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FIXED POINT APPLICATIONS OF $\mathbb{S}$-COUPLED CYCLIC MAPPINGS IN BIPOLAR PARAMETRIC METRIC SPACES

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Abstract: In this paper, common strong coupled fixed points in Bipolar parametric metric space are established and $\mathbb{S}$-coupled cyclic contraction is presented. Many pertinent findings from the current literature are expanded upon and generalised by our findings. We also provided one example to bolster our main conclusions. Lastly, we apply the findings to homotopy and integral equations.

Keywords: bipolar parametric metric space; covariant map; strong coupled fixed point; $\mathbb{S}$-type cyclic contraction.

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1. INTRODUCTION

The study of coupled fixed points has rapidly advanced within metric fixed point theory. The concept was first introduced by Guo *et al.* in 1987 [1]. In 2003, Kirk *et al.* [2] introduced cyclic contractions, proving that such contractions possess fixed points. Following this, Bhaskar *et al.* [3] established the coupled contraction mapping theorem.

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In 2017, S. Binayak Choudhury *et al.* [4] introduced the concept of coupling between two non-empty subsets in a metric space. They demonstrated that these couplings possess strong unique fixed points, provided they satisfy Banach-type or Chatterjea-type contractive inequalities. This concept was later generalized by G. V. R. Babu *et al.* [6] and S. Mary Anushia *et al.* [7], who derived strong coupled fixed points in the complete S-metric space.

The study of fixed points and associated properties for couplings meeting different kinds of inequalities was presented as an open problem by Choudhury *et al.* [4, 5]. For (ϕ,ψ)-contraction type coupling in complete partial metric spaces, Aydi *et al.* [8] demonstrated the existence and uniqueness of a strong coupled fixed point. By introducing SCC-Map and ϕ-contraction type T-coupling, as well as generalising the ϕ-contraction type coupling provided by Aydi *et al.* [8] to ϕ-contraction type T-coupling, Rashid *et al.*[9] and D. Eshi *et al.*[10] attempted to address this open problem and demonstrated the existence theorem of coupled coincidence point for metric spaces. Subsequently, Fuad Abdulkerim *et al.* [11] established coupled fixed point results by using (ϕ,ψ)-contraction type T-coupling mappings.

N. Hussain *et al.* recently introduced and researched the idea that parametric metric spaces are a natural generalization of metric spaces ([12, 13]). As a generalization of parametric metric space, Kumar *et al.* [14] established fixed point theorems and introduced the idea of binary operation at the place non-negative parameter $\mathfrak{t}$. The concept of BPPMS, or bipolar parametric metric space, was introduced and some FPT were proved on this space in 2024 by M. I. Pasha *et.al.*[15].

In this paper, we use $\mathbb{S}$-coupled cyclic contraction to provide various strong coupled fixed point theorems in the context of complete bipolar parametric metric space. Additionally, we are able to give appropriate instances that are pertinent to homotopy and integral equations.

What follows is in our subsequent conversations, we compile a few suitable definitions.

2. Preliminaries

Definition 2.1: ([15]) Suppose $\rho_c : \pounds \times \Im \times (0,\infty) \to \mathbb{R}^+$ is a function defined on two non empty sets $\pounds$ and $\Im$ such that.

(a) If $\rho_c(\mathfrak{x},\mathfrak{y},c) = 0$ for all $c > 0$ then $\mathfrak{x} = \mathfrak{y}$, for all $(\mathfrak{x},\mathfrak{y}) \in \pounds \times \Im$.

(b) If $\mathfrak{x}=\mathfrak{y}$, then $\rho_c(\mathfrak{x},\mathfrak{y},c)=0$, for all $c>0$ and $(\mathfrak{x},\mathfrak{y})\in \pounds\times\Im$

(c) If $\mathfrak{x},\mathfrak{y}\in \pounds\cap\Im$ then $\rho_c(\mathfrak{x},\mathfrak{y},c)=\rho_c(\mathfrak{y},\mathfrak{x},c)$, for all $c>0$

(d) $\rho_c(\mathfrak{x},\mathfrak{y},c)\leq \rho_c(\mathfrak{x},\mathfrak{z},c)+\rho_c(\mathfrak{d},\mathfrak{z},c)+\rho_c(\mathfrak{d},\mathfrak{y},c)$, for all $c>0$, $\mathfrak{x},\mathfrak{d}\in\pounds$ and $\mathfrak{y},\mathfrak{z}\in\Im$.

The triplet $(\pounds,\Im,\rho_c)$ is called a BPPMSs.

Example 2.2: ([15]) For all $\mathfrak{x}\in\pounds$, $\mathfrak{y}\in\Im$, and $c>0$, let $\pounds=[-1,0]$ and $\Im=[0,1]$ be equipped with $\rho_c(\mathfrak{x},\mathfrak{y},c)=c|\mathfrak{x}-\mathfrak{y}|$. Consequently, $(\pounds,\Im,\rho_c)$ is a complete BPPMS.

Definition 2.3: ([15]) Let $(\pounds_1,\Im_1,\rho_{c1})$ and $(\pounds_2,\Im_2,\rho_{c2})$ be BPPMSs and $\Omega:\pounds_1\cup\Im_1\to\pounds_2\cup\Im_2$ be a function. If $\Omega(\pounds_1)\subseteq\pounds_2$ and $\Omega(\Im_1)\subseteq\Im_2$, then Ω is called a covariant map, or a map from $(\pounds_1,\Im_1,\rho_{c1})$ to $(\pounds_2,\Im_2,\rho_{c2})$ and this is written as $\Omega:(\pounds_1,\Im_1,\rho_{c1})\rightrightarrows(\pounds_2,\Im_2,\rho_{c2})$. If $\Omega:(\pounds_1,\Im_1,\rho_{c1})\rightrightarrows(\Im_2,\pounds_2,\overline{\rho_{c2}})$ is a map, then Ω is called a contravariant map from $(\pounds_1,\Im_1,\rho_{c1})$ *to* $(\pounds_2,\Im_2,\rho_{c2})$ and this is denoted as $\Omega:(\pounds_1,\Im_1,\rho_{c1})\rightleftarrows(\pounds_2,\Im_2,\rho_{c2})$.

Definition 2.4: ([15]) In a BPPMS $(\pounds,\Im,\rho_c)$ for any $\xi\in\pounds\cup\Im$ is left point if $\xi\in\pounds$, is right point if $\xi\in\Im$ and is central point if $\xi\in\pounds\cap\Im$. Also, $\{\mathfrak{x}_a\}$ in $\pounds$ and $\{\mathfrak{z}_a\}$ in $\Im$ are left and right sequence respectively. In a BPPMS, we call a sequence, a left or a right one. A sequence $\{\mathfrak{y}_a\}$ is said to be convergent to ξ iff either $\{\mathfrak{y}_a\}$ is a left sequence, ξ is a right point and $\lim\limits_{a\to\infty}\rho_c(\mathfrak{y}_a,\xi,c)=0$, or $\{\mathfrak{y}_a\}$ is a right sequence, ξ is a left point and $\lim\limits_{a\to\infty}\rho_c(\xi,\mathfrak{y}_a,c)=0$. The bisequence $(\{\mathfrak{x}_a\},\{\mathfrak{z}_a\})\subseteq(\pounds,\Im)$ is a sequence on $(\pounds,\Im,\rho_c)$. In the case where $\{\mathfrak{x}_a\}$ and $\{\mathfrak{z}_a\}$ are both convergent, then $(\{\mathfrak{x}_a\},\{\mathfrak{z}_a\})$ is convergent.

The bi-sequence $(\{\mathfrak{x}_a\},\{\mathfrak{z}_a\})$ is a Cauchy bisequence if $\lim\limits_{a,b\to\infty}\rho_c(\mathfrak{x}_a,\mathfrak{z}_b,c)=0$.

Note that every convergent Cauchy bisequence is biconvergent.The BPPMS is complete, if each Cauchy bisequence is convergent (and so it is biconvergent).

3. Main Results

Let $\mathscr{K}$ denote the class of all functions $\mathscr{G}:[0,\infty)\to[0,\infty)$ such that $\mathscr{G}$ is increasing, continuous, $\mathscr{G}(\mathfrak{z})<\mathfrak{z}$ for all $\mathfrak{z}>0$ and $\mathscr{G}(0)=0$.

Definition 3.1: Let $(\pounds,\Im,\rho_c)$ be a BPPMS, $\mathscr{A},\mathscr{B}$ and $\mathscr{P},\mathscr{Q}$ be a nonempty subsets of $\pounds$ and $\Im$ respectively. Then

(i) a covariant map $\Omega:\left(\pounds^2,\Im^2\right)\rightrightarrows(\pounds,\Im)$ is said to be a cyclic with respect to $(\mathscr{A},\mathscr{P})$ and $(\mathscr{B},\mathscr{Q})$ if $\Omega\left(\mathscr{A}\cup\mathscr{B},\mathscr{P}\cup\mathscr{Q}\right)\subseteq\mathscr{P}\cup\mathscr{Q}$ and $\Omega\left(\mathscr{P}\cup\mathscr{Q},\mathscr{A}\cup\mathscr{B}\right)\subseteq\mathscr{A}\cup\mathscr{B}$.

(ii) a covariant map $\Lambda : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ is said to be a cyclic with respect to $(\mathscr{A},\mathscr{P})$ and $(\mathscr{B},\mathscr{Q})$ if $\Lambda(\mathscr{A}\cup\mathscr{B}) \subseteq \mathscr{P}\cup\mathscr{Q}$ and $\Lambda(\mathscr{P}\cup\mathscr{Q}) \subseteq \mathscr{A}\cup\mathscr{B}$.

Definition 3.2: Let $\mathscr{A},\mathscr{B}$ and $\mathscr{P},\mathscr{Q}$ be any nonempty subsets of a BPPMS $(\pounds,\Im,\rho_c)$ and $\mathbb{S} : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ be a self-covariant map. Then $\mathbb{S}$ is said to be SCC-covariant map with respect to $(\mathscr{A},\mathscr{P})$ and $(\mathscr{B},\mathscr{Q})$, if

(i) $\mathbb{S}(\mathscr{A}\cup\mathscr{B}) \subseteq \mathscr{A}\cup\mathscr{B}$ and $\mathbb{S}(\mathscr{P}\cup\mathscr{Q}) \subseteq \mathscr{P}\cup\mathscr{Q}$;

(ii) $\mathbb{S}(\mathscr{A}\cup\mathscr{B})$ and $\mathbb{S}(\mathscr{P}\cup\mathscr{Q})$ are closed in $\pounds\cup\Im$.

Definition 3.3: Let $(\pounds,\Im,\rho_c)$ be a BPPMS and a pair $(\wp,\varpi)$ is called

(a) a coupled fixed point of covariant mapping $\Omega : \left(\pounds^2,\Im^2\right) \rightrightarrows (\pounds,\Im)$ if $\Omega(\wp,\varpi) = \wp$, $\Omega(\varpi,\wp) = \varpi$ for $(\wp,\varpi) \in \pounds^2\cup\Im^2$;

(a_i) a strong coupled fixed point of covariant mapping $\Omega : \left(\pounds^2,\Im^2\right) \rightrightarrows (\pounds,\Im)$ if $(\wp,\varpi)$ is coupled fixed point and $\wp = \varpi$ *i.e* $\Omega(\wp,\wp) = \wp$;

(b) a coupled coincident point of $\Omega : \left(\pounds^2,\Im^2\right) \rightrightarrows (\pounds,\Im)$ and $\Lambda : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ if $\Omega(\wp,\varpi) = \Lambda\wp$, $\Omega(\varpi,\wp) = \Lambda\varpi$;

(b_i) a strong coupled coincident point of $\Omega : \left(\pounds^2,\Im^2\right) \rightrightarrows (\pounds,\Im)$ and $\Lambda : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ if $\wp = \varpi$. *i.e* $\Omega(\wp,\wp) = \Lambda\wp$;

(c) a coupled common fixed point of $\Omega : \left(\pounds^2,\Im^2\right) \rightrightarrows (\pounds,\Im)$ and $\Lambda : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ if $\Omega(\wp,\varpi) = \Lambda\wp = \wp$, $\Omega(\varpi,\wp) = \Lambda\varpi = \varpi$;

(c_i) a strong coupled common fixed point of $\Omega : \left(\pounds^2,\Im^2\right) \rightrightarrows (\pounds,\Im)$ and $\Lambda : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ if $\wp = \varpi$. *i.e* $\Omega(\wp,\wp) = \Lambda\wp = \wp$;

(d) the pair (Ω,Λ) is weakly compatible if $\Lambda(\Omega(\wp,\varpi)) = \Omega(\Lambda\wp,\Lambda\varpi)$ and $\Lambda(\Omega(\varpi,\wp)) = \Omega(\Lambda\varpi,\Lambda\wp)$ whenever $\Omega(\wp,\varpi) = \Lambda\wp$, $\Omega(\varpi,\wp) = \Lambda\varpi$

Definition 3.4: Let $\mathscr{A},\mathscr{B}$ and $\mathscr{P},\mathscr{Q}$ be a nonempty subsets of a BPPMS $(\pounds,\Im,\rho_c)$ and $\mathbb{S} : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ is a SCC-covariant map on $\pounds\cup\Im$ with respect to $(\mathscr{A},\mathscr{P})$ and $(\mathscr{B},\mathscr{Q})$. The contravariant map $\Omega : \left(\pounds^2,\Im^2\right) \rightrightarrows (\pounds,\Im)$ which is cyclic with respect to $(\mathscr{A},\mathscr{P})$ and $(\mathscr{B},\mathscr{Q})$ is said to be $\mathbb{S}$-coupled cyclic mapping if it satisfies the following inequality.

$$\rho_c(\Omega(\mathfrak{z},\mathfrak{e}),\Omega(\mathfrak{x},\mathfrak{y}),c) \leq \frac{\ell}{2}\mathscr{G}(\rho_c(\mathbb{S}\mathfrak{x},\mathbb{S}\mathfrak{z},c)+\rho_c(\mathbb{S}\mathfrak{y},\mathbb{S}\mathfrak{e},c)) \tag{1}$$

for all $\mathscr{G} \in \mathscr{K}$, $\ell \in (0,1)$, $\mathfrak{x} \in \mathscr{A}, \mathfrak{z} \in \mathscr{P}$ and $\mathfrak{y} \in \mathscr{B}, \mathfrak{e} \in \mathscr{Q}$.

Theorem 3.5: Let $\mathscr{A}, \mathscr{B}$ and $\mathscr{P}, \mathscr{Q}$ be a nonempty closed subsets of a complete BPPMS $(\pounds, \mathfrak{I}, \rho_c)$, suppose $\mathbb{S} : (\pounds, \mathfrak{I}) \rightrightarrows (\pounds, \mathfrak{I})$ be a SCC-covariant map on $\pounds \cup \mathfrak{I}$ with respect to $(\mathscr{A}, \mathscr{P})$ and $(\mathscr{B}, \mathscr{Q})$ and $\Omega : \left(\pounds^2, \mathfrak{I}^2\right) \rightrightarrows (\pounds, \mathfrak{I})$ be a $\mathbb{S}$-coupled cyclic mapping with respect to $(\mathscr{A}, \mathscr{P})$ and $(\mathscr{B}, \mathscr{Q})$), then

(i) $\mathbb{S}(\mathscr{A} \cup \mathscr{B}) \cap \mathbb{S}(\mathscr{P} \cup \mathscr{Q}) \neq \emptyset$;

(ii) Ω and $\mathbb{S}$ have a coupled coincidence point in $(\mathscr{A} \cup \mathscr{B})^2 \cup (\mathscr{P} \cup \mathscr{Q})^2$;

(iii) If Ω and $\mathbb{S}$ are weakly compatible, then Ω and $\mathbb{S}$ have a unique strong coupled common fixed point in $(\mathscr{A} \cup \mathscr{B})^2 \cap (\mathscr{P} \cup \mathscr{Q})^2$.

Proof Since $\mathscr{A}, \mathscr{B}$ and $\mathscr{P}, \mathscr{Q}$ are non-empty subsets of $(\pounds, \mathfrak{I})$ and Ω is $\mathbb{S}$-coupled cyclic mapping with respect to $(\mathscr{A}, \mathscr{P})$ and $(\mathscr{B}, \mathscr{Q})$, then for $\mathfrak{a}_0 \in \mathscr{A}, \mathfrak{b}_0 \in \mathscr{B}$ and $\mathfrak{c}_0 \in \mathscr{P}, \mathfrak{d}_0 \in \mathscr{Q}$. For each $\kappa \in \mathbb{N}$, define

$$\Omega(\mathfrak{a}_\kappa, \mathfrak{b}_\kappa) = \mathbb{S}\mathfrak{c}_\kappa = \mathfrak{z}_\kappa, \quad \Omega(\mathfrak{c}_\kappa, \mathfrak{d}_\kappa) = \mathbb{S}\mathfrak{a}_{\kappa+1} = \mathfrak{x}_{\kappa+1}$$
$$\Omega(\mathfrak{b}_\kappa, \mathfrak{a}_\kappa) = \mathbb{S}\mathfrak{d}_\kappa = \mathfrak{e}_\kappa, \quad \Omega(\mathfrak{d}_\kappa, \mathfrak{c}_\kappa) = \mathbb{S}\mathfrak{b}_{\kappa+1} = \mathfrak{y}_{\kappa+1}$$

Then $(\{\mathfrak{x}_\kappa\}, \{\mathfrak{z}_\kappa\}), (\{\mathfrak{y}_\kappa\}, \{\mathfrak{e}_\kappa\})$ are bisequences in $(\mathscr{A}, \mathscr{P})$ and $(\mathscr{B}, \mathscr{Q})$.

For all $\kappa, \iota \in \mathbb{Z}^+$ and from (1), we get

$$\begin{aligned}
\rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_\kappa, c) &= \rho_c(\Omega(\mathfrak{c}_{\kappa-1}, \mathfrak{d}_{\kappa-1}), \Omega(\mathfrak{a}_\kappa, \mathfrak{b}_\kappa), c) \\
&\leq \frac{\ell}{2}\mathscr{G}(\rho_c(\mathbb{S}\mathfrak{a}_\kappa, \mathbb{S}\mathfrak{c}_{\kappa-1}, c) + \rho_c(\mathbb{S}\mathfrak{b}_\kappa, \mathbb{S}\mathfrak{d}_{\kappa-1}, c)) \\
&\leq \frac{\ell}{2}\mathscr{G}(\rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c) + \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c)) \qquad (2)
\end{aligned}$$

Similarly, we can prove

$$\rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_\kappa, c) \leq \ \leq \ \frac{\ell}{2}\mathscr{G}(\rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c) + \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c)) \qquad (3)$$

Combining (2) and (3), we have

$$\begin{aligned}
\mathfrak{D}_\kappa = \rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_\kappa, c) + \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_\kappa, c) &\leq \ell\mathscr{G}(\rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c) + \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c)) \\
&< \ell(\rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c) + \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c)) \\
&\leq \ell^2(\rho_c(\mathfrak{x}_{\kappa-1}, \mathfrak{z}_{\kappa-1}, c) + \rho_c(\mathfrak{y}_{\kappa-1}, \mathfrak{e}_{\kappa-1}, c)) \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&\leq \ell^{2\kappa}\left(\rho_c\left(\mathfrak{x}_0,\mathfrak{z}_0,c\right)+\rho_c\left(\mathfrak{y}_0,\mathfrak{e}_0,c\right)\right)\\
&\leq \ell^{2\kappa}\mathfrak{D}_0 \to 0 \text{ as } \kappa\to\infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_\kappa,c) &= \rho_c(\Omega\left(\mathfrak{c}_\kappa,\mathfrak{d}_\kappa\right),\Omega\left(\mathfrak{a}_\kappa,\mathfrak{b}_\kappa\right),c)\\
&\leq \frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c\right)+\rho_c\left(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c\right)\right)\\
&< \frac{\ell}{2}\left(\rho_c\left(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c\right)+\rho_c\left(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c\right)\right). \qquad (4)
\end{aligned}$$

Similarly, we can prove

$$\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_\kappa,c) < \frac{\ell}{2}\left(\rho_c\left(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c\right)+\rho_c\left(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c\right)\right). \qquad (5)$$

Combining (4) and (5), we have

$$\begin{aligned}
\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_\kappa,c)+\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_\kappa,c) &\leq \ell\left(\rho_c\left(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c\right)+\rho_c\left(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c\right)\right)\\
&\leq \ell^{2\kappa+1}\mathfrak{D}_0 \to 0 \text{ as } \kappa\to\infty.
\end{aligned}$$

Let $\kappa,\iota\in\mathbb{N}$ with $\iota>\kappa$. By Axiom (d) of definition (2), we have

$$\begin{aligned}
&\rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_\kappa,c)+\rho_c(\mathfrak{y}_{\kappa+\iota},\mathfrak{e}_\kappa,c)\\
\leq\; &\rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_\kappa,c)\\
&+\rho_c(\mathfrak{y}_{\kappa+\iota},\mathfrak{e}_{\kappa+1},c)+\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_{\kappa+1},c)+\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_\kappa,c)\\
\leq\; &(\rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{y}_{\kappa+\iota},\mathfrak{e}_{\kappa+1},c))\\
&+(\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_{\kappa+1},c))\\
&+(\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_\kappa,c)+\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_\kappa,c))\\
\leq\; &(\rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+2},c)+\rho_c(\mathfrak{y}_{\kappa+\iota},\mathfrak{e}_{\kappa+2},c))\\
&+(\rho_c(\mathfrak{x}_{\kappa+2},\mathfrak{z}_{\kappa+2},c)+\rho_c(\mathfrak{y}_{\kappa+2},\mathfrak{e}_{\kappa+2},c))\\
&+(\rho_c(\mathfrak{x}_{\kappa+2},\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{y}_{\kappa+2},\mathfrak{e}_{\kappa+1},c))+(\ell^{2\kappa+2}+\ell^{2\kappa+1})\mathfrak{D}_0\\
\leq\; &(\rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+2},c)+\rho_c(\mathfrak{y}_{\kappa+\iota},\mathfrak{e}_{\kappa+2},c))\\
&+(\ell^{2\kappa+4}+\ell^{2\kappa+3}+\ell^{2\kappa+2}+\ell^{2\kappa+1})\mathfrak{D}_0
\end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&\leq (\rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+\iota-1},c)+\rho_c(\mathfrak{y}_{\kappa+\iota},\mathfrak{e}_{\kappa+\iota-1},c))\\
&\quad +(\ell^{2\kappa+2\iota-2}+\cdots+\ell^{2\kappa+1})\mathfrak{D}_0\\
&\leq (\ell^{2\kappa+2\iota-1}+\ell^{2\kappa+2\iota-2}+\ell^{2\kappa+2\iota-3}+\cdots+\ell^{2\kappa+1})\mathfrak{D}_0\\
&\leq \ell^{2\kappa+1}\sum_{i=0}^{\infty}\ell^{i}\mathfrak{D}_0 \to 0 \text{ as } i\to\infty.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\rho_c(\mathfrak{x}_{\kappa},\mathfrak{z}_{\kappa+\iota},c)+\rho_c(\mathfrak{y}_{\kappa},\mathfrak{e}_{\kappa+\iota},c)\\
\leq\ & (\rho_c(\mathfrak{x}_{\kappa},\mathfrak{z}_{\kappa},c)+\rho_c(\mathfrak{y}_{\kappa},\mathfrak{e}_{\kappa},c))+(\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_{\kappa},c)+\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_{\kappa},c))\\
&+(\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_{\kappa+\iota},c)+\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_{\kappa+\iota},c))\\
\leq\ & (\ell^{2\kappa}+\ell^{2\kappa+1})\mathfrak{D}_0+(\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_{\kappa+1},c))\\
&+(\rho_c(\mathfrak{x}_{\kappa+2},\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{y}_{\kappa+2},\mathfrak{e}_{\kappa+1},c))\\
&+(\rho_c(\mathfrak{x}_{\kappa+2},\mathfrak{z}_{\kappa+\iota},c)+\rho_c(\mathfrak{y}_{\kappa+2},\mathfrak{e}_{\kappa+\iota},c))\\
\leq\ & (\ell^{2\kappa}+\ell^{2\kappa+1}+\ell^{2\kappa+2}+\ell^{2\kappa+3})\mathfrak{D}_0\\
&+(\rho_c(\mathfrak{x}_{\kappa+2},\mathfrak{z}_{\kappa+\iota},c)+\rho_c(\mathfrak{y}_{\kappa+2},\mathfrak{e}_{\kappa+\iota},c))\\
\vdots\ &\\
\leq\ & (\ell^{2\kappa}+\ell^{2\kappa+1}+\cdots+\ell^{2\kappa+2\iota-1})\mathfrak{D}_0\\
&+(\rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+\iota},c)+\rho_c(\mathfrak{y}_{\kappa+\iota},\mathfrak{e}_{\kappa+\iota},c))\\
\leq\ & (\ell^{2\kappa}+\ell^{2\kappa+1}+\cdots+\ell^{2\kappa+2\iota-1}+\ell^{2\kappa+2\iota})\mathfrak{D}_0\\
\leq\ & \ell^{2\kappa}\sum_{i=0}^{\infty}\ell^{i}\mathfrak{D}_0 \to 0 \text{ as } i\to\infty,
\end{aligned}
$$

since $0<\ell<1$. Thus, $(\{\mathfrak{x}_\kappa\},\{\mathfrak{z}_\kappa\})$ and $(\{\mathfrak{y}_\kappa\},\{\mathfrak{e}_\kappa\})$ are Cauchy bisequence in $(\mathscr{A},\mathscr{P})$ and $(\mathscr{B},\mathscr{Q})$ respectively. Since $\mathbb{S}(\mathscr{A}\cup\mathscr{B})$ and $\mathbb{S}(\mathscr{P}\cup\mathscr{Q})$ are closed subset of a complete BPPMS $(\pounds,\mathfrak{I},\rho_c)$. Then the sequences $\{\mathfrak{x}_\kappa\}$, $\{\mathfrak{y}_\kappa\}\subseteq\mathbb{S}(\mathscr{A}\cup\mathscr{B})$ and $\{\mathfrak{z}_\kappa\}$, $\{\mathfrak{e}_\kappa\}\subseteq\mathbb{S}(\mathscr{P}\cup\mathscr{Q})$ are convergence in complete BPPMS $(\mathbb{S}(\mathscr{A}\cup\mathscr{B}),\mathbb{S}(\mathscr{P}\cup\mathscr{Q}),\rho_c)$. Therefore, there exist $\mathfrak{a},\mathfrak{b}\in\mathbb{S}(\mathscr{A}\cup\mathscr{B})$

and $\mathfrak{l},\mathfrak{m} \in \mathbb{S}(\mathscr{P} \cup \mathscr{Q})$ such that

$$\lim_{\kappa\to\infty} \mathfrak{x}_\kappa = \lim_{\kappa\to\infty} \mathbb{S}\mathfrak{a}_\kappa = \mathfrak{l},\ \lim_{\kappa\to\infty} \mathfrak{y}_\kappa = \lim_{\kappa\to\infty} \mathbb{S}\mathfrak{b}_\kappa = \mathfrak{m},$$

$$\lim_{\kappa\to\infty} \mathfrak{z}_\kappa = \lim_{\kappa\to\infty} \mathbb{S}\mathfrak{c}_\kappa = \mathfrak{a},\ \lim_{\kappa\to\infty} \mathfrak{e}_\kappa = \lim_{\kappa\to\infty} \mathbb{S}\mathfrak{d}_\kappa = \mathfrak{b}. \tag{6}$$

Then there exists $\kappa_1 \in \mathbb{N}$ with $\rho_c(\mathfrak{x}_\kappa, \mathfrak{l}, c) < \frac{\varsigma}{3}$, $\rho_c(\mathfrak{y}_\kappa, \mathfrak{m}, c) < \frac{\varsigma}{3}$, $\rho_c(\mathfrak{a}, \mathfrak{z}_\kappa, c) < \frac{\varsigma}{3}$ and $\rho_c(\mathfrak{b}, \mathfrak{e}_\kappa, c) < \frac{\varsigma}{3}$ for all $\kappa \geq \kappa_1$ and every $\varsigma > 0$. Since $(\{\mathfrak{x}_\kappa\}, \{\mathfrak{z}_\kappa\})$ and $(\{\mathfrak{y}_\kappa\}, \{\mathfrak{e}_\kappa\})$ are Cauchy bisequence, we get $\rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_\kappa, c) < \frac{\varsigma}{3}$ and $\rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_\kappa, c) < \frac{\varsigma}{3}$. Now consider,

$$\rho_c(\mathfrak{a}, \mathfrak{l}, c) \leq \rho_c(\mathfrak{a}, \mathfrak{z}_\kappa, c) + \rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_\kappa, c) + \rho_c(\mathfrak{x}_\kappa, \mathfrak{l}, c) < \varsigma \tag{7}$$

and

$$\rho_c(\mathfrak{b}, \mathfrak{m}, c) \leq \rho_c(\mathfrak{b}, \mathfrak{e}_\kappa, c) + \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_\kappa, c) + \rho_c(\mathfrak{y}_\kappa, \mathfrak{m}, c) < \varsigma \tag{8}$$

Therefore, from (7) and (8), we have

$$\mathfrak{a} = \mathfrak{l} \text{ and } \mathfrak{b} = \mathfrak{m} \tag{9}$$

it follows that $\mathfrak{a}, \mathfrak{b} \in \mathbb{S}(\mathscr{A} \cup \mathscr{B}) \cap \mathbb{S}(\mathscr{P} \cup \mathscr{Q}) \neq \emptyset$.

Now, since $\mathfrak{a}, \mathfrak{b} \in \mathbb{S}(\mathscr{A} \cup \mathscr{B})$ and $\mathfrak{l}, \mathfrak{m} \in \mathbb{S}(\mathscr{P} \cup \mathscr{Q})$, there exist $\mathfrak{f}, \mathfrak{g} \in \mathscr{A} \cup \mathscr{B}$ and $\mathfrak{p}, \mathfrak{q} \in \mathscr{P} \cup \mathscr{Q}$ such that $\mathbb{S}\mathfrak{f} = \mathfrak{a}, \mathbb{S}\mathfrak{g} = \mathfrak{b}$ and $\mathbb{S}\mathfrak{p} = \mathfrak{l}, \mathbb{S}\mathfrak{q} = \mathfrak{m}$.

From (6) and (9), we get

$$\mathbb{S}\mathfrak{a}_\kappa \to \mathbb{S}\mathfrak{p},\ \mathbb{S}\mathfrak{b}_\kappa \to \mathbb{S}\mathfrak{q}, \mathbb{S}\mathfrak{c}_\kappa \to \mathbb{S}\mathfrak{f},\ \mathbb{S}\mathfrak{d}_\kappa \to \mathbb{S}\mathfrak{g}$$

$$\mathbb{S}\mathfrak{p} = \mathbb{S}\mathfrak{f} \text{ and } \mathbb{S}\mathfrak{q} = \mathbb{S}\mathfrak{g} \tag{10}$$

Claim that $\Omega(\mathfrak{f}, \mathfrak{g}) = \mathfrak{a}, \Omega(\mathfrak{g}, \mathfrak{f}) = \mathfrak{b}$ and $\Omega(\mathfrak{p}, \mathfrak{q}) = \mathfrak{l}, \Omega(\mathfrak{q}, \mathfrak{p}) = \mathfrak{m}$.

By using (1), (10), (d), and the properties of $\mathscr{G}$, we have

$$\rho_c(\mathfrak{a}, \Omega(\mathfrak{f}, \mathfrak{g}), c) \leq \rho_c(\mathfrak{a}, \mathbb{S}\mathfrak{c}_\kappa, c) + \rho_c(\mathbb{S}\mathfrak{a}_\kappa, \mathbb{S}\mathfrak{c}_\kappa, c) + \rho_c(\mathbb{S}\mathfrak{a}_\kappa, \Omega(\mathfrak{f}, \mathfrak{g}), c).$$

Letting $\kappa \to \infty$, we get

$$\begin{aligned} \rho_c(\mathfrak{a}, \Omega(\mathfrak{f}, \mathfrak{g}), c) &\leq \lim_{\kappa\to\infty} \rho_c(\mathbb{S}\mathfrak{a}_\kappa, \Omega(\mathfrak{f}, \mathfrak{g}), c) \\ &\leq \lim_{\kappa\to\infty} \rho_c(\Omega(\mathfrak{c}_{\kappa-1}, \mathfrak{d}_{\kappa-1}), \Omega(\mathfrak{f}, \mathfrak{g}), c) \end{aligned}$$

$$\begin{aligned} &\leq \lim_{\kappa\to\infty} \frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathbb{S}\mathfrak{f},\mathbb{S}\mathfrak{c}_{\kappa-1},c\right)+\rho_c\left(\mathbb{S}\mathfrak{g},\mathbb{S}\mathfrak{d}_{\kappa-1},c\right)\right)\\ &\leq \frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathbb{S}\mathfrak{f},\mathfrak{a},c\right)+\rho_c\left(\mathbb{S}\mathfrak{g},\mathfrak{b},c\right)\right).\end{aligned}$$

Similarly, we have

$$\rho_c\left(\mathfrak{b},\Omega(\mathfrak{g},\mathfrak{f}),c\right) \leq \frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathbb{S}\mathfrak{f},\mathfrak{a},c\right)+\rho_c\left(\mathbb{S}\mathfrak{g},\mathfrak{b},c\right)\right).$$

Therefore,

$$\begin{aligned}\rho_c\left(\mathfrak{a},\Omega(\mathfrak{f},\mathfrak{g}),c\right)+\rho_c\left(\mathfrak{b},\Omega(\mathfrak{g},\mathfrak{f}),c\right) &\leq \ell\mathscr{G}\left(\rho_c\left(\mathbb{S}\mathfrak{f},\mathfrak{a},c\right)+\rho_c\left(\mathbb{S}\mathfrak{g},\mathfrak{b},c\right)\right).\\ &\leq \ell\mathscr{G}\left(0\right)=0.\end{aligned}$$

Consequently, $\Omega(\mathfrak{f},\mathfrak{g})=\mathfrak{a}$ and $\Omega(\mathfrak{g},\mathfrak{f})=\mathfrak{b}$.

Similarly, we can prove $\Omega(\mathfrak{p},\mathfrak{q})=\mathfrak{l},\Omega(\mathfrak{q},\mathfrak{p})=\mathfrak{m}$. Hence, from (9), we get

$$\Omega(\mathfrak{f},\mathfrak{g})=\mathbb{S}\mathfrak{f}=\mathfrak{a}=\mathfrak{l}=\mathbb{S}\mathfrak{p}=\Omega(\mathfrak{p},\mathfrak{q}) \text{ and } \Omega(\mathfrak{g},\mathfrak{f})=\mathbb{S}\mathfrak{g}=\mathfrak{b}=\mathfrak{m}=\mathbb{S}\mathfrak{q}=\Omega(\mathfrak{q},\mathfrak{p}).$$

Therefore,$(\mathfrak{f},\mathfrak{g})\in(\mathscr{A}\cup\mathscr{B})^2\cap(\mathscr{P}\cup\mathscr{Q})^2$ is the coupled coincidence point and $(\mathbb{S}(\mathfrak{f}),\mathbb{S}(\mathfrak{g}))$ is the coupled point of coincidence of Ω and $\mathbb{S}$. Let $(\mathfrak{f}^\star,\mathfrak{g}^\star)$ be another coupled coincidence point of Ω and $\mathbb{S}$. So, we will prove that $\mathbb{S}(\mathfrak{f})=\mathbb{S}(\mathfrak{f}^\star)$ and $\mathbb{S}(\mathfrak{g})=\mathbb{S}(\mathfrak{g}^\star)$. From (1), we have

$$\begin{aligned}\rho_c(\mathbb{S}(\mathfrak{f}),\mathbb{S}(\mathfrak{f}^\star),c) &= \rho_c(\Omega\left(\mathfrak{f},\mathfrak{g}\right),\Omega\left(\mathfrak{f}^\star,\mathfrak{g}^\star\right),c)\\ &\leq \frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathbb{S}\mathfrak{f},\mathbb{S}\mathfrak{f}^\star,c\right)+\rho_c\left(\mathbb{S}\mathfrak{g},\mathbb{S}\mathfrak{g}^\star,c\right)\right).\end{aligned}$$

Therefore,

$$\rho_c\left(\mathbb{S}\mathfrak{f},\mathbb{S}\mathfrak{f}^\star,c\right)+\rho_c\left(\mathbb{S}\mathfrak{g},\mathbb{S}\mathfrak{g}^\star,c\right) \leq \ell\mathscr{G}\left(\rho_c\left(\mathbb{S}\mathfrak{f},\mathbb{S}\mathfrak{f}^\star,c\right)+\rho_c\left(\mathbb{S}\mathfrak{g},\mathbb{S}\mathfrak{g}^\star,c\right)\right).$$

Using the property of $\mathscr{G}$ and $\ell\in(0,1)$, we get $\mathbb{S}(\mathfrak{f})=\mathbb{S}(\mathfrak{f}^\star)$ and $\mathbb{S}(\mathfrak{g})=\mathbb{S}(\mathfrak{g}^\star)$. Hence, the coupled point of coincidence of Ω and $\mathbb{S}$ is unique. Finally, we prove $\mathbb{S}(\mathfrak{f})=\mathbb{S}(\mathfrak{g})$.

$$\begin{aligned}\rho_c\left(\mathbb{S}(\mathfrak{f}),\mathbb{S}(\mathfrak{g}),c\right) &= \rho_c(\Omega\left(\mathfrak{f},\mathfrak{g}\right),\Omega\left(\mathfrak{g},\mathfrak{f}\right),c)\\ &\leq \frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathbb{S}\mathfrak{f},\mathbb{S}\mathfrak{g},c\right)+\rho_c\left(\mathbb{S}\mathfrak{g},\mathbb{S}\mathfrak{f},c\right)\right)\end{aligned}$$

by the property of $\mathscr{G}$, we have

$$\rho_c\left(\mathbb{S}(\mathfrak{f}),\mathbb{S}(\mathfrak{g}),c\right)\leq\ell\rho_c\left(\mathbb{S}(\mathfrak{f}),\mathbb{S}(\mathfrak{g}),c\right)$$

Since, $\ell \in (0,1)$. Thus, $(\mathbb{S}(\mathfrak{f}),\mathbb{S}(\mathfrak{f}))$ is the unique coupled point of coincidence of the mappings Ω and $\mathbb{S}$ with respect to $\mathscr{A}\cup\mathscr{B}$ and $\mathscr{P}\cup\mathscr{Q}$. Now, we show that Ω and $\mathbb{S}$ have unique strong coupled common fixed point. For this let $\mathbb{S}(\mathfrak{f})=\mathfrak{h}$, then, we have $\mathfrak{h}=\mathbb{S}(\mathfrak{f})=\Omega(\mathfrak{f},\mathfrak{f})$ by the weakly compatibility of Ω and $\mathbb{S}$, we have

$$\mathbb{S}\mathfrak{h}=\mathbb{S}(\mathbb{S}(\mathfrak{f}))=\mathbb{S}\Omega(\mathfrak{f},\mathfrak{f})=\Omega(\mathbb{S}\mathfrak{f},\mathbb{S}\mathfrak{f})=\Omega(\mathfrak{h},\mathfrak{h})$$

Thus, $(\mathbb{S}\mathfrak{h},\mathbb{S}\mathfrak{h})$ is coupled point of coincidence of Ω and $\mathbb{S}$. By the uniqueness of coupled point of coincidence of Ω and $\mathbb{S}$, we have $\mathbb{S}\mathfrak{h}=\mathbb{S}\mathfrak{f}$. Thus, we obtain $\mathfrak{h}=\mathbb{S}\mathfrak{h}=\Omega(\mathfrak{h},\mathfrak{h})$. Therefore, $(\mathfrak{h},\mathfrak{h})$ is the unique strong coupled common fixed point of Ω and $\mathbb{S}$.

Corollary 3.6: Let $\mathscr{A},\mathscr{B}$ and $\mathscr{P},\mathscr{Q}$ be a nonempty closed subsets of a complete BPPMS $(\pounds,\Im,\rho_c)$, a contravariant map $\Omega:\left(\pounds^2,\Im^2\right)\rightrightarrows(\pounds,\Im)$ be satisfying coupled cyclic type contraction with respect to $(\mathscr{A},\mathscr{P})$ and $(\mathscr{B},\mathscr{Q})$, then Ω has a unique strong coupled fixed point in $(\mathscr{A}\cup\mathscr{B})^2\cap(\mathscr{P}\cup\mathscr{Q})^2$.

Proof Following the lines of Theorem (3.5), by taking as a $\mathbb{S}=I_{\pounds\cup\Im}$.

Example 3.7: Let $\pounds=\{(0,0),(0,1),(1,0),(1,1)\}$ and $\Im=\{(0,0),(0,2),(2,0),(2,2)\}$ be equipped with $\rho_c(\sigma,\eta,c)=c\,(|\mathfrak{x}-\mathfrak{z}|+|\mathfrak{y}-\mathfrak{e}|)$ for all $\sigma=(\mathfrak{x},\mathfrak{y})\in\pounds$, $\eta=(\mathfrak{z},\mathfrak{e})\in\Im$ and $c>0$. Then, $(\pounds,\Im,\aleph)$ is a complete BPPMS.

Let $\mathscr{A}=\{(0,0),(0,1)\},\mathscr{B}=\{(0,0),(1,0)\}$, $\mathscr{P}=\{(0,0),(0,2)\}$ and $\mathscr{Q}=\{(0,0),(2,0)\}$. Then $\mathscr{A},\mathscr{B}$ and $\mathscr{P},\mathscr{Q}$ are closed subsets of $\pounds$ and $\Im$ respectively.

Define $\Omega:\pounds^2\cup\Im^2\rightleftarrows\pounds\cup\Im$ given by

$$\Omega((a,b),(x,y))=\begin{cases}(0,0) & \text{if } (a,b),(x,y)\in\pounds^2\cap\Im^2,\\ (0,1) & \text{Otherwise,}\end{cases}$$

Let $\mathbb{S}:\pounds\cup\Im\rightleftarrows\pounds\cup\Im$ as $\mathbb{S}((\mathfrak{x},\mathfrak{z}))=\begin{cases}(0,0) & \text{if } \mathfrak{x}=\mathfrak{z}\in\pounds\cup\Im,\\ (0,1) & \text{if } (\mathfrak{x},\mathfrak{z})\in\pounds\times\pounds,\\ (0,2) & \text{if } (\mathfrak{x},\mathfrak{z})\in\Im\times\Im.\end{cases}$

Observer that $\mathbb{S}(\mathscr{A}\cup\mathscr{B})=\{(0,0),(0,1)\}\subseteq\mathscr{A}\cup\mathscr{B}$ and $\mathbb{S}(\mathscr{P}\cup\mathscr{Q})=\{(0,0),(0,2)\}\subseteq\mathscr{P}\cup\mathscr{Q}$ are closed in $\pounds\cup\Im$. Hence $\mathbb{S}$ is a SCC-map. For all $(\mathfrak{x},\mathfrak{z}),(\varpi,\wp)\in(\mathscr{A}\cup\mathscr{B})^2\cap(\mathscr{P}\cup\mathscr{Q})^2$, we have $\Omega(\mathfrak{x},\mathfrak{z}) = (0,0) \in \mathscr{P} \cup \mathscr{Q}$ and $\Omega(\varpi,\wp) = (0,0) \in \mathscr{A} \cup \mathscr{B}$. Which show that Ω is $\mathbb{S}$-coupled cyclic with respect to $(\mathscr{A},\mathscr{P})$ and $(\mathscr{B},\mathscr{Q})$.

Obviously, $\Omega((0,0),(0,0)) = (0,0) = \mathbb{S}((0,0))$ implies that $((0,0),(0,0))$ is a strong coupled coincidence point of Ω and $\mathbb{S}$

also $\Omega((0,0),(0,0)) = \Omega(\mathbb{S}((0,0)),\mathbb{S}((0,0))) = \mathbb{S}\Omega((0,0),(0,0)) = \mathbb{S}((0,0)) = (0,0)$ then $(\Omega,\mathbb{S})$ is ω-compt. After that, $\mathscr{G} : [0,\infty) \to [0,\infty)$ as $\mathscr{G}(t) = \frac{27}{2}\log(\frac{2t}{3}+1)$, $c = 2 > 0$ and $\ell = \frac{1}{2}$. Since $(\Omega((a,b),(x,y)),\Omega((p,q),(r,s)))$, $(\mathbb{S}((p,q)),\mathbb{S}((a,b)))$, $(\mathbb{S}((r,s)),\mathbb{S}((x,y))) \in (\mathscr{A}\cup\mathscr{B})^2 \cup (\mathscr{P}\cup\mathscr{Q})^2$ then $\rho_c(\Omega((a,b),(x,y)),\Omega((p,q),(r,s)),c)$, $\rho_c(\mathbb{S}((p,q)),\mathbb{S}((a,b)),c)$, $\rho_c(\mathbb{S}((r,s)),\mathbb{S}((x,y)),c) \in \{(0,0),(0,1),(0,2)\}$. Hence, one can easily check that the contractive condition Eq. (1) is satisfied for every $\mathfrak{x} \in \mathscr{A}, \mathfrak{z} \in \mathscr{P}$ and $\mathfrak{y} \in \mathscr{B}, \mathfrak{e} \in \mathscr{Q}$. Thus, all the conditions of Theorem 3.5 are fulfilled and Ω and $\mathbb{S}$ have a strong unique common fixed point in $(\mathscr{A}\cup\mathscr{B}) \cap (\mathscr{P}\cup\mathscr{Q})$. Here $\Omega((0,0),(0,0)) = \mathbb{S}((0,0)) = (0,0)$.

3.1. Application to Integral Equations.

In this section, we investigate the existence of a unique solution to a Differential Equation as an application of Corollary 3.6.

Think about the Differential Equation.

$$\frac{d^2\mathfrak{y}}{d\delta^2} + \mathscr{F}(\delta,\mathfrak{y}(\delta),\mathfrak{y}(\delta)) = 0,\ \delta \in \mathscr{E}_1 \cup \mathscr{E}_2 \tag{11}$$

The associated Green's function $\mathscr{G} : \mathscr{E}_1^2 \cup \mathscr{E}_2^2 \to \mathbb{R}^+$ to Eq. 11, can be defined as follows:

$$\mathscr{G}(s,t) = \begin{cases} s(1-t) & \text{if} \quad s \le t \in \mathscr{E}_1 \cup \mathscr{E}_2 \\ t(1-s) & \text{if} \quad t \le s \in \mathscr{E}_1 \cup \mathscr{E}_2 \end{cases}$$

We have the following properties of the Green's function $\mathscr{G}$:

(*a*) $\mathscr{G}(s,t) \ge 0$ for all $s,t \in \mathscr{E}_1 \cup \mathscr{E}_2$;

(*b*) $\sup_{t\in\mathscr{E}_1\cup\mathscr{E}_2} \int_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(s,\tau)d\tau \le 1$.

Let $\pounds = C(\mathscr{L}^\infty(\mathscr{E}_1)), \Im = C(\mathscr{L}^\infty(\mathscr{E}_2))$ be the set of essential bounded measurable continuous functions on Lebesgue measurable sets $\mathscr{E}_1$ and $\mathscr{E}_2$ with $m(\mathscr{E}_1 \cup \mathscr{E}_2) < \infty$.

Define $\rho_c : \pounds \times \Im \times (0,\infty) \to \mathbb{R}^+$ as $\rho_c(\ell,\sigma,c) = c.\sup_{\mathfrak{v}\in\mathscr{E}_1\cup\mathscr{E}_2} |\ell(\mathfrak{v}) - \sigma(\mathfrak{v})|$ for all $\ell \in \pounds, \sigma \in \Im$, $c \in (0,\infty)$. Therefore, $(\pounds,\Im,\rho_c)$ is a complete bipolar parametric metric space.

Let $\mathscr{G} : [0,\infty) \to [0,\infty)$ as $\mathscr{G}(\mathfrak{x}) = \mathfrak{x}$ be a continuous function. The associated integral operator $\Gamma : \left(\pounds^2, \mathfrak{I}^2\right) \rightrightarrows (\pounds, \mathfrak{I})$ to Eq. 11 is defined by

$$\Gamma(\mathfrak{z},\mathfrak{w})(\mathfrak{v}) = \int\limits_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(\mathfrak{v},\mathfrak{u})\mathscr{F}(\mathfrak{u},\mathfrak{z}(\mathfrak{u}),\mathfrak{w}(\mathfrak{u}))d\mathfrak{u},\ \mathfrak{u} \in \mathscr{E}_1 \cup \mathscr{E}_2.$$

It is noted that the operator Γ has a coupled fixed point that solves Eq.11. The condition under which the Differential Equation has a solution is given by the following theorem.

Theorem 3.1.1: Let the function $\mathscr{F} : (\mathscr{E}_1 \cup \mathscr{E}_2) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the following condition:

$|\mathscr{F}(\mathfrak{u},\mathfrak{z}(\mathfrak{u}),\mathfrak{w}(\mathfrak{u})) - \mathscr{F}(\mathfrak{u},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))| \leq \frac{\ell}{2c}\left(\rho_c(\mathfrak{x},\mathfrak{z},c) + \rho_c(\mathfrak{y},\mathfrak{w},c)\right)$ where $\ell \in (0,1)$, $c > 0$ for all $\mathfrak{x} \in \mathscr{A}, \mathfrak{y} \in \mathscr{B}, \mathfrak{z} \in \mathscr{P}, \mathfrak{w} \in \mathscr{Q}$ where $\mathscr{A},\mathscr{B}$ and $\mathscr{P},\mathscr{Q}$ are closed subsets of $\pounds$ and $\mathfrak{I}$ respectively. Then the Differential Equation 11 has a solution.

Proof To accomplish this proof, Corollary 3.6 will be used. The operator $\Gamma : \left(\pounds^2, \mathfrak{I}^2\right) \rightrightarrows (\pounds, \mathfrak{I})$ defined above is continuous since the function $\mathscr{F}$ is continuous. We continue as follows to demonstrate that the mapping Γ forms a coupled cyclic contraction: Using the inequalities, $(a),(b)$ and for every $\mathfrak{v} \in \mathscr{E}_1 \cup \mathscr{E}_2$, we have

$$\begin{aligned}
|\Gamma(\mathfrak{z},\mathfrak{w})(\mathfrak{v}) - \Gamma(\mathfrak{x},\mathfrak{y})(\mathfrak{v})| &= |\int\limits_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(\mathfrak{v},\mathfrak{u})\left(\mathscr{F}(\mathfrak{u},\mathfrak{z}(\mathfrak{u}),\mathfrak{w}(\mathfrak{u})) - \mathscr{F}(\mathfrak{u},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))\right)d\mathfrak{u}| \\
&\leq \int\limits_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(\mathfrak{v},\mathfrak{u})|\mathscr{F}(\mathfrak{u},\mathfrak{z}(\mathfrak{u}),\mathfrak{w}(\mathfrak{u})) - \mathscr{F}(\mathfrak{u},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))|d\mathfrak{u} \\
&\leq \int\limits_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(\mathfrak{v},\mathfrak{u})\frac{\ell}{2c}\left(\rho_c(\mathfrak{x},\mathfrak{z},c) + \rho_c(\mathfrak{y},\mathfrak{w},c)\right)d\mathfrak{u} \\
&\leq \frac{\ell}{2c}\left(\rho_c(\mathfrak{x},\mathfrak{z},c) + \rho_c(\mathfrak{y},\mathfrak{w},c)\right)\int\limits_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(\mathfrak{v},\mathfrak{u})d\mathfrak{u}.
\end{aligned}$$

Therefore,

$$c.\sup_{\mathfrak{v}\in\mathscr{E}_1\cup\mathscr{E}_2} |\Gamma(\mathfrak{z},\mathfrak{w})(\mathfrak{v}) - \Gamma(\mathfrak{x},\mathfrak{y})(\mathfrak{v})| \leq \frac{\ell}{2}\left(\rho_c(\mathfrak{x},\mathfrak{z},c) + \rho_c(\mathfrak{y},\mathfrak{w},c)\right)\sup_{\mathfrak{v}\in\mathscr{E}_1\cup\mathscr{E}_2}\int\limits_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(\mathfrak{v},\mathfrak{u})d\mathfrak{u}.$$

which implies that

$$\rho_c(\Gamma(\mathfrak{z},\mathfrak{w})(\mathfrak{v}),\Gamma(\mathfrak{x},\mathfrak{y})(\mathfrak{v}),c) \leq \frac{\ell}{2}\mathscr{G}\left(\rho_c(\mathfrak{x},\mathfrak{z},c) + \rho_c(\mathfrak{y},\mathfrak{w},c)\right).$$

Hence, all the conditions of Corollary (3.6) hold, we conclude that Γ has a unique coupled solution in $(\mathscr{A}\cup\mathscr{B})^2\cap(\mathscr{P}\cup\mathscr{Q})^2$ to the integral equation (11).

3.2. Applications to Homotopy.

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 3.2.1: Let $(\pounds,\mathfrak{I},\rho_c)$ be a complete BPPMS, $((\mathscr{S},\mathscr{G}),(\mathscr{I},\mathscr{L}))$ and $\left((\overline{\mathscr{S}},\overline{\mathscr{G}}),(\overline{\mathscr{I}},\overline{\mathscr{L}})\right)$ be an open and closed subsets of $(\pounds,\mathfrak{I})$ such that $((\mathscr{S},\mathscr{G}),(\mathscr{I},\mathscr{L}))\subseteq\left((\overline{\mathscr{S}},\overline{\mathscr{G}}),(\overline{\mathscr{I}},\overline{\mathscr{L}})\right)$. Suppose $\mathscr{H}_c:(\overline{\mathscr{S}}\cup\overline{\mathscr{G}})^2\cup(\overline{\mathscr{I}}\cup\overline{\mathscr{L}})^2\times[0,1]\to\pounds\cup\mathfrak{I}$ be an operator with following conditions are satisfying,

$\star_1$) $\mathfrak{x}\neq\mathscr{H}_c(\mathfrak{x},\mathfrak{e},s)$, $\mathfrak{e}\neq\mathscr{H}_c(\mathfrak{e},\mathfrak{x},s)$, for each $\mathfrak{x}\in\partial(\mathscr{S}\cup\mathscr{G}),\mathfrak{e}\in\partial(\mathscr{I}\cup\mathscr{L})$ and $s\in[0,1]$ (Here $\partial(\mathscr{S}\cup\mathscr{G})\cup\partial(\mathscr{I}\cup\mathscr{L})$ is boundary of $(\mathscr{S}\cup\mathscr{G})\cup(\mathscr{I}\cup\mathscr{L})$ in $\pounds\cup\mathfrak{I}$);

$\star_2$) for all $\mathfrak{x}\in\overline{\mathscr{S}},\mathfrak{e}\in\overline{\mathscr{G}},\imath\in\overline{\mathscr{I}},\varkappa\in\overline{\mathscr{L}}$, $s\in[0,1]$ and $\ell\in(0,1)$ such that

$$\rho_c\left(\mathscr{H}_c(\mathfrak{x},\mathfrak{e},s),\mathscr{H}_c(\imath,\varkappa,s),c\right)\leq\frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathfrak{x},\imath,c\right)+\rho_c\left(\mathfrak{e},\varkappa,c\right)\right)$$

$\star_3$) $\exists\, M\geq0\ni\rho_c\left(\mathscr{H}_c(\mathfrak{x},\mathfrak{e},s),\mathscr{H}_c(\imath,\varkappa,t),c\right)\leq Mc|s-t|$

for every $\mathfrak{x}\in\overline{\mathscr{S}},\mathfrak{e}\in\overline{\mathscr{G}},\imath\in\overline{\mathscr{I}},\varkappa\in\overline{\mathscr{L}}$ and $s,t\in[0,1]$.

Then $\mathscr{H}_c(.,0)$ has a coupled fixed point $\iff$ $\mathscr{H}_c(.,1)$ has a coupled fixed point.

Proof Let the set

$$\Theta=\left\{\ s\in[0,1]:\mathscr{H}_c(\mathfrak{x},\mathfrak{e},s)=\mathfrak{x},\mathscr{H}_c(\mathfrak{e},\mathfrak{x},s)=\mathfrak{e}\text{ for some }(\mathfrak{x},\mathfrak{e})\in(\mathscr{S}\cup\mathscr{G})^2\cup(\mathscr{I}\cup\mathscr{L})^2\ \right\}$$
$$\Upsilon=\left\{\ t\in[0,1]:\mathscr{H}_c(\imath,\varkappa,t)=\imath,\mathscr{H}_c(\varkappa,\imath,t)=\varkappa\text{ for some }(\imath,\varkappa)\in(\mathscr{S}\cup\mathscr{G})^2\cup(\mathscr{I}\cup\mathscr{L})^2\ \right\}.$$

Suppose that $\mathscr{H}_c(.,0)$ has a coupled fixed point in $(\mathscr{S}\cup\mathscr{G})^2\cup(\mathscr{I}\cup\mathscr{L})^2$, we have that $(0,0)\in\Theta^2\cap\Upsilon^2$. Now we show that $\Theta^2\cap\Upsilon^2$ is both closed and open in $[0,1]$ and hence by the connectedness $\Theta=\Upsilon=[0,1]$. As a result, $\mathscr{H}_c(.,1)$ has a coupled fixed point in $\Theta^2\cap\Upsilon^2$. First we show that $\Theta^2\cap\Upsilon^2$ closed in $[0,1]$. To see this, Let $(\{a_p\}_{p=1}^{\infty},\{x_p\}_{p=1}^{\infty})\subseteq(\Theta,\Upsilon)$ with $(a_p,x_p)\to(\alpha,\alpha)\in[0,1]$ as $p\to\infty$. We must show that $(\alpha,\alpha)\in\Theta^2\cap\Upsilon^2$.

Since $(a_p,x_p)\in(\Theta,\Upsilon)$ for $p=0,1,2,3,\cdots$, there exists sequences $\left(\{\mathfrak{x}_p\},\{\imath_p\}\right)$ and $\left(\{\mathfrak{e}_p\},\{\varkappa_p\}\right)$ with $\mathfrak{x}_p=\mathscr{H}_c(\mathfrak{x}_p,\mathfrak{e}_p,a_p)$, $\mathfrak{e}_p=\mathscr{H}_c(\mathfrak{e}_p,\mathfrak{x}_p,a_p)$ and $\imath_p=\mathscr{H}_c(\imath_p,\varkappa_p,x_p)$, $\varkappa_p=\mathscr{H}_c(\varkappa_p,\varkappa_p,x_p)$.

Consider

$$\begin{aligned}
& \rho_c(\mathfrak{x}_p,\iota_{p+1},c) \\
= \ & \rho_c\left(\mathscr{H}_c(\mathfrak{x}_p,\mathfrak{e}_p,a_p),\mathscr{H}_c(\iota_{p+1},\varkappa_{p+1},x_{p+1}),c\right) \\
\leq \ & \rho_c\left(\mathscr{H}_c(\mathfrak{x}_p,\mathfrak{e}_p,a_p),\mathscr{H}_c(\iota_{p+1},\varkappa_{p+1},a_p),c\right) \\
& +\rho_c\left(\mathscr{H}_c(\mathfrak{x}_{p+1},\mathfrak{e}_{p+1},a_{p+1}),\mathscr{H}_c(\iota_{p+1},\varkappa_{p+1},a_p),c\right) \\
& +\rho_c\left(\mathscr{H}_c(\mathfrak{x}_{p+1},\mathfrak{e}_{p+1},a_{p+1}),\mathscr{H}_c(\iota_{p+1},\varkappa_{p+1},x_{p+1}),c\right) \\
\leq \ & \rho_c\left(\mathscr{H}_c(\mathfrak{x}_p,\mathfrak{e}_p,a_p),\mathscr{H}_c(\iota_{p+1},\varkappa_{p+1},a_p),c\right)+Mc|a_{p+1}-a_p|+Mc|a_{p+1}-x_{p+1}|.
\end{aligned}$$

Letting $p\to\infty$, we get

$$\begin{aligned}
\lim_{p\to\infty}\rho_c(\mathfrak{x}_p,\iota_{p+1},c) &\leq \lim_{p\to\infty}\rho_c\left(\mathscr{H}_c(\mathfrak{x}_p,\mathfrak{e}_p,a_p),\mathscr{H}_c(\iota_{p+1},\varkappa_{p+1},a_p),c\right) \\
&\leq \lim_{p\to\infty}\frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathfrak{x}_p,\iota_{p+1},c\right)+\rho_c\left(\mathfrak{e}_p,\varkappa_{p+1},c\right)\right).
\end{aligned}$$

Similarly, we can prove

$$\lim_{p\to\infty}\rho_c(\mathfrak{e}_p,\varkappa_{p+1},c) \leq \lim_{p\to\infty}\frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathfrak{x}_p,\iota_{p+1},c\right)+\rho_c\left(\mathfrak{e}_p,\varkappa_{p+1},c\right)\right).$$

Therefore,

$$\lim_{p\to\infty}\left(\rho_c(\mathfrak{x}_p,\iota_{p+1},c)+\rho_c(\mathfrak{e}_p,\varkappa_{p+1},c\right) \leq \lim_{p\to\infty}\ell\mathscr{G}\left(\rho_c\left(\mathfrak{x}_p,\iota_{p+1},c\right)+\rho_c\left(\mathfrak{e}_p,\varkappa_{p+1},c\right)\right).$$

By the definition of $\mathscr{G}$, it follows that $(1-\ell)\lim_{p\to\infty}\left(\rho_c(\mathfrak{x}_p,\iota_{p+1},c)+\rho_c(\mathfrak{e}_p,\varkappa_{p+1},c\right)=0$.
So that $\lim_{p\to\infty}\rho_c\left(\mathfrak{x}_p,\iota_{p+1},c\right)=0$ and $\lim_{p\to\infty}\rho_c\left(\mathfrak{e}_p,\varkappa_{p+1},c\right)=0$.
We will prove $\left(\{\mathfrak{x}_p\},\{\iota_p\}\right)$ and $\left(\{\mathfrak{e}_p\},\{\varkappa_p\}\right)$ are a Cauchy bisequence. Assume there are $\varepsilon>0$ and $\{q_k\},\{p_k\}$ so that for $p_k>q_k>k$,

$$(12)\quad \rho_c(\mathfrak{x}_{p_k},\iota_{q_k},c)\geq\varepsilon,\ \ \rho_c(\mathfrak{x}_{p_{k-1}},\iota_{q_k},c)<\varepsilon,\rho_c(\mathfrak{e}_{p_k},\varkappa_{q_k},c)\geq\varepsilon,\ \ \rho_c(\mathfrak{e}_{p_{k-1}},\varkappa_{q_k},c)<\varepsilon$$

and

$$(13)\quad \rho_c(\mathfrak{x}_{q_k},\iota_{p_k},c)\geq\varepsilon,\ \ \rho_c(\mathfrak{x}_{q_k},\iota_{p_{k-1}},c)<\varepsilon,\rho_c(\mathfrak{e}_{q_k},\varkappa_{p_k},c)\geq\varepsilon,\ \ \rho_c(\mathfrak{e}_{q_k},\varkappa_{p_{k-1}},c)<\varepsilon$$

By view of (12) and triangle inequality, we get

$$\varepsilon \leq \rho_c(\mathfrak{x}_{p_k},\iota_{q_k},c)$$

$$
\begin{aligned}
&\leq \rho_c(\mathfrak{x}_{p_k},\iota_{p_{k-1}},c)+\rho_c(\mathfrak{x}_{p_{k-1}},\iota_{p_{k-1}},c)+\rho_c(\mathfrak{x}_{p_{k-1}},\iota_{q_k},c)\\
&< \rho_c(\mathfrak{x}_{p_k},\iota_{p_{k-1}},c)+\rho_c\left(\mathscr{H}_c(\mathfrak{x}_{p_{k-1}},\mathfrak{e}_{p_{k-1}},a_{p_{k-1}}),\mathscr{H}_c(\iota_{p_{k-1}},\varkappa_{p_{k-1}},x_{p_{k-1}}),c\right)+\varepsilon\\
&< \rho_c(\mathfrak{x}_{p_k},\iota_{p_{k-1}},c)+Mc|a_{p_{k-1}}-x_{p_{k-1}}|+\varepsilon.
\end{aligned}
$$

Letting $k\to\infty$ in the above inequality, we obtain

$$
\lim_{k\to\infty}\rho_c(\mathfrak{x}_{p_k},\iota_{q_k},c)=\varepsilon \tag{14}
$$

Using (13), one can prove

$$
\lim_{k\to\infty}\rho_c(\mathfrak{x}_{q_k},\iota_{p_k},c)=\varepsilon \tag{15}
$$

Similarly, we can prove

$$
\lim_{k\to\infty}\rho_c(\mathfrak{e}_{p_k},\varkappa_{q_k},c)=\varepsilon,\;\;\lim_{k\to\infty}\rho_c(\mathfrak{e}_{q_k},\varkappa_{p_k},c)=\varepsilon.
$$

For all $k\in\mathbb{N}$, by $(\star_2)$ we have

$$
\begin{aligned}
\lim_{k\to\infty}\left(\rho_c(\mathfrak{x}_{p_k},\iota_{q_k},c)+\rho_c\left(\mathfrak{e}_{p_k},\varkappa_{q_k},c\right)\right) &\leq \ell\lim_{k\to\infty}\mathscr{G}\left(\rho_c(\mathfrak{x}_{p_k},\iota_{q_k},c)+\rho_c\left(\mathfrak{e}_{p_k},\varkappa_{q_k},c\right)\right)\\
&< \ell\lim_{k\to\infty}\left(\rho_c(\mathfrak{x}_{p_k},\iota_{q_k},c)+\rho_c\left(\mathfrak{e}_{p_k},\varkappa_{q_k},c\right)\right).
\end{aligned}
$$

By the definition of $\mathscr{G}$, it follows that $(1-\ell)\lim_{k\to\infty}\left(\rho_c(\mathfrak{x}_{p_k},\iota_{q_k},c)+\rho_c\left(\mathfrak{e}_{p_k},\varkappa_{q_k},c\right)\right)=0$ implies that $\lim_{k\to\infty}\rho_c(\mathfrak{x}_{p_k},\iota_{q_k},c)=0$ and $\lim_{k\to\infty}\rho_c(\mathfrak{e}_{p_k},\varkappa_{q_k},c)=0$ then $\varepsilon=0$ which is contradictory, by applying (14) and (15). Therefore, $\left(\{\mathfrak{x}_p\},\{\iota_p\}\right)\subseteq(\mathscr{S},\mathscr{I})$ and $\left(\{\mathfrak{e}_p\},\{\varkappa_p\}\right)\subseteq(\mathscr{G},\mathscr{L})$ are Cauchy bi-sequences. By completeness, there exist $(a,x)\in\mathscr{S}\times\mathscr{I}$ and $(y,b)\in\mathscr{G}\times\mathscr{L}$ with

$$
\lim_{p\to\infty}\mathfrak{x}_{p+1}=x\quad\lim_{p\to\infty}\iota_{p+1}=a\quad\lim_{p\to\infty}\mathfrak{e}_{p+1}=b\quad\lim_{p\to\infty}\varkappa_{p+1}=y
$$

we have

$$
\begin{aligned}
\rho_c\left(\mathscr{H}_c(a,y,\alpha),x,c\right) &\leq \rho_c\left(\mathscr{H}_c(a,y,\alpha),\iota_p,c\right)+\rho_c(\mathfrak{x}_p,\iota_p,c)+\rho_c(\mathfrak{x}_p,x,c)\\
&\leq \rho_c\left(\mathscr{H}_c(a,y,\alpha),\mathscr{H}_c(\iota_p,\varkappa_p,x_p),c\right)+Mc|a_p-x_p|+\rho_c(\mathfrak{x}_p,x,c).
\end{aligned}
$$

Letting $p\to\infty$ in the above inequality, we have

$$
\begin{aligned}
\rho_c\left(\mathscr{H}_c(a,y,\alpha),x,c\right) &\leq \lim_{k\to\infty}\rho_c\left(\mathscr{H}_c(a,y,\alpha),\mathscr{H}_c(\iota_p,\varkappa_p,\alpha),c\right)\\
&\leq \lim_{k\to\infty}\frac{\ell}{2}\mathscr{G}\left(\rho_c\left(a,\iota_p,c\right)+\rho_c\left(y,\varkappa_p,c\right)\right)
\end{aligned}
$$

$$\leq \frac{\ell}{2}\mathscr{G}(0) = 0$$

it follows that $\rho_c\left(\mathscr{H}_c(a,y,\alpha),x,c\right)=0$ implies that $\mathscr{H}_c(a,y,\alpha)=x$. Similarly, we can prove that $\mathscr{H}_c(y,a,\alpha)=b$ and $\mathscr{H}_c(x,b,\alpha)=a$, $\mathscr{H}_c(b,x,\alpha)=y$.

On the other hand,

$$\rho_c\left(a,x,c\right)=\rho_c\left(\lim_{p\to\infty}\iota_{p+1},\lim_{p\to\infty}\mathfrak{x}_{p+1},c\right)=\lim_{p\to\infty}\rho_c(\mathfrak{x}_{p+1},\iota_{p+1},c)=0$$

and

$$\rho_c\left(y,b,c\right)=\rho_c\left(\lim_{p\to\infty}\varkappa_{p+1},\lim_{p\to\infty}\mathfrak{e}_{p+1},c\right)=\lim_{p\to\infty}\rho_c(\mathfrak{e}_{p+1},\varkappa_{p+1},c)=0$$

Therefore, $x=a$, $y=b$ and hence $(\alpha,\alpha)\in\Theta^2\cap\Upsilon^2$.Clearly $\Theta^2\cap\Upsilon^2$ is closed in $[0,1]$. Let $(\alpha_0,\beta_0)\in\Theta^2\cap\Upsilon^2$, there exists bisequences $(\mathfrak{x}_0,\iota_0)$ and $(\mathfrak{e}_0,\varkappa_0)$ with $\mathfrak{x}_0=\mathscr{H}_c(\mathfrak{x}_0,\mathfrak{e}_0,\alpha_0)$, $\mathfrak{e}_0=\mathscr{H}_c(\mathfrak{e}_0,\mathfrak{x}_0,\alpha_0)$ and $\iota_0=\mathscr{H}_c(\iota_0,\varkappa_0,\beta_0)$, $\varkappa_0=\mathscr{H}_c(\varkappa_0,\iota_0,\beta_0)$. Since $(\mathscr{S}\cup\mathscr{G})^2\cup(\mathscr{I}\cup\mathscr{L})^2$ is open, then there exist $\delta>0$ such that $B_{\rho_c}(\mathfrak{x}_0,\delta)$, $B_{\rho_c}(\mathfrak{e}_0,\delta)$, $B_{\rho_c}(\iota_0,\delta)$ and $B_{\rho_c}(\varkappa_0,\delta)\subseteq(\mathscr{S}\cup\mathscr{G})^2\cup(\mathscr{I}\cup\mathscr{L})^2$. Choose $\alpha\in(\alpha_0-\varepsilon,\alpha_0+\varepsilon)$, $\beta\in(\beta_0-\varepsilon,\beta_0+\varepsilon)$ such that $|\alpha-\alpha_0|\leq\frac{1}{M^p}<\frac{\varepsilon}{2}$, $|\alpha_0-\beta_0|\leq\frac{1}{M^p}<\frac{\varepsilon}{2}$.

Then for, $\iota\in\overline{B}_{\pounds\cup\mathfrak{I}}(\mathfrak{x}_0,\delta)=\{\iota,\iota_0\in\mathfrak{I}/\ell\rho_c(\mathfrak{x}_0,\iota,c)\leq\rho_c(\mathfrak{x}_0,\iota_0,c)+\frac{\delta}{2}\}$,
$\varkappa\in\overline{B}_{\pounds\cup\mathfrak{I}}(\mathfrak{e}_0,\delta)=\{\varkappa,\varkappa_0\in\mathfrak{I}/\ell\rho_c(\mathfrak{e}_0,\varkappa,c)\leq\rho_c(\mathfrak{e}_0,\varkappa_0,c)+\frac{\delta}{2}\}$
$\mathfrak{x}\in\overline{B}_{\pounds\cup\mathfrak{I}}(\delta,\iota_0)=\{\mathfrak{x},\mathfrak{x}_0\in\pounds/\ell\rho_c(\mathfrak{x},\iota_0,c)\leq\rho_c(\mathfrak{x}_0,\iota_0,c)+\frac{\delta}{2}\}$
$\mathfrak{e}\in\overline{B}_{\pounds\cup\mathfrak{I}}(\delta,\varkappa_0)=\{\mathfrak{e},\mathfrak{e}_0\in\pounds/\ell\rho_c(\mathfrak{e},\varkappa_0,c)\leq\rho_c(\mathfrak{e}_0,\varkappa_0,c)+\frac{\delta}{2}\}$

$$\begin{aligned}
\rho_c\left(\mathscr{H}_c(\mathfrak{x},\mathfrak{e},\alpha),\iota_0,c\right) &= \rho_c\left(\mathscr{H}_c(\mathfrak{x},\mathfrak{e},\alpha),\mathscr{H}_c(\iota_0,\varkappa_0,\beta_0),c\right)\\
&\leq \rho_c\left(\mathscr{H}_c(\mathfrak{x},\mathfrak{e},\alpha),\mathscr{H}_c(\iota,\varkappa,\alpha_0),c\right)+\rho_c\left(\mathscr{H}_c(\mathfrak{x}_0,\mathfrak{e}_0,\alpha_0),\mathscr{H}_c(\iota,\varkappa,\alpha_0),c\right)\\
&\qquad \rho_c\left(\mathscr{H}_c(\mathfrak{x}_0,\mathfrak{e}_0,\alpha_0),\mathscr{H}_c(\iota_0,\varkappa_0,\beta_0),c\right)\\
&\leq Mc|\alpha-\alpha_0|+\rho_c\left(\mathscr{H}_c(\mathfrak{x}_0,\mathfrak{e}_0,\alpha_0),\mathscr{H}_c(\iota,\varkappa,\alpha_0),c\right)+Mc|\alpha_0-\beta_0|\\
&\leq \frac{2c}{M^{p-1}}+\rho_c\left(\mathscr{H}_c(\mathfrak{x}_0,\mathfrak{e}_0,\alpha_0),\mathscr{H}_c(\iota,\varkappa,\alpha_0),c\right)
\end{aligned}$$

Letting $p\to\infty$, then we have

$$\begin{aligned}
\rho_c\left(\mathscr{H}_c(\mathfrak{x},\mathfrak{e},\alpha),\iota_0,c\right) &\leq \rho_c\left(\mathscr{H}_c(\mathfrak{x}_0,\mathfrak{e}_0,\alpha_0),\mathscr{H}_c(\iota,\varkappa,\alpha_0),c\right)\\
&\leq \frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathfrak{x}_0,\iota,c\right)+\rho_c\left(\mathfrak{e}_0,\varkappa,c\right)\right)
\end{aligned}$$

Similarly we can prove

$$\rho_c\left(\mathscr{H}_c(\mathfrak{e},\mathfrak{x},\alpha),\varkappa_0,c\right) \leq \frac{\ell}{2}\mathscr{G}\left(\rho_c\left(\mathfrak{x}_0,\iota,c\right)+\rho_c\left(\mathfrak{e}_0,\varkappa,c\right)\right)$$

Therefore,

$$\begin{aligned}\rho_c\left(\mathscr{H}_c(\mathfrak{x},\mathfrak{e},\alpha),\iota_0,c\right)+\rho_c\left(\mathscr{H}_c(\mathfrak{e},\mathfrak{x},\alpha),\varkappa_0,c\right) &\leq \ell\mathscr{G}\left(\rho_c\left(\mathfrak{x}_0,\iota,c\right)+\rho_c\left(\mathfrak{e}_0,\varkappa,c\right)\right)\\ &\leq \ell\left(\rho_c\left(\mathfrak{x}_0,\iota,c\right)+\rho_c\left(\mathfrak{e}_0,\varkappa,c\right)\right)\\ &\leq \rho_c\left(\mathfrak{x}_0,\iota_0,c\right)+\rho_c\left(\mathfrak{e}_0,\varkappa_0,c\right)+\delta \qquad (16)\end{aligned}$$

Similarly, we have

$$\rho_c\left(\mathscr{H}_c(\iota,\varkappa,\beta),\mathfrak{x}_0,c\right)+\rho_c\left(\mathscr{H}_c(\varkappa,\iota,\beta),\mathfrak{e}_0,c\right) \leq \rho_c\left(\mathfrak{x}_0,\iota_0,c\right)+\rho_c\left(\mathfrak{e}_0,\varkappa_0,c\right)+\delta \qquad (17)$$

On the other hand,

$$\rho_c(\mathfrak{x}_0,\iota_0,c)=\rho_c\left(\mathscr{H}_c(\mathfrak{x}_0,\mathfrak{e}_0,\alpha_0),\mathscr{H}_c(\iota_0,\varkappa_0,\beta_0)\right)\leq Mc|\alpha_0-\beta_0|<\frac{c}{M^{p-1}}\to 0 \text{ as } p\to\infty.$$

and

$$\rho_c(\mathfrak{e}_0,\varkappa_0,c)=\rho_c\left(\mathscr{H}_c(\mathfrak{e}_0,\mathfrak{x}_0,\alpha_0),\mathscr{H}_c(\varkappa_0,\iota_0,\beta_0)\right)\leq Mc|\alpha_0-\beta_0|<\frac{c}{M^{p-1}}\to 0 \text{ as } p\to\infty.$$

So $\mathfrak{x}_0=\iota_0$ and $\varkappa_0=\mathfrak{e}_0$ and hence $\alpha=\beta$. Thus for each fixed $\alpha\in(\alpha_0-\varepsilon,\alpha_0+\varepsilon)$, $\mathscr{H}_c(.,\alpha):\overline{B}_{\Theta\cup\Upsilon}(\mathfrak{x}_0,\delta)\to\overline{B}_{\Theta\cup\Upsilon}(\mathfrak{x}_0,\delta)$ and $\mathscr{H}_c(.,\alpha):\overline{B}_{\Theta\cup\Upsilon}(\iota_0,\delta)\to\overline{B}_{\Theta\cup\Upsilon}(\iota_0,\delta)$. Thus, we conclude that $\mathscr{H}_c(.,\alpha)$ has a coupled fixed point in $(\overline{\mathscr{S}}\cup\overline{\mathscr{G}})^2\cup(\overline{\mathscr{I}}\cup\overline{\mathscr{L}})^2$. But this must be in $(\mathscr{S}\cup\mathscr{G})^2\cup(\mathscr{I}\cup\mathscr{L})^2$. Therefore, $(\alpha,\alpha)\in\Theta^2\cap\Upsilon^2$ for $\alpha\in(\alpha_0-\varepsilon,\alpha_0+\varepsilon)$.Hence, $(\alpha_0-\varepsilon,\alpha_0+\varepsilon)\subseteq\Theta^2\cap\Upsilon^2$. Clearly, $\Theta^2\cap\Upsilon^2$ is open in $[0,1]$. For the reverse implication, we use the same strategy.

4. Conclusions

This paper uses contractive mappings of the 𝕊-coupled cyclic contraction functions to demonstrate certain CFPT in the context of complete BPPMS, along with appropriate examples that highlight the main findings. Applications for integral equations and homotopy are also given.

Conflict of Interests

The authors declare that there is no conflict of interests.

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