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# THEORETICAL APPROACHES FOR GENERALIZED PROXIMAL $\mathscr{Z}$-CONTRACTION MAPS IN $b$-METRIC SPACES WITH APPLICATIONS

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**Abstract.** This paper explores the simulation function and the concept of $\mathscr{Z}$-contraction concerning $\zeta$, which serve as a generalization of the Banach contraction principle. These ideas unify various existing contraction types by incorporating both $d(Ta,Tb)$ and $d(a,b)$. Our findings extend and generalize the work of Olgun et al. [27], Abbas et al. [1], and Goswami et al. [21], transitioning from metric spaces to the framework of $b$-metric spaces. We propose the concept of a generalized proximal $\mathscr{Z}$-contraction for pairs of non-self mappings and demonstrate the existence and uniqueness of common best proximity points in complete $b$-metric spaces. Additionally, we present supporting examples and illustrate some applications of our findings.

**Keywords:** common best proximity points; $b$-metric space; integral equation; functional equation.

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## 1. Introduction

The field of nonlinear functional analysis is greatly impacted by fixed point theory, which provides essential tools for solving various mathematical problems. The notion of a $b$-metric space, was first proposed by Czerwik [19] as an extension of metric space. The existence of fixed points for single-valued and multi-valued mappings in $b$-metric spaces under different contraction conditions has been studied by a number of academics since its invention. However, solving the equation $Ta = a$ is not always feasible, particularly in the case of non-self-mappings. In such scenarios, an alternative approach is to determine a point $a$ that minimizes the distance $d(a, Ta)$, leading to the emergence of the best proximity point concept. Over the past few decades, the study of best proximity points has gained significant attention in mathematical research [3, 14, 17, 18, 20, 24, 26, 31, 32, 33, 36, 37].

## 2. Preliminaries

**Definition 2.1** ([19]). *Let $X$ be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied: for any $x, y, z \in X$*

$(i)$ $0 \leq d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$,

$(ii)$ $d(x,y) = d(y,x)$,

$(iii)$ $d(x,z) \leq s[d(x,y) + d(y,z)]$.

*In this case, the pair $(X,d)$ is called a b-metric space with coefficient s.*

Every metric space is a $b$-metric space with $s = 1$. In general, every $b$-metric space is not a metric space. In this paper, we denote $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{N}$ is the set of all natural numbers.

The following lemmas are useful in proving our main results.

**Lemma 2.1** ([29]). *Suppose $(X,d)$ is a b-metric space with coefficient $s \geq 1$ and $\{a_n\}$ be a sequence in $X$ such that $d(a_n, a_{n+1}) \to 0$ as $n \to \infty$. If $\{a_n\}$ is a not Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(a_{m_k}, a_{n_k}) \geq \varepsilon$. For each $k > 0$, corresponding to $m_k$, we can choose $n_k$ to be the smallest positive integer such that $d(a_{m_k}, a_{n_k}) \geq \varepsilon, d(a_{m_k}, a_{n_k - 1}) < \varepsilon$ and*

(i) $\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} d(a_{m_k-1}, a_{n_k-1}) \leq \limsup_{k\to\infty} d(a_{m_k-}, a_{n_k-1}) \leq s\varepsilon$,

(ii) $\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(a_{m_k+1}, a_{n_k}) \leq \limsup_{k\to\infty} d(a_{m_k+1}, a_{n_k}) \leq s^2\varepsilon$,

(iii) $\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(a_{m_k}, a_{n_k+1}) \leq \limsup_{k\to\infty} d(a_{m_k}, a_{n_k+1}) \leq s^2\varepsilon$,

(iv) $\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} d(a_{m_k+1}, a_{n_k+1}) \leq \limsup_{k\to\infty} d(a_{m_k+1}, a_{n_k+1}) \leq s^3\varepsilon$.

**Lemma 2.2** ([2]). *Let $(X,d)$ be a b-metric space with coefficient $s \geq 1$.*

*Suppose that $\{a_n\}$ and $\{b_n\}$ are b-convergent to $x$ and $y$ respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(a_n, b_n) \leq \limsup_{n\to\infty} d(a_n, b_n) \leq s^2 d(x,y).$$

*In particular, if $x = y$, then we have $\lim_{n\to\infty} d(a_n, b_n) = 0$. Moreover for each $z \in X$ we have*

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(a_n, z) \leq \limsup_{n\to\infty} d(a_n, z) \leq s d(x,z).$$

In 2015, Khojasteh et al. [22] introduced simulation function and defined $\mathscr{Z}$-contraction with respect to a simulation function as follows.

**Definition 2.2** ([22]). *A simulation function is a mapping $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ satisfying the following conditions:*

($\zeta_1$) $\zeta(0,0) = 0$;

($\zeta_2$) $\zeta(t,s) < s - t$ for all $s,t > 0$;

($\zeta_3$) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

**Remark 2.1** ([12]). *Let $\zeta$ be a simulation function. If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(kt_n, s_n) < 0$ for any $k > 1$.*

The following are examples of simulation functions.

**Exercise 2.1** ([12]). *Let $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ be defined by*

(i) $\zeta(t,s) = \lambda s - t$ for all $t,s \in \mathbb{R}^+$, where $\lambda \in [0,1)$;

(ii) $\zeta(t,s) = \frac{s}{1+s} - t$ for all $s,t \in \mathbb{R}^+$;

(iii) $\zeta(t,s) = s - kt$ for all $t,s \in \mathbb{R}^+$, where $k > 1$;

(iv) $\zeta(t,s) = \frac{1}{1+s} - (1+t)$ for all $s,t \in \mathbb{R}^+$;

$(v)$ $\zeta(t,s) = \frac{1}{k+s} - t$ for all $s,t \in \mathbb{R}^+$ where $k > 1$.

The set of all simulation functions is denoted by $\mathscr{Z}$.

Using the simulation function approach, the notion of $\mathscr{Z}$-contraction was introduced in [22] which is a generalization of Banach contraction. It also unified various existing types of contraction mappings. The advantage of this notion is in providing a unique point of view for several fixed point problems.

**Definition 2.3** ([22])**.** *Let $(X,d)$ be a metric space. A self mapping $f$ on $X$ is called a $\mathscr{Z}$-contraction if for some simulation function $\zeta \in \mathscr{Z}, T$ satisfies $\zeta(d(fx,fy),d(x,y)) \geq 0$, for all $x,y \in X$.*

It should be observed that all $\mathscr{Z}$-contraction mappings are continuous and contractive. Olgun et al. [27] relaxed this continuity, defining a generalized $\mathscr{Z}$-contraction mapping which is not necessarily continuous.

**Definition 2.4** ([27])**.** *Let $(X,d)$ be a metric space. A self mapping $f$ on $X$ is called a generalized $\mathscr{Z}$-contraction if for some simulation function $\zeta \in \mathscr{Z}$, and for all $x,y \in X, T$ satisfies*

$$\zeta(d(fx,fy),M(x,y)) \geq 0,$$

*where*

$$M(x,y) = \max\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\}.$$

A novel kind of mapping was defined by Kumam et al. [23] by combining $\mathscr{Z}$-contraction and Suzuki type contraction, as explained below.

**Definition 2.5** ([23])**.** *Let $(X,d)$ be a metric space. A self mapping $f$ on $X$ is called a Suzuki type $\mathscr{Z}$-contraction if for some simulation function $\zeta \in \mathscr{Z}, T$ satisfies*

$$\frac{1}{2}d(x,fx) < d(x,y) \quad \Rightarrow \quad \zeta(d(fx,fy),d(x,y)) \geq 0, \quad \textit{for all} \quad x,y \in X.$$

In 2018, Padcharoen et al. [28] proved the following theorem in complete metric spaces.

**Theorem 2.1** ([28])**.** *Let $(X,d)$ be a complete metric space and $T : X \to X$ be a self-map on $X$. If there exists a simulation function $\zeta$ such that*

$$\frac{1}{2}d(x, fx) < d(x,y) \Rightarrow \zeta(d(Tx,Ty), M(x,y)) \geq 0,$$

*for all $x,y \in X$, where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$. Then $T$ has a unique fixed point in $X$.*

The following theorems are due to Babu et al. [7, 9] in complete $b$-metric spaces.

**Theorem 2.2** ([7])**.** *Let $(X,d)$ be a complete $b$-metric space and $T : X \to X$ be a self-map on $X$. If there exists a simulation function $\zeta$ such that*

$$\frac{1}{2s}d(x, fx) < d(x,y) \Rightarrow \zeta(s^4 d(Tx,Ty), M(x,y)) \geq 0,$$

*for all $x,y \in X$, where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$. Then $T$ has a unique fixed point in $X$.*

**Theorem 2.3** ([9])**.** *Let $(X,d)$ be a complete $b$-metric space and $T : X \to X$ be a self-map on $X$. If there exists a simulation function $\zeta$ such that*

$$\frac{1}{2s}\min\{d(x, fx), d(y, gy)\} \leq d(x,y) \Rightarrow \zeta(s^4 d(fx, gy), M(x,y)) \geq 0,$$

*for all $x,y \in X$, where $M(x,y) = \max\{d(x,y), d(x, fx), d(y, gy), \frac{d(x,gy)+d(y,fx)}{2s}\}$. Then $T$ has a unique fixed point in $X$.*

For more Suzuki-type contractions see [5, 8, 35].

Let $A$ and $B$ be two non-empty subsets of a metric space $(X,d)$. Define

$$d(A,B) = \inf\{d(a,b) : a \in A \text{ and } b \in B\}.$$

$$A_0 = \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\},$$

$$B_0 = \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\}.$$

**Definition 2.6** ([15])**.** *An element $a$ in $A$ is said to be a common best proximity point of the non-self mappings $S : A \to B$ and $T : A \to B$ if it satisfies the condition that $d(a,Sa) = d(a,Ta) = d(A,B)$.*

**Definition 2.7** ([25]). *The mappings $S : A \to B$ and $T : A \to B$ are said to be commute proximally if they satisfy the condition that $d(u,Sa) = d(v,Ta) = d(A,B) \Rightarrow Sv = Tu$.*

**Definition 2.8** ([30]). *If $A_0 \neq \emptyset$ then the pair $(A,B)$ is said to have P-property if and only if for any $a_1,a_2 \in A_0$ and $b_1,b_2 \in B_0$,*

$$d(a_1,b_1) = d(A,B), d(a_2,b_2) = d(A,B) \Rightarrow d(a_1,a_2) = d(b_1,b_2).$$

In [1], Abbas et al. introduced the concept of proximal simulative contraction of first kind and second kind.

**Definition 2.9** ([1]). *For two non-empty subsets A and B of a metric space $(X,d)$, a mapping $T : A \to B$ is said to be proximal simulative contraction of first kind if there exists a simulation function $\zeta$ such that*

$$d(a_1,Tb_1) = d(A,B) = d(a_2,Tb_2) \text{ implies}$$

$$\zeta(d(a_1,a_2),d(b_1,b_2)) \geq 0, \textit{ for all } a_1,a_2,b_1,b_2 \in A.$$

**Definition 2.10** ([1]). *For two non-empty subsets A and B of a metric space $(X,d)$, a mapping $T : A \to B$ is said to be proximal simulative contraction of second kind if there exists a simulation function $\zeta$ such that*

$$d(a_1,Tb_1) = d(A,B) = d(a_2,Tb_2) \text{ implies}$$

$$\zeta(d(Ta_1,Ta_2),d(Tb_1,Tb_2)) \geq 0, \textit{ for all } a_1,a_2,b_1,b_2 \in A.$$

Recenrly, Goswami et al. [21] proved the following theorem in complete metric spaces.

**Theorem 2.4** ([21]). *Let A and B be two non-empty subsets of a complete metric space $(X,d)$. Suppose $T : A \to B$ be a map with $T(A_0) \subseteq B_0$, where $A_0,B_0$ are non-empty and $A_0$ is closed. If there exists a simulation function $\zeta$ such that*

$$d(a_1,Tb_1) = d(A,B) = d(a_2,Tb_2) \text{ implies}$$

$$\zeta(d(a_1,a_2),M(b_1,b_2,a_1,a_2)) \geq 0, \textit{ for all } a_1,a_2,b_1,b_2 \in A,$$

*where $M(b_1,b_2,a_1,a_2) = \max\{d(b_1,b_2),d(b_1,a_1),d(b_2,a_2),\frac{d(b_1,a_2)+d(b_2,a_1)}{2}\}$.*
*Then $T$ has a unique best proximity point in $A_0$.*

Recently, Babu [4] proved the following theorems in complete $b$-metric spaces.

**Theorem 2.5** ([4])**.** *Let $A$ and $B$ be two non-empty subsets of a complete $b$-metric space $(X,d)$. Suppose $T : A \to B$ be a map with $T(A_0) \subseteq B_0$, where $A_0, B_0$ are non-empty and $A_0$ is closed. If there exists a simulation function $\zeta$ such that $d(a_1, Tb_1) = d(A,B) = d(a_2, Tb_2)$ then $\zeta(d(a_1,a_2), M(b_1,b_2,a_1,a_2)) \geq 0$ for all $a_1,a_2,b_1,b_2 \in A$, where*

$$M(b_1,b_2,a_1,a_2) = \max\left\{d(b_1,b_2), d(b_1,a_1), d(b_2,a_2), \frac{d(b_1,a_2)+d(b_2,a_1)}{2s}\right\}$$

*Then $T$ has a unique best proximity point in $A_0$.*

**Theorem 2.6** ([4])**.** *Let $A$ and $B$ be two non-empty subsets of a complete $b$-metric space $(X,d)$. Suppose $T : A \to B$ be a map with $T(A_0) \subseteq B_0$, where $A_0, B_0$ are non-empty and $B_0$ is closed subset of $B$. If there exists a simulation function $\zeta$ such that*

$$d(a_1, Tb_1) = d(A,B) = d(a_2, Tb_2)$$

*then*

$$\zeta(d(Ta_1,Ta_2), M(Tb_1,Tb_2,Ta_1,Ta_2)) \geq 0, \quad \forall a_1,a_2,b_1,b_2 \in A,$$

*where*

$$M(Tb_1,Tb_2,Ta_1,Ta_2) = \max\left\{d(Tb_1,Tb_2), d(Tb_1,Ta_1), d(Tb_2,Ta_2), \frac{d(Tb_1,Ta_2)+d(Tb_2,Ta_1)}{2s}\right\}.$$

*Then $T$ has a unique best proximity point in $A_0$.*

Motivated by all the above works, we introduce generalized proximal $\mathscr{Z}$-contraction of first kind and second kind for pair of nonself-maps, we extend Theorem 2.5 and Theorem 2.6 to pair of maps, and we extend Theorem 2.4 to $b$-metric spaces for the maps satisfying generalized proximal $\mathscr{Z}$-contraction of first kind.

## 3. Main Results

We introduce generalized proximal $\mathscr{Z}$-contraction for the pair $(S,T)$ as follows:

**Definition 3.1.** For two non-empty subsets $A$ and $B$ of a $b$-metric space $(X,d)$, the mappings $S,T : A \to B$ are said to be generalized proximal $\mathscr{Z}$-contraction of first kind if there exists a simulation function $\zeta$ such that

$$d(a_1,Sx_1) = d(a_2,Sx_2) = d(A,B) = d(b_1,Tx_1) = d(b_2,Tx_2) \text{ implies}$$

$$\zeta(s^4 d(a_1,a_2), M(b_1,b_2,a_1,a_2)) \geq 0, \text{ for all } b_1,b_2,a_1,a_2,x_1,x_2 \in A, \tag{3.1}$$

where

$$M(b_1,b_2,a_1,a_2) = \max\{d(b_1,b_2), d(b_1,a_1), d(b_2,a_2), \frac{d(b_2,a_1)+d(b_1,a_2)}{2s}\}.$$

**Definition 3.2.** For two non-empty subsets $A$ and $B$ of a $b$-metric space $(X,d)$, the mappings $S,T : A \to B$ are said to be generalized proximal $\mathscr{Z}$-contraction of second kind if there exists a simulation function $\zeta$ such that

$$d(a_1,Sx_1) = d(a_2,Sx_2) = d(A,B) = d(b_1,Tx_1) = d(b_2,Tx_2) \text{ implies}$$

$$\zeta(s^4 d(Sa_1,Sa_2), M(b_1,b_2,a_1,a_2)) \geq 0, \text{ for all } b_1,b_2,a_1,a_2,x_1,x_2 \in A, \tag{3.2}$$

where

$$M(b_1,b_2,a_1,a_2) = \max\{d(Tb_1,Tb_2), \frac{d(Tb_1,Sa_1)}{2s}, \frac{d(Tb_2,Sa_2)}{2s}, \frac{d(Tb_2,Sa_1)+d(Tb_1,Sa_2)}{1+s+2s^2}\}.$$

**Theorem 3.1.** *Let $A$ and $B$ be two non-empty subsets of a complete $b$-metric space $(X,d)$. Suppose $S,T : A \to B$ be generalized proximal $\mathscr{Z}$-contraction of first kind with $S(A_0) \subseteq B_0, S(A_0) \subseteq T(A_0)$, where $A_0, B_0$ are non-empty and $A_0$ is closed subset of $A$. Moreover, assume that $S$ and $T$ are commute proximally and $S$ and $T$ are $b$-continuous. Then $S$ and $T$ have a unique common best proximity point in $A_0$.*

*Proof.* Suppose $a_0 \in A_0$. Since, $S(A_0) \subseteq T(A_0)$, there exists an element $a_1 \in A_0$ such that $Sa_0 = Ta_1$. Continuing in this way, we can choose an element $a_{n+1} \in A_0$ satisfying

$$Sa_n = Ta_{n+1}, \text{ for each } n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Since, $S(A_0) \subseteq T(A_0)$, there exists $x_n \in A_0$ such that

$$d(x_n, Sa_n) = d(A,B), \text{ for each } n \geq 0. \tag{3.4}$$

From (3.3), (3.4), and from the choice of $a_n, x_n$, we conclude that

$$d(x_n, Sa_n) = d(x_{n+1}, Sa_{n+1}) = d(A,B) = d(x_n, Ta_{n+1}), \text{ for each } n \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

Suppose $x_{n_0} = x_{n_0+1}$ for some $n_0$, then from (3.5), we get

$$d(A,B) = d(x_{n_0+1}, Sa_{n_0+1}) = d(x_{n_0}, Sa_{n_0}) = d(x_{n_0}, Ta_{n_0+1}).$$

By the commute proximal of $S$ and $T$, we have

$$Sx_{n_0} = Tx_{n_0+1} = Tx_{n_0}.$$

Since, $S(A_0) \subseteq B_0$, there exists $y_{n_0} \in A_0$ such that

$$d(y_{n_0}, Sx_{n_0}) = d(A,B) = d(y_{n_0}, Tx_{n_0}). \tag{3.6}$$

As $S$ and $T$ are commute proximally, we have

$$Sy_{n_0} = Ty_{n_0}.$$

Since, $S(A_0) \subseteq B_0$, there exists $z_{n_0} \in A_0$ such that

$$d(z_{n_0}, Sy_{n_0}) = d(A,B) = d(z_{n_0}, Ty_{n_0}). \tag{3.7}$$

If $d(y_{n_0}, z_{n_0}) > 0$ then from (3.6), (3.7), using (3.1) and ($\zeta_2$), we have

$$0 \leq \zeta(s^4 d(z_{n_0}, y_{n_0}), M(z_{n_0}, y_{n_0}, z_{n_0}, y_{n_0})) \leq d(y_{n_0}, z_{n_0}) - s^4 d(y_{n_0}, z_{n_0}),$$

a contradiction. Therefore, $y_{n_0}$ is a common best proximity point of $S$ and $T$.

Now, suppose that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$.

From (3.5), we have

$$d(x_n, Sa_n) = d(x_{n+1}, Sa_{n+1}) = d(A,B) = d(x_{n-1}, Ta_n) = d(x_n, Ta_{n+1}). \tag{3.8}$$

Since, $(S,T)$ is a generalized proximal $\mathscr{Z}$-contraction of first kind, we have

$$\zeta(s^4 d(x_n, x_{n+1}), M(x_{n-1}, x_n, x_n, x_{n+1})) \geq 0, \text{ for all } n \in \mathbb{N}, \tag{3.9}$$

where

$$\begin{aligned} M(x_{n-1},x_n,x_n,x_{n+1}) &= \max\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),\tfrac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2s}\} \\ &= \max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}. \end{aligned}$$

If $d(x_n,x_{n-1}) < d(x_n,x_{n+1})$ then $M(x_{n-1},x_n,x_n,x_{n+1}) = d(x_n,x_{n+1})$.

Now, from (3.9) using ($\zeta_2$), we get,

$$\begin{aligned} 0 &\leq \zeta(s^4 d(x_n,x_{n+1}),M(x_{n-1},x_n,x_n,x_{n+1})) \\ &< M(x_{n-1},x_n,x_n,x_{n+1}) - s^4 d(x_n,x_{n+1})) < 0, \end{aligned}$$

which is a contradiction. Therefore, $d(x_n,x_{n+1}) \leq d(x_{n-1},x_n),$ for all $n \in \mathbb{N}$,

i.e., $\{d(x_n,x_{n+1})\}$ is a decreasing sequence of positive real numbers and so there exists a real number $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = r$.

If $r > 0$, then from (3.9), and using ($\zeta_3$), we get,

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}[\zeta(s^4 d(x_n,x_{n+1}),M(x_{n-1},x_n,x_n,x_{n+1}))] \\ &= \limsup_{n\to\infty}(s^4 d(x_n,x_{n+1}),d(x_n,x_{n+1}) < 0, \end{aligned}$$

a contradiction. Therefore, $r = 0$, i.e., $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = 0$.

Now, we prove that $\{x_n\}$ is a $b$-Cauchy sequence. On the contrary, suppose that $\{x_n\}$ is not $b$-Cauchy. By Lemma 2.1, there exist an $\varepsilon > 0$ and sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with

$n_k > m_k \geq k$ such that

$$d(x_{m_k},x_{n_k}) \geq \varepsilon \text{ and } d(x_{m_k},x_{n_k-1}) < \varepsilon$$

satisfying $(i)-(iv)$ of Lemma 2.1.

As $\{x_{m_k}\}$ and $\{x_{n_k}\}$ satisfy (3.8), we get

$$d(x_{m_k+1},Sa_{m_k+1}) = d(x_{n_k+1},Sa_{n_k+1}) = d(x_{m_k},Ta_{m_k+1}) = d(x_{n_k},Ta_{n_k+1}) = d(A,B), \text{ for all } k.$$

From the inequality (3.1), we have

$$\zeta(s^4 d(x_{m_k+1},x_{n_k+1}),M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1})) \geq 0, \tag{3.10}$$

where

$$M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1})=\max\{d(x_{m_k},x_{n_k}),\quad d(x_{m_k},x_{m_k+1}),d(x_{n_k},x_{n_k+1}),\ \tfrac{d(x_{m_k},x_{n_k+1})+d(x_{m_k+1},x_{n_k})}{2s}\}.$$

On taking limit superior as $k\to\infty$, we get,

$$\limsup_{k\to\infty} M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1})\leq \max\{s\varepsilon,0,0,s\varepsilon\}=s\varepsilon.$$

Now, from (3.10), and using $(\zeta_2)$, we get,

$$\begin{aligned}
0 &\leq \limsup_{k\to\infty}[\zeta(s^4 d(x_{m_k+1},x_{n_k+1}),M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1}))]\\
&\leq \limsup_{k\to\infty}[M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1})-s^4 d(x_{m_k+1},x_{n_k+1})]\\
&= \limsup_{k\to\infty} M(x_{m_k},x_{n_k},x_{m_k+1},x_{n_k+1})-s^4\liminf_{k\to\infty} d(x_{m_k+1},x_{n_k+1})\\
&= s\varepsilon - s^4(\tfrac{\varepsilon}{s^2})<0,
\end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ is a $b$-Cauchy sequence in $A_0$. Since, $A_0$ is closed, there exists some $x\in A_0$ such that $x_n\to x$ as $n\to\infty$.

Because of the fact the mappings $S$ and $T$ are commuting proximally and using (3.5), we get

$$Sx_n = Tx_{n+1},\ \text{for every } n\in\mathbb{N}\cup\{0\}.$$

Therefore, the continuity of the mappings $S$ and $T$ ensures that

$$Tx=\lim_{n\to\infty} Tx_n=\lim_{n\to\infty} Sx_{n-1}=Sx.$$

Since, $S(A_0)\subseteq B_0$, there exists an element $u\in A_0$ such that

$$d(u,Sx)=d(A,B)=d(u,Tx). \tag{3.11}$$

As $S$ and $T$ are commuting proximally, we get $Su=Tu$. Again, since, $S(A_0)\subseteq B_0$, there exists an element $v\in A_0$ such that

$$d(v,Su)=d(A,B)=d(v,Tu). \tag{3.12}$$

From (3.11), (3.12) and the pair $(S,T)$ is generalized proximal $\mathscr{Z}$-contraction of first kind, we have

$$\zeta(s^4 d(u,v),M(u,v,u,v))\geq 0. \tag{3.13}$$

where $M(u,v,u,v) = d(u,v)$.

From (3.13), and using $(\zeta_2)$, we get

$$0 \leq \zeta(s^4 d(u,v), M(u,v,u,v)) < M(u,v,u,v) - s^4 d(u,v) = d(u,v) - s^4 d(u,v),$$

it is a contradiction. Therefore, $u$ is a common best proximity point of $S$ and $T$.

Thus, it follows that

$$d(u,Su) = d(A,B) = d(u,Tu). \tag{3.14}$$

Let $u'(\neq u)$ be another common best proximity point of $S$ and $T$, so that

$$d(u',Su') = d(A,B) = d(u',Tu'). \tag{3.15}$$

Since, $(S,T)$ is generalized proximal $\mathscr{Z}$-contraction of first kind, from (3.14) and (3.15), we have

$$0 \leq \zeta(s^4 d(u,u'), M(u,u',u,u')) < M(u,u',u,u') - s^4 d(u,u') = d(u,u') - s^4 d(u,u'),$$

which is a contradiction. Hence, $u$ is a unique common best proximity point of $S$ and $T$. $\square$

**Theorem 3.2.** *Let $A$ and $B$ be two non-empty subsets of a complete b-metric space $(X,d)$. Suppose $S,T : A \to B$ be generalized proximal $\mathscr{Z}$-contraction of second kind such that*

*(i) $S(A_0) \subseteq B_0, S(A_0) \subseteq T(A_0)$, where $A_0, B_0$ are non-empty and $A_0$ is closed subset of $A$;*

*(ii) $S$ and $T$ are proximally commute;*

*(iii) $S$ and $T$ are b-continuous;*

*(iv) the pair $(A,B)$ have P-property.*

*Then $S$ and $T$ have a unique common best proximity point in $A_0$.*

*Proof.* In a manner akin to Theorem 3.1, and using the condition of generalized proximal $\mathscr{Z}$-contraction of second kind, we can show that $\{d(Sx_n, Sx_{n+1})\}$ is a decreasing sequence of positive real numbers and $\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0$.

Now, we prove that $\{Sx_n\}$ is a $b$-Cauchy sequence. On the contrary, suppose that $\{Sx_n\}$ is not

$b$-Cauchy. By Lemma 2.1, there exist an $\varepsilon > 0$ and sequences of positive integers $\{Sn_k\}$ and $\{Sm_k\}$ with $n_k > m_k \geq k$ such that

$$d(Sx_{m_k}, Sx_{n_k}) \geq \varepsilon \text{ and } d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon$$

satisfying $(i)-(iv)$ of Lemma 2.1. As $\{Sx_{m_k}\}$ and $\{Sx_{n_k}\}$ satisfy (3.8), we get

$$d(x_{m_k+1}, Sa_{m_k+1}) = d(x_{n_k+1}, Sa_{n_k+1}) = d(x_{m_k}, Ta_{m_k+1}) = d(x_{n_k}, Ta_{n_k+1}) = d(A,B), \text{ for all } k.$$

From the inequality (3.2), we have

$$\zeta(s^4 d(Sx_{m_k+1}, Sx_{n_k+1}), M(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1})) \geq 0, \tag{3.16}$$

where

$$\begin{aligned} M(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1}) &= \max\{d(Tx_{m_k}, Tx_{n_k}), \tfrac{d(Tx_{m_k}, Sx_{m_k+1})}{2s}, \tfrac{d(Tx_{n_k}, Sx_{n_k+1})}{2s}, \\ &\qquad \tfrac{d(Tx_{m_k}, Sx_{n_k+1}) + d(Tx_{n_k}, Sx_{m_k+1})}{1+s+2s^2}\} \\ &= \max\{d(Sx_{m_k-1}, Sx_{n_k-1}), \tfrac{d(Sx_{m_k-1}, Sx_{m_k+1})}{2s}, \tfrac{d(Sx_{n_k-1}, Sx_{n_k+1})}{2s}, \\ &\qquad \tfrac{d(Sx_{m_k-1}, Sx_{n_k+1}) + d(Sx_{n_k-1}, Sx_{m_k+1})}{1+s+2s^2}\}. \end{aligned}$$

On taking limit superior as $k \to \infty$, we get,

$$\limsup_{k\to\infty} M(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1}) \leq \max\{s\varepsilon, 0, 0, \tfrac{2s^2\varepsilon}{1+s+2s^2}\} = s\varepsilon.$$

Now, from (3.16), and using $(\zeta_2)$, we get,

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} [\zeta(s^4 d(Sx_{m_k+1}, Sx_{n_k+1}), M(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1}))] \\ &\leq \limsup_{k\to\infty} [M(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1}) - s^4 d(Sx_{m_k+1}, Sx_{n_k+1})] \\ &= \limsup_{k\to\infty} M(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1}) - s^4 \liminf_{k\to\infty} d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &= s\varepsilon - s^4(\tfrac{\varepsilon}{s^2}) < 0, \end{aligned}$$

which is a contradiction. Therefore, $\{Sx_n\}$ is a $b$-Cauchy sequence. Let $\{u_n\}$ be a sequence of elements in $A_0$ such that $d(u_n, Sx_n) = d(A,B)$ for all $n \geq 0$.

By the $P$-property, we obtain $d(u_m, u_n) = d(Sx_m, Sx_n)$, for all $m, n \in \mathbb{N} \cup \{0\}$. Clearly, the sequence $\{u_n\}$ is a $b$-Cauchy sequence in $A_0$.

Since, $A_0$ is closed, there exists some $u \in A_0$ such that $u_n \to u$ as $n \to \infty$.

Proceeding similar to Theorem 3.1, and using the condition of generalized proximal $\mathscr{Z}$-contraction of second kind, we can easily see that $S$ and $T$ have a unique common best proximity point. $\square$

The following are the examples in support of our results.

**Exercise 3.1.** *Let* $X=\mathbb{R}^2, A=[0,\infty)\times\{1\}, B=[0,\infty)\times\{0\}$,
$A_0=[0,1]\times\{1\}$ *and* $B_0=[0,1]\times\{0\}$.
*We define* $d: X\times X\to\mathbb{R}^+$ *by*
$d((a_1,a_2),(b_1,b_2))=|a_1-b_1|^2+|a_2-b_2|^2$ *for all* $(a_1,a_2),(b_1,b_2)\in X$.
*Then, clearly* $(X,d)$ *is a b-metric space with* $s=2$.
*We define the map* $S,T: A\to B$ *by*
$S(x,1)=\begin{cases}(\log(\frac{x^2}{25}+1),0) & \text{if } x\in[0,1],\\ (\log\frac{26}{25},0) & \text{if } x\geq 1\end{cases}$, $T(x,1)=\begin{cases}(\frac{x}{5},0) & \text{if } x\in[0,1],\\ (\frac{1}{5},0) & \text{if } x\geq 1\end{cases}$
*and* $\zeta:\mathbb{R}^+\times\mathbb{R}^+\to(-\infty,\infty)$ *by* $\zeta(t,s)=\frac{9}{10}s-t, s,t\in\mathbb{R}^+$.
*Clearly,* $S,T$ *are continuous.*
*It easy to see that* $S(A_0)\subseteq T(A_0), S(A_0)\subseteq B_0, d(A,B)=1$ *and* $\zeta$ *is a simulation function.*
*Now, let* $(x,1),(y,1),(u,1),(v,1)\in A$ *such that*

$$\begin{aligned} d((u,1),S(x,1)) &= d((u,1),T(x,1))=d(A,B)=1\\ d((v,1),S(y,1)) &= d((v,1),T(y,1))=d(A,B)=1. \end{aligned} \tag{3.17}$$

*Without loss of generality, we assume that* $u\geq v$.
*From (3.17), we have,* $x\in[0,1], y\in[0,1], u=\log(\frac{x^2}{16}+1)\in[0,\log\frac{26}{25}], v=\frac{y}{5}\in[0,\frac{1}{5}]$.
*Now,* $s^4d((u.1),(v,1))=16|u-v|^2=16(u-v)^2$ *and*

$$\begin{aligned} M((x,1),(y,1),(u,1),\quad(v,1)) &= \max\{|x-y|^2,|x-u|^2,|y-v|^2,\tfrac{|x-v|^2+|y-u|^2}{4}\}\\ &= \max\{(5\sqrt{e^u-1}-5v)^2,(5\sqrt{e^u-1}-u)^2,4v,\tfrac{(5\sqrt{e^u-1}-5v)^2+|5v-u|^2}{4}\}. \end{aligned}$$

*We consider*

$$\begin{aligned} &\zeta(s^4d((u.1),(v,1)),M((x,1),(y,1),(u,1),(v,1)))\\ &=\tfrac{9}{10}M((x,1),(y,1),(u,1),(v,1))-s^4d((u.1),(v,1))\\ &=\tfrac{9}{10}\max\{(5\sqrt{e^u-1}-5v)^2,(5\sqrt{e^u-1}-u)^2,4v,\tfrac{(5\sqrt{e^u-1}-5v)^2+|5v-u|^2}{4}\}-16(u-v)^2\geq 0. \end{aligned}$$

*Therefore, the pair $(S,T)$ is a generalized proximal $\mathscr{Z}$-contraction of first kind. Hence, the mappings $S$ and $T$ satisfy all the hypotheses of Theorem 3.1 and $(0,1)$ is the unique best proximity point in $A_0$.*

**Exercise 3.2.** *Let $X=\mathbb{R}^2, A=\{(x,0): x\geq 0\}, B=\{(x,1): x\geq 0\}$,
$A_0=\{(x,0): x\in[0,1]\}$ and $B_0=\{(x,1): x\in[0,1]\}$.
We define $d: X\times X\to\mathbb{R}^+$ by
$d((a_1,a_2),(b_1,b_2))=|a_1-b_1|^2+|a_2-b_2|^2$ for all $(a_1,a_2),(b_1,b_2)\in X$.
Then, clearly $(X,d)$ is a b-metric space with $s=2$.
We define the map $S,T: A\to B$ by*

$$S(x,0)=\begin{cases}(\frac{x^3}{2},1) & \text{if } x\in[0,1],\\ (\frac{1}{2},1) & \text{if } x\geq 1\end{cases},\ T(x,0)=\begin{cases}(\frac{2x}{3},1) & \text{if } x\in[0,1],\\ (\frac{2}{3},1) & \text{if } x\geq 1\end{cases}$$

*Clearly, $S,T$ are continuous, $S(A_0)\subseteq T(A_0)$, $S(A_0)\subseteq B_0$, $d(A,B)=1$.
Now, let $(x,0),(y,0),(u,0),(v,0)\in A$ such that*

$$\begin{aligned} d((u,0),S(x,0)) &= d((u,0),T(x,0))=d(A,B)=1\\ d((v,0),S(y,0)) &= d((v,0),T(y,0))=d(A,B)=1. \end{aligned} \tag{3.18}$$

*From (3.18), we have, $x\in[0,1], y\in[0,1], u=\frac{x^3}{2}\in[0,\frac{1}{2}], v=\frac{2y}{3}\in[0,\frac{2}{3}]$.
Now, $s^4d(S(u.0),S(v,0))=4|u^3-v^3|^2$ and*

$$\begin{aligned} & d(T(x,0),T(y,0))=\tfrac{4}{9}|x-y|^2=\tfrac{4}{9}|(2u)^{\frac{1}{3}}-\tfrac{3v}{2}|^2, d(T(x,0),S(u,0))=|\tfrac{2}{3}(2u)^{\frac{1}{3}}-\tfrac{u^3}{2}|^2,\\ & d(T(y,0),S(v,0))=|v-\tfrac{v^3}{2}|^2, d(T(x,0),S(v,0))=|\tfrac{2}{3}(2u)^{\frac{1}{3}}-\tfrac{v^3}{2}|^2, d(T(y,0),S(u,0))=|v-\tfrac{u^3}{2}|^2,\\ & M((x,0),(y,0),(u,0),(v,0))=\max\{\tfrac{4}{9}|(2u)^{\frac{1}{3}}-\tfrac{3v}{2}|^2, \tfrac{|\frac{2}{3}(2u)^{\frac{1}{3}}-\frac{u^3}{2}|^2}{4}, \tfrac{|v-\frac{v^3}{2}|^2}{4}, \tfrac{|\frac{2}{3}(2u)^{\frac{1}{3}}-\frac{v^3}{2}|^2+|v-\frac{u^3}{2}|^2}{11}\} \end{aligned}$$

*We consider*

$$\begin{aligned} & \zeta(s^4d(S(u.0),S(v,0)),M((x,0),(y,0),(u,0),(v,0)))\\ &= \tfrac{99}{100}M((x,0),(y,0),(u,0),(v,0))-s^4d(S(u.0),S(v,0))\\ &= \tfrac{99}{100}\max\{\tfrac{4}{9}|(2u)^{\frac{1}{3}}-\tfrac{3v}{2}|^2, \tfrac{|\frac{2}{3}(2u)^{\frac{1}{3}}-\frac{u^3}{2}|^2}{4}, \tfrac{|v-\frac{v^3}{2}|^2}{4}, \tfrac{|\frac{2}{3}(2u)^{\frac{1}{3}}-\frac{v^3}{2}|^2+|v-\frac{u^3}{2}|^2}{11}\}-4|u^3-v^3|^2\geq 0. \end{aligned}$$

*Therefore, the pair $(S,T)$ is a generalized proximal $\mathscr{Z}$-contraction of second kind.
Hence, the mappings $S$ and $T$ satisfy all the hypotheses of Theorem 3.2 and $(0,0)$ is the unique best proximity point.*

**Corollary 3.1.** *Let $A$ and $B$ be two non-empty subsets of a complete b-metric space $(X,d)$. Suppose $T: A \to B$ be a mapping with $T(A_0) \subseteq B_0$ where $A_0, B_0$ are non-empty and $A_0$ is closed subset of $A$ such that $d(a_1, Tb_1) = d(A,B) = d(a_2, Tb_2)$ implies*

$$\zeta(s^4 d(a_1,a_2), M(b_1,b_2,a_1,a_2)) \geq 0, \textit{ for all } b_1,b_2,a_1,a_2 \in A,$$

*where* $M(b_1,b_2,a_1,a_2) = \max\{d(b_1,b_2), d(b_1,a_1), d(b_2,a_2), \frac{d(b_2,a_1)+d(b_1,a_2)}{2s}\}$.
*Then $T$ has a unique best proximity point.*

**Corollary 3.2.** *Let $A$ and $B$ be two non-empty subsets of a complete b-metric space $(X,d)$. Suppose $T: A \to B$ be a mapping with $T(A_0) \subseteq B_0$ where $A_0, B_0$ are non-empty and $A_0$ is closed subset of $A$ such that $d(a_1, Tb_1) = d(A,B) = d(a_2, Tb_2)$ implies*

$$\zeta(s^4 d(a_1,a_2), M(b_1,b_2,a_1,a_2)) \geq 0, \textit{ for all } b_1,b_2,a_1,a_2 \in A,$$

*where* $M(b_1,b_2,a_1,a_2) = \max\{d(b_1,b_2), \frac{d(b_1,a_1)+d(b_2,a_2)}{2}, \frac{d(b_2,a_1)+d(b_1,a_2)}{2s}\}$.
*Then $T$ has a unique best proximity point.*

**Remark 3.1.** *Corollary 3.1 extend and generalize Theorem 2.4 to b-metric spaces.*

**Remark 3.2.** *Taking $A = B = X$ and $s = 1$ in Corollary 3.1, we get Theorem 2 of* [27] *as a particular case.*

**Remark 3.3.** *In Corollary 3.1, the mapping $T$ is not necessarily continuous. Moreover, the sets $A$ and $B$ are not required to be closed. Thus, for $M(b_1,b_2,a_1,a_2) = d(b_1,b_2)$, (when the mapping $T$ reduces to proximal simulative contraction of first kind) Corollary 3.1 improves Theorem 1 of* [1] *in b-metric spaces.*

## 4. APPLICATIONS

### 4.1. Application to nonlinear integral equations.

In this section, we obtain the solution of integral equation as an application of our obtained results. If we take $A = B = X$ in Theorem 3.1, we obtain the solution of nonlinear integral equation. Let $\Omega = C[a,b]$ be a set of real valued continuous functions on $[a,b]$,where $[a,b]$ is closed and bounded integral in $\mathbb{R}$. we define $d: \Omega \times \Omega \to \mathbb{R}^+$ by $d(\xi,\eta) = \max\limits_{t \in [a,b]} |\xi(t) - \eta(t)|^p$, where $p > 1$ a real number, for all $\xi, \eta \in \Omega$.

Therefore $(\Omega, d)$ is a complete $b$-metric space with $s = 2^{p-1}$. Many author's studied unique solution of a system of nonlinear Integral equations [4, 6, 10, 11, 13, 34]. Here, we employ a technique in fixed-point theory for the existence of a solution. In this section, we establish the existence of unique common solution of a system of two nonlinear integral equations of Fredholm type defined by

$$\begin{cases} \xi(t) = f(t) + \mu \int_a^b \mathscr{D}_1(t, r, \xi(r))dr, \\ \zeta(t) = f(t) + \mu \int_a^b \mathscr{D}_2(t, r, \zeta(r))dr, \end{cases} \tag{4.1}$$

where $\xi \in C[a,b]$ is the unknown function, $\mu \in \mathbb{R}, t, r \in [a,b], \mathscr{D}_1, \mathscr{D}_2 : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ are continuous functions.

Let $\mathscr{F}_1, \mathscr{F}_2 : \Omega \to \Omega$ be two mappings defined by

$$\begin{cases} \mathscr{F}_1(\xi(t)) = f(t) + \mu \int_a^b \mathscr{D}_1(t, r, \xi(r))dr, \\ \mathscr{F}_2(\xi(t)) = f(t) + \mu \int_a^b \mathscr{D}_2(t, r, \xi(r))dr. \end{cases} \tag{4.2}$$

Assume the following:

$(i)$ there exists a continuous function $\gamma : [a,b] \times [a,b] \to \mathbb{R}^+$, such that $\max\limits_{r \in [a,b]} \int_a^b \gamma(t,r)dr \leq 1$;

$(ii)$ $\mathscr{F}_1(\Omega) \subseteq \mathscr{F}_2(\Omega)$;

$(iii)$ there exists a constant $K \in (0,1)$ such that for all $t, r \in [a,b]$ and $\xi, \zeta \in \mathbb{R}$, the following condition is satisfied: $|\mathscr{D}_1(t,r,\xi_1(r)) - \mathscr{D}_2(t,r,\xi_2(r)|^p \leq \frac{K}{(b-a)^{p-1} 2^{6p-6}} \gamma(t,r) \Delta(\eta_1, \eta_2, \xi_1, \xi_2)$,

where

$\Delta(\eta_1, \eta_2, \xi_1, \xi_2) = \max\{|\eta_1(r) - \eta_2(r)|^p, |\eta_1(r) - \xi_1(r)|^p, |\eta_2(r) - \xi_2(r)|^p, \frac{|\eta_1(r) - \xi_2(r)|^p + |\eta_2(r) - \xi_1(r)|^p}{2^p}\}$;

$(iv)$ $|\mu| \leq 1$.

**Theorem 4.1.** *Let $\mathscr{F}_1, \mathscr{F}_2 : \Omega \to \Omega$ be defined by* (4.2) *for which the conditions $(i)-(iv)$ hold. Then, the system of nonlinear integral equations* (4.1) *has a unique common solution in $\Omega$.*

*Proof.* Let $\xi, \eta \in \Omega$ and let $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$ using Hölder's inequality and from the conditions $(i)-(iv)$, for all $t$, we have

$$
\begin{aligned}
d(\xi_1,\xi_2) &= \max_{t\in[a,b]} |\xi_1(t)-\xi_2(t)|^p \\
&= |\mu|^p \max_{t\in[a,b]} \left| \int_a^b \mathscr{D}_1(t,r,\xi_1(r) - \int_a^b \mathscr{D}_2(t,r,\xi_2(r)dr \right|^p \\
&= |\mu|^p \max_{t\in[a,b]} \left| \int_a^b (\mathscr{D}_1(t,r,\xi_1(r) - \mathscr{D}_2(t,r,\xi_2(r))dr \right|^p \\
&\leq \left[ |\mu|^p \max_{t\in[a,b]} \left( \int_a^b 1^p dr \right)^{\frac{1}{q}} \left( \int_a^b |(\mathscr{D}_1(t,r,\xi_1(r) - \mathscr{D}_2(t,r,\xi_2(r))|^p dr \right)^{\frac{1}{p}} \right]^p \\
&\leq (b-a)^{\frac{p}{q}} \max_{t\in[a,b]} \left( \int_a^b |(\mathscr{D}_1(t,r,\xi_1(r) - \mathscr{D}_2(t,r,\xi_2(r))|^p dr \right) \\
&= (b-a)^{p-1} \max_{t\in[a,b]} \left( \int_a^b |(\mathscr{D}_1(t,r,\xi_1(r) - \mathscr{D}_2(t,r,\xi_2(r))|^p dr \right) \\
&\leq (b-a)^{p-1} \max_{t\in[a,b]} \int_a^b \tfrac{K}{(b-a)^{p-1}2^{6p-6}} \gamma(t,r)\Delta(\eta_1,\eta_2,\xi_1,\xi_2)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
s^4 d(\xi_1,\xi_2) &\leq \tfrac{K}{s^2} \max\{d(\eta_1,\eta_2), d(\eta_1,\xi_1), d(\eta_2,\xi_2), \tfrac{d(\eta_1,\xi_2)+d(\eta_2,\xi_1)}{2s}\} \\
&\leq \lambda\Delta(\eta_1,\eta_2,\xi_1,\xi_2)
\end{aligned}
$$

where $\lambda = \frac{K}{s^2} \in (0,1)$.

Therefore, by taking $\zeta(t,s) = \lambda s - t$, all the conditions of Theorem 3.1 are satisfied, and hence $\mathscr{F}_1, \mathscr{F}_2$ have a unique common solution of the system of nonlinear integral equations (4.1). $\square$

**4.2. Application to dynamic programming.** In this section, we assume that $\mathscr{X}_1$ and $\mathscr{X}_2$ be two Banach spaces; $\mathscr{D} \subseteq \mathscr{X}_1$ is the decision space; $\mathscr{S} \subseteq \mathscr{X}_2$ is the state space; $\Omega(\mathscr{S})$ is the Banach space of all bounded real valued functions on $\mathscr{S}$ with b-metric defined by;

$d(\xi,\zeta) = \sup\limits_{t\in\mathscr{S}} \mid \xi(t) - \zeta(t) \mid^p$, for all $\xi,\zeta \in \Omega(\mathscr{S})$ with coefficient $s = 2^{p-1}$ and the norm is defined as $\|\mathscr{F}\| = \sup\{\mid \mathscr{F}(t) \mid : t \in \mathscr{S}\}$, where $\mathscr{F} \in \Omega(\mathscr{S})$.

It is clear that $\Omega(\mathscr{S},d)$ is a complete b-metric space. The basic form of the functional equation in dynamic programming is given by Bellman and Lee [16] as follows;

$f(\xi) = \underset{\zeta\in\tilde{\mathscr{D}}}{H}(\xi,\zeta,f(T(\xi,\zeta))), \xi \in \mathscr{S}$, where $\xi$ and $\zeta$ denotes the state and decision vectors, respectively. T denotes the transformation of the process, $f(\xi)$ denotes the optimal return function with the initial state $\xi$ and opt represents Sup of inf.

We consider the system of functional equations

$$
\begin{aligned}
f_1(\nu_s) &= \operatorname*{opt}_{\nu_d \in \tilde{\mathscr{D}}} \eta_1(\nu_s, \nu_d)) + \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) \forall \nu_s \in \mathscr{S}, \\
f_2(\nu_s) &= \operatorname*{opt}_{\nu_d \in \tilde{\mathscr{D}}} \eta_2(\nu_s, \nu_d)) + \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \forall \nu_s \in \mathscr{S}
\end{aligned} \tag{4.3}
$$

where $\nu_s$ is a state vector, $\nu_d$ is a decision vector,$\rho_1, \rho_2$ represents the transformations of the process, and $f_1(\nu_s), f_2(\nu_s)$ denotes the optimal return functions with initial state $\nu_s$.
Let $\mathscr{F}_1, \mathscr{F}_2 : \Omega(\mathscr{S}) \to \Omega(\mathscr{S})$ be two mappings defined by;

$$
\begin{aligned}
\mathscr{F}_1 f_1(\nu_s) &= \operatorname*{opt}_{\nu_d \in \tilde{\mathscr{D}}} \eta_1(\nu_s, \nu_d)) + \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) \forall \nu_s \in \mathscr{S}, \\
\mathscr{F}_2 f_2(\nu_s) &= \operatorname*{opt}_{\nu_d \in \tilde{\mathscr{D}}} \eta_2(\nu_s, \nu_d)) + \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \forall \nu_s \in \mathscr{S}
\end{aligned} \tag{4.4}
$$

Assume the following:

($\mathscr{D}_a$) $\mathscr{F}_1(\Omega(\mathscr{S})) \subseteq \mathscr{F}_2(\Omega(\mathscr{S}))$;

($\mathscr{D}_b$) for all $(\nu_s, \nu_d, f_1, f_2, g_1, g_2) \in \mathscr{S} \times \mathscr{D} \times \Omega(\mathscr{S}) \times \Omega(\mathscr{S}) \times \Omega(\mathscr{S}) \times \Omega(\mathscr{S})$ and there exists

$0 < h < 1$, we have;

$| \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) | + | \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) |$
$\leq [\frac{h}{2^{4p-4}} M(g_1, g_2, f_1, f_2)]^{\frac{1}{p}}$, where
$M(g_1, g_2, f_1, f_2) = \max\{|\mathscr{F}_2 g_1 - \mathscr{F}_2 g_2|^p, \frac{|\mathscr{F}_2 g_1 - \mathscr{F}_1 f_1|^p}{2^p}, \frac{|\mathscr{F}_2 g_2 - \mathscr{F}_1 f_2|^p}{2^p}, \frac{|\mathscr{F}_2 g_1 - \mathscr{F}_1 f_2|^p + |\mathscr{F}_2 g_2 - \mathscr{F}_1 f_1|^p}{1 + 2^{p-1} + 2^{2p-1}}\}$;

($\mathscr{D}_c$) $\rho_i, \xi_i$ are bounded $i = 1, 2$.

**Theorem 4.2.** *Let $\mathscr{F}_1, \mathscr{F}_2 : \Omega(\mathscr{S}) \to \Omega(\mathscr{S})$ be defined by* (4.4) *for which the conditions $\mathscr{D}_a - \mathscr{D}_c$ hold. Then, the system of functional equations given by* (4.3) *has a unique bounded common solution in* $\Omega(S)$.

*Proof.* Let $\nu_s \in \mathscr{S}, f_1, f_2 \in \Omega(\mathscr{S})$ and $\varepsilon > 0$.
Since $\rho_i, \xi_i$ are bounded for $i = 1, 2$ there exists $M \geq 0$ such that

$$
\sup\{||\rho_1(\nu_s, \nu_d)||, ||\rho_2(\nu_s, \nu_d)||, ||\xi_2(\nu_s, \nu_d, t)|| : (\nu_s, \nu_d, t) \in \mathscr{S} \times \mathscr{D} \times \mathbb{R}\} \leq M. \tag{4.5}
$$

From the inequalities (4.4) and (4.5), we conclude that $\mathscr{F}_1, \mathscr{F}_2$ are self mappings of $\Omega(\mathscr{S})$
First assume that $\operatorname*{opt}_{\nu_s \in \tilde{\mathscr{D}}} = \inf_{\nu_d \in \mathscr{D}}$.

From the inequality (4.4), we can find $\nu_d \in \mathscr{D}$ and $(\nu_s, f_1, f_2) \in \mathscr{S} \times \Omega(\mathscr{S}) \times \Omega(\mathscr{S})$ such that

$$\mathscr{F}_1 f_1(\nu_s) > \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d)) + \eta_1(\nu_s, \nu_d) - \varepsilon \tag{4.6}$$

$$\mathscr{F}_1 f_2(\nu_s) > \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d)) + \eta_2(\nu_s, \nu_d) - \varepsilon \tag{4.7}$$

$$\mathscr{F}_1 f_1(\nu_s) \leq \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d)) + \eta_1(\nu_s, \nu_d) \tag{4.8}$$

$$\mathscr{F}_1 f_2(\nu_s) \leq \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d)) + \eta_2(\nu_s, \nu_d) \tag{4.9}$$

By using the inequalities (4.6) and (4.9), we get that

$$\begin{cases} \mathscr{F}_1 f_1(\nu_s) - \mathscr{F}_1 f_2(\nu_s) > \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ \qquad +\eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) - \varepsilon \\ \qquad \geq -\{\mid \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \mid \\ \qquad + \mid \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) \mid + \varepsilon\} \end{cases} \tag{4.10}$$

Also, from (4.7) and (4.8), we have

$$\begin{cases} \mathscr{F}_1 f_1(\nu_s) - \mathscr{F}_1 f_2(\nu_s) \leq \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ \qquad +\eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) + \varepsilon \\ \qquad \leq \mid \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \mid \\ \qquad + \mid \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) \mid + \varepsilon \end{cases} \tag{4.11}$$

By using (4.10) and (4.11), we get that

$$\begin{aligned} \mid \mathscr{F}_1 f_1(\nu_s) - \mathscr{F}_1 f_2(\nu_s) \mid &< \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ &\quad +\eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) + \varepsilon \\ &\leq \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ &\quad +\eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) + \varepsilon \end{aligned}$$

Now, we support that $\operatorname*{opt}_{\nu_d \in \tilde{\mathscr{D}}} = \sup_{\nu_d \in \mathscr{D}}$.

Again, using the inequality (4.4), we can find $\nu_d \in \mathscr{D}$ and $(\nu_s, f, g) \in \mathscr{S} \times \Omega(\mathscr{S}) \times \Omega(\mathscr{S})$ such that

$$\mathscr{F}_1 f_1(\nu_s) < \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) + \eta_1(\nu_s, \nu_d) + \varepsilon \tag{4.12}$$

$$\mathscr{F}_1 f_2(\nu_s) < \xi_2(\nu_s,\nu_d,f_2(\rho_2(\nu_s,\nu_d))) + \eta_2(\nu_s,\nu_d) + \varepsilon \tag{4.13}$$

$$\mathscr{F}_1 f_1(\nu_s) < \xi_1(\nu_s,\nu_d,f_1(\rho_1(\nu_s,\nu_d))) + \eta_1(\nu_s,\nu_d) \tag{4.14}$$

$$\mathscr{F}_1 f_2(\nu_s) < \xi_2(\nu_s,\nu_d,f_2(\rho_2(\nu_s,\nu_d))) + \eta_2(\nu_s,\nu_d) \tag{4.15}$$

Using the inequalities (4.12) and (4.15), we have

$$\begin{cases} \mathscr{F}_1 f_1(\nu_s) - \mathscr{F}_1 f_2(\nu_s) < \xi_1(\nu_s,\nu_d,f_1(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d,f_2(\rho_2(\nu_s,\nu_d))) \\ \qquad +\eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) + \varepsilon \\ \qquad \le \mid \xi_1(\nu_s,\nu_d,f_1(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d,f_2(\rho_2(\nu_s,\nu_d))) \mid + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) \mid + \varepsilon \end{cases} \tag{4.16}$$

Also, from the inequalities (4.13) and (4.14), we get that

$$\begin{cases} \mathscr{F}_1 f_1(\nu_s) - \mathscr{F}_1 f_2(\nu_s) \ge \xi_1(\nu_s,\nu_d,f_1(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d,f_2(\rho_2(\nu_s,\nu_d))) \\ \qquad +\eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) - \varepsilon \\ \qquad \ge -\{\mid \xi_1(\nu_s,\nu_d,f_1(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d,f_2(\rho_2(\nu_s,\nu_d))) \mid + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) + \varepsilon\} \end{cases} \tag{4.17}$$

From (4.16) and (4.17), we have

$$\begin{cases} \mid \mathscr{F}_1 f_1(\nu_s) - \mathscr{F}_1 f_2(\nu_s) \mid < \xi_1(\nu_s,\nu_d,f_1(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d,f_2(\rho_2(\nu_s,\nu_d))) \\ \qquad +\eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) - \varepsilon \\ \le \mid \xi_1(\nu_s,\nu_d,f_1(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d,f_2(\rho_2(\nu_s,\nu_d))) \mid + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) + \varepsilon \end{cases} \tag{4.18}$$

On taking $\varepsilon \to 0$ in (4.18), we obtain that

$$\begin{aligned} \mid \mathscr{F}_1 f_1(\nu_s) - \mathscr{F}_1 f_2(\nu_s) \mid \le \mid \xi_1(\nu_s,\nu_d,f_1(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d,f_2(\rho_2(\nu_s,\nu_d))) \mid \\ + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) \mid \end{aligned}$$

From the condition $(\mathscr{D}_b)$, we have

$$\begin{aligned} \mid \mathscr{F}_1 f_1(\nu_s) - \mathscr{F}_1 f_2(\nu_s) \mid &\le \mid \xi_1(\nu_s,\nu_d,f_1(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d,f_2(\rho_2(\nu_s,\nu_d))) \mid \\ &\quad + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) \mid \\ &\le [\tfrac{h}{2^{4p-4}} M(g_1,g_2,f_1,f_2)]^{\frac{1}{p}} \\ &\le [\sup_{\nu_s\in\mathscr{S}} (\tfrac{h}{2^{6p-6}} \max\{|\mathscr{F}_2 g_1 - \mathscr{F}_2 g_2|^p, \tfrac{|\mathscr{F}_2 g_1 - \mathscr{F}_1 f_1|^p}{2^p}, \tfrac{|\mathscr{F}_2 g_2 - \mathscr{F}_1 f_2|^p}{2^p}, \tfrac{|\mathscr{F}_2 g_1 - \mathscr{F}_1 f_2|^p + |\mathscr{F}_2 g_2 - \mathscr{F}_1 f_1|^p}{1+2^{p-1}+2^{2p-1}}\})]^{\frac{1}{p}} \end{aligned}$$

which implies that

$$\sup_{\nu_s\in\mathscr{S}} \mid \mathscr{F}_1 f_1(\nu_s)-\mathscr{F}_1 f_2(\nu_s)\mid^p \leq \frac{h}{2^{4p-4}}\max\{[\sup_{\nu_s\in\mathscr{S}}(|\mathscr{F}_2 g_1-\mathscr{F}_2 g_2|^p, \frac{|\mathscr{F}_2 g_1-\mathscr{F}_1 f_1|^p}{2^p}, \frac{|\mathscr{F}_2 g_2-\mathscr{F}_1 f_2|^p}{2^p}, \frac{|\mathscr{F}_2 g_1-\mathscr{F}_1 f_2|^p+|\mathscr{F}_2 g_2-\mathscr{F}_1 f_1|^p}{1+2^{p-1}+2^{2p-1}})]\}.$$

Now, for all $f_1, f_2, g_1, g_2 \in \Omega(\mathscr{S})$, we have

$$s^4 d(\mathscr{F}_1 f_1, \mathscr{F}_1 f_2) \leq h\max\{d(\mathscr{F}_2 g_1, \mathscr{F}_2 g_2), \frac{d(\mathscr{F}_2 g_1, \mathscr{F}_1 f_1)}{2s}, \frac{d(\mathscr{F}_2 g_2, \mathscr{F}_1 f_2)}{2s}, \frac{d(\mathscr{F}_2 g_1, \mathscr{F}_1 f_2)+d(\mathscr{F}_2 g_2, \mathscr{F}_1 f_1)}{1+s+2s^2}\}.$$

Therefore, by taking $\zeta(t,s) = hs-t$, all the conditions of Theorem 3.2 are satisfied with $A = B = X$, and hence $\mathscr{F}_1, \mathscr{F}_2$ have a unique bounded common solution of the system of functional equations (4.3). □

## 5. Conclusion and Future Work

In this paper, we introduced generalized proximal $\mathscr{Z}$-contraction of the first kind and the second kind and obtained some common best proximity points via simulation functions. Using similar approaches, it can be studied new best proximity points result in metric and some generalized metric spaces. The investigation of certain circumstances to exclude the identity map of $X$ from Theorem 3.1 and Theorem 3.2 and related results is a worthwhile problem for future efforts.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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