &\leqslant \psi(D(x_{p(n)}, \omega)(D(x_{p(n)}, Tx_{p(n)}) + D(\omega, T\omega)) \\ &\leqslant q(D(x_{p(n)}, Tx_{p(n)}) + D(\omega, T\omega)) \end{aligned}$$

Letting $n \to \infty$, we find that

$$D(\omega, T\omega) \leqslant qD(\omega, T\omega)$$

This implies

$$(1-q)D(\omega, T\omega) \leqslant 0$$

Since $q < 1$, we deduce $D(\omega, T\omega) = 0$

Thus

$$T\omega = \omega.$$

Suppose that $T\alpha = \alpha$, we have

$$0 = \frac{1}{\|\theta\|} D(\omega, T\omega) \leqslant D(\omega, \alpha)$$

By hypothesis, we have

$$D(\omega, \alpha) = D(T\omega, T\alpha) \leqslant \psi(D(\omega, \alpha))(D(\omega, T\omega) + D(\alpha, T\alpha)) = 0.$$

Then $\alpha = \omega$.

Finally, T has a unique fixed point $\omega \in X$. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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