Available online at http://scik.org
Adv. Fixed Point Theory, 2025, 15:42
https://doi.org/10.28919/afpt/9512
ISSN: 1927-6303

APPLICATIONS OF $\mathscr{C}_G$-CLASS MAPPINGS WITH $\mathscr{F}_G(\varphi, \wp, \varpi)$-CYCLIC COUPLED RATIONAL CONTRACTIONS IN PARTIAL b-METRIC SPACES

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Abstract: The primary goal of this research is to derive generalized $\mathscr{C}_G$-class functions and prove that a common coupled fixed point exists for $\mathscr{F}_G(\varphi, \wp, \varpi)$-cyclic rational contraction in partial b-metric spaces(PbMS). In our study, we use some properties of different control functions. Our findings broaden and unify a number of previous results in the literature. The conclusions are supported by examples. In addition, we provide applications for integral equations and homotopy as well as an explanation of the significance of the obtained results.

Keywords: $\mathscr{C}_G$-class functions; ω-compatible; $\mathscr{F}_G(\varphi, \wp, \varpi)$-cyclic coupled rational contraction; cyclic maps.
2020 AMS Subject Classification: 54H25, 47H10, 54E50.

1. INTRODUCTION

The renowned Banach's contraction principle is a foundational result in fixed point theory, serving as a widely-utilized tool in addressing numerous problems across various branches of mathematics. Over time, numerous extensions of this principle have been documented in different literature. The study of coupled fixed points has seen rapid advancements within metric

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Received July 25, 2025

fixed point theory. The concept was first introduced by Guo et al. in 1987 [1]. In 2003, Kirk et al. [2] introduced cyclic contractions, demonstrating that such contractions possess fixed points. In 2014, Choudhury et al. [3] introduced the concept of strong coupled fixed points and proved some cyclic coupled fixed point results using Kannan-type contractions. In 2017, S. Binayak Choudhury et al. [4] introduced the notion of coupling between two non-empty subsets in a metric space. They showed that these couplings possess strong unique fixed points, provided they satisfy Banach-type or Chatterjea-type contractive inequalities. This concept was later generalized by G. V. R. Babu et al. [5], S. Mary Anushia et al. [6], and Youssef El Bekri et al. [7].

Shukla [11] employed the concepts of b-metric [8, 9] and partial metric spaces [10] to define partial b-metric spaces (PbMS) as a generalization. In PbMS, he demonstrated Kannan-type fixed point theorems and the Banach contraction principle. Subsequently, Mustafa et al. [12] established several common fixed point theorems within this framework. Recently, several authors have obtained fixed point results in PbMS (see [13]-[17]).

In 2014, Ansari [18] introduced the concept of C-class functions and provided proofs for unique fixed point theorems for certain contractive mappings. This initiated significant research in this area (see [19, 25]). In 2022, Leta Bekere Kumssa [26] established fixed point theorems for Suzuki contractions of rational type within the $b_v(s)$-metric space and introduced a new class of functions known as generalized C_G-class functions.

This paper establishes a common coupled fixed point (CCFP) theorem for two mappings that conform to $\mathscr{F}_G(\varphi, \wp, \varpi)$-cyclic rational type contractive constraints using generalized $\mathscr{C}_G$-class functions within PbMS. Furthermore, it delves into a system of non-linear integral equations and examines their applications to homotopy, illustrated with pertinent examples.

What follows is in our subsequent conversations, we compile a few suitable definitions.

2. Preliminaries

Definition 2.1:([11]) Let $\pounds$ be a nonempty set and $\mathfrak{v} \geq 1$ be a given real number. If the following conditions are satisfied for each $æ_1, æ_2, æ_3 \in \pounds$, then a partial b- metric is defined as a function $\varsigma_\rho : \pounds \times \pounds \to [0, \infty)$:

(ς_{ρ_1}) $æ_1 = æ_2$ if and only if $\varsigma_\rho(æ_1,æ_1) = \varsigma_\rho(æ_1,æ_2) = \varsigma_\rho(æ_2,æ_2)$;

(ς_{ρ_2}) $\varsigma_\rho(æ_1,æ_1) \leq \varsigma_\rho(æ_1,æ_2)$;

(ς_{ρ_3}) $\varsigma_\rho(æ_1,æ_2) = \varsigma_\rho(æ_2,æ_1)$;

(ς_{ρ_4}) $\varsigma_\rho(æ_1,æ_2) \leq \mathfrak{v}\left(\varsigma_\rho(æ_1,æ_3) + \varsigma_\rho(æ_3,æ_2) - \varsigma_\rho(æ_3,æ_3)\right)$.

A partial b-metric space is the pair $(£,\varsigma_\rho)$.

Remark: Since a partial metric space is a particular case of a PbMS $(£,\varsigma_\rho)$ when $\mathfrak{v} = 1$, the class of PbMS $(£,\varsigma_\rho)$ is more general than the class of partial metric spaces. Furthermore, since a b-metric space is a particular instance of a PbMS $(£,\varsigma_\rho)$ where the self-distance $P(æ_1;æ_1)$ equals 0, the class of PbMS, represented as $(£,\varsigma_\rho)$, is larger than the class of b-metric spaces. Both a b-metric on £ and a PbMS on £ do not necessarily need to meet the requirements listed in ([11],[12]), as the examples illustrate.

Example 2.2:([11]) Assume that $£ = [0,1)$. $\varsigma_\rho(\mathfrak{z}_1;\mathfrak{z}_2) = [\max\{\mathfrak{z}_1,\mathfrak{z}_2\}]^2 + |\mathfrak{z}_1 - \mathfrak{z}_2|^2$ is the formula to create a function ς_ρ. For every $\mathfrak{z}_1,\mathfrak{z}_2 \in £$. The pair $(£,\varsigma_\rho)$ is called a partial b-metric space when $\mathfrak{v} = 2 > 1$. However, ς_ρ is neither a b-metric nor a partial metric on £.

Definition 2.3:([12]) Every PbMS ς_ρ defines a b-metric d_{ς_ρ}, where

$d_{\varsigma_\rho}(\mathfrak{z}_1,\mathfrak{z}_2) = 2\varsigma_\rho(\mathfrak{z}_1,\mathfrak{z}_2) - \varsigma_\rho(\mathfrak{z}_1,\mathfrak{z}_1) - \varsigma_\rho(\mathfrak{z}_2,\mathfrak{z}_2)$, for all $\mathfrak{z}_1,\mathfrak{z}_2 \in £$

Definition 2.4:([12]) In a PbMS $(£,\varsigma_\rho)$, a sequence $\{æ_p\}$ is defined as follows:

(i) The ς_ρ-convergent toward a target if $\lim_{p\to\infty} \varsigma_\rho(æ,æ_p) = \varsigma_\rho(æ,æ)$ then $æ \in £$.

(ii) In ς_ρ if $\lim_{p,q\to\infty} \varsigma_\rho(æ_p,æ_q)$ exists and is finite, then ς_ρ-Cauchy sequence

(iii) A $(£,\varsigma_\rho)$ PbMS ς_ρ is said to be ς_ρ-complete if and only if, for each ς_ρ-Cauchy sequence $\{æ_p\}$ in £, there is a convergence to a point $æ \in £$ such that

$$\lim_{p,q\to\infty} \varsigma_\rho(æ_p,æ_q) = \lim_{p\to\infty} \varsigma_\rho(æ,æ_p) = \varsigma_\rho(æ,æ)$$

Lemma 2.5:([12]) A PbMS $(£,\varsigma_\rho)$ has a ς_ρ-Cauchy sequence $\{æ_n\}$ solely in the event that it is a ς_ρ-Cauchy sequence in b-metric space $(£,d_{\varsigma_\rho})$.

Lemma 2.6:([12]) If and only if the b-metric space $(£,d_{\varsigma_\rho})$ is ς_ρ-complete, a PbMS $(£,\varsigma_\rho)$ qualifies as ς_ρ-complete. Additionally,

$\lim_{p,q\to\infty} d_{\varsigma_\rho}(æ_p,æ_q) = 0 \iff \lim_{q\to\infty} \varsigma_\rho(æ_q,æ) = \lim_{p\to\infty} \varsigma_\rho(æ_p,æ) = \varsigma_\rho(æ,æ)$.

Definition 2.7:[18] A continuous mapping $\mathscr{F} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a $\mathscr{C}$-class function if for all $\mathfrak{e},\mathfrak{y} \in [0,\infty)$

(1) $\mathscr{F}(\mathfrak{e},\mathfrak{y}) \leq \mathfrak{e}$;
(2) $\mathscr{F}(\mathfrak{e},\mathfrak{y}) = \mathfrak{e} \Rightarrow$ either $\mathfrak{e} = 0$ or $\mathfrak{y} = 0$.

$\mathfrak{C}$ represents the family of all $\mathscr{C}$-class functions.

Definition 2.8:[26] A mapping $\mathscr{F}_G : [0,\infty)\times[0,\infty)\times[0,\infty) \to \mathbb{R}$ is called a generalized $\mathscr{C}_G$-class function if for all $\mathfrak{e},\mathfrak{y},\mathfrak{z} \in [0,\infty)$

(i) $\mathscr{F}_G$ is continuous;
(ii) $\mathscr{F}_G(\mathfrak{e},\mathfrak{y},\mathfrak{z}) \leq \max\{\mathfrak{e},\mathfrak{y}\}$;
(iii) $\mathscr{F}_G(\mathfrak{e},\mathfrak{y},\mathfrak{z}) = \mathfrak{e}$ or $\mathfrak{y} \Rightarrow$ either of $\mathfrak{e},\mathfrak{y}$ or $\mathfrak{z}$ is zero.

$\mathfrak{C}_G$ represents the family of all $\mathscr{C}_G$-class functions.

Example 2.9:[26] In the following, we give some members of $\mathscr{C}_G$ where $\mathscr{F}_G : [0,\infty)\times[0,\infty)\times[0,\infty) \to \mathbb{R}$ is a mapping:

(1) $\mathscr{F}_G(\mathfrak{e},\varsigma,\rho) = \varsigma - \mathfrak{e} - \rho$ where $\frac{\varsigma-\rho}{2} > \mathfrak{e}$. This implies $\mathfrak{e} < \varsigma$ and $\mathscr{F}_G(\mathfrak{e},\varsigma,\rho) = \varsigma$ implies $\mathfrak{e} = 0 = \rho$.
(2) $\mathscr{F}_G(\mathfrak{e},\varsigma,\rho) = \varsigma - \frac{\mathfrak{e}}{1+\rho}$ whenever $\mathfrak{e} < \frac{\varsigma(1+\rho)}{2+\rho}$, $\mathscr{F}_G(\mathfrak{e},\varsigma,\rho) = \varsigma$ implies $\mathfrak{e} = 0$.
(3) $\mathscr{F}_G(\mathfrak{e},\varsigma,\rho) = \varsigma - \Xi(\varsigma) - \Xi(\mathfrak{e})$ for all $\mathfrak{e},\varsigma,\rho \in [0,\infty)$ where $\Xi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous function such that $\varsigma - \Xi(\varsigma) > \mathfrak{e} + \Xi(\mathfrak{e})$, $\Xi(\varsigma) = 0$ and $\Xi(\mathfrak{e}) = 0$ only when $\varsigma = \mathfrak{e} = 0$. $\mathscr{F}_G(\mathfrak{e},\varsigma,\rho) = \varsigma$ implies $\Xi(\varsigma) + \Xi(\mathfrak{e}) = 0 \iff \varsigma = \mathfrak{e} = 0$.

A new category of contractive fixed point results was addressed by Khan et al. [27] and A. H Ansari et al.[18, 19]. In their study they introduced the notion of an altering distance function and ultra altering distance functions which are control functions that alters distance between two points in a metric space.

Definition 2.10:[27] The mapping $\varphi : [0,\infty) \to [0,\infty)$ is called the altering distance function if the following properties are met:

(1) φ is non-decreasing and continuous;
(2) $\varphi(\mathfrak{e}) = 0 \iff \mathfrak{e} = 0$;

$\mathfrak{A}$ represents the family of all altering distance functions.

Definition 2.11:[18, 19] The mapping $\varpi : [0,\infty) \to [0,\infty)$ is called the ultra altering distance function if the following properties are met:

(1) ϖ is continuous;

(2) $\varpi(\mathfrak{e}) > 0 \; \forall \; \mathfrak{e} > 0$;

$\mathfrak{K}$ represents the class of all ultra-altering distance functions.

Definition 2.12:[2] Let $\mathscr{S}$ and $\mathscr{T}$ be nonempty subsets of a metric space $(\pounds, d)$ and $\mathscr{A} : \pounds \to \pounds$. Then $\mathscr{A}$ is called a cyclic map (w.r.t.$\mathscr{S}$ and $\mathscr{T}$) if $\mathscr{A}(\mathscr{S}) \subseteq \mathscr{T}$ and $\mathscr{A}(\mathscr{T}) \subseteq \mathscr{S}$.

Definition 2.13:[3, 7] Let $\mathscr{S}$ and $\mathscr{T}$ be nonempty subsets of a metric space $(\pounds, d)$. We call any function $\mathscr{A} : \pounds \times \pounds \to \pounds$ such that $\mathscr{A}(æ, œ) \in \mathscr{T}$ if $æ \in \mathscr{S}$, $œ \in \mathscr{T}$ and $\mathscr{A}(æ, œ) \in \mathscr{S}$ if $æ \in \mathscr{T}$, $œ \in \mathscr{S}$ a cyclic mapping with respect to $\mathscr{S}$ and $\mathscr{T}$.

Now we prove our main result.

3. Main Results

For convenience, we set: $\Delta = \{\wp / \wp : [0,\infty) \to [0,\infty)\}$ be a family of functions that satisfy the following properties;

(i) $\wp$ is a non-decreasing, upper semi-continuous from the right;

(ii) $\wp(\mathfrak{e}) = 0$ if and only if $\mathfrak{e} = 0$;

Definition 3.1: Let $(\pounds, \varsigma_\rho)$ be a PbMS and a pair $(æ, œ) \in \pounds \times \pounds$ is called

(a) a coupled fixed point (CFP) of mapping $\mathscr{A} : \pounds \times \pounds \to \pounds$ if $æ = \mathscr{A}(æ, œ)$ and $œ = \mathscr{A}(œ, æ)$

(a_i) a strong coupled fixed point (SCFP) of mapping $\mathscr{A} : \pounds \times \pounds \to \pounds$ if $(æ, œ)$ is coupled fixed point and $æ = œ$ *i.e* $æ = \mathscr{A}(æ, æ)$

(b) a coupled coincident point (CCIP) of $\mathscr{A} : \pounds^2 \to \pounds$ and $\mathfrak{f} : \pounds \to \pounds$ if $\mathscr{A}(æ, œ) = \mathfrak{f}æ$, $\mathscr{A}(œ, æ) = \mathfrak{f}œ$.

(b_i) a strong coupled coincident point (SCCIP) of $\mathscr{A} : \pounds^2 \to \pounds$ and $\mathfrak{f} : \pounds \to \pounds$ if $æ = œ$. *i.e* $\mathscr{A}(æ, æ) = \mathfrak{f}æ$

(c) a coupled common fixed point (CCFP) of $\mathscr{A} : \pounds^2 \to \pounds$ and $\mathfrak{f} : \pounds \to \pounds$ if $\mathscr{A}(æ, œ) = \mathfrak{f}æ = æ$, $\mathscr{A}(œ, æ) = \mathfrak{f}œ = œ$

(c_i) a strong coupled common fixed point (SCCFP) of $\mathscr{A} : \pounds^2 \to \pounds$ and $\mathfrak{f} : \pounds \to \pounds$ if $æ = œ$. *i.e* $\mathscr{A}(æ, æ) = \mathfrak{f}æ = æ$

(d) the pair $(\mathscr{A},\mathfrak{f})$ is weakly compatible if $\mathfrak{f}(\mathscr{A}(æ,œ)) = \mathscr{A}(\mathfrak{f}æ,\mathfrak{f}œ)$ and $\mathfrak{f}(\mathscr{A}(œ,æ)) = \mathscr{A}(\mathfrak{f}œ,\mathfrak{f}æ)$ whenever $\mathscr{A}(æ,œ) = \mathfrak{f}æ$, $\mathscr{A}(œ,æ) = \mathfrak{f}œ$ for all $æ,œ \in £$.

Definition 3.2: Let $\mathscr{S}$ and $\mathscr{T}$ be nonempty subsets of a PbMS $(£,\varsigma_\rho)$ with coefficient $\mathfrak{v} \geq 1$. We call a mappings $\mathscr{A} : £^2 \to £$ and $\mathfrak{f} : £ \to £$ are $\mathscr{F}_G(\varphi,\wp,\varpi)$-cyclic coupled rational type contraction with respect to $\mathscr{S}$ and $\mathscr{T}$ if $\mathscr{A}$ and $\mathfrak{f}$ are cyclic with respect to $\mathscr{S}$ and $\mathscr{T}$ satisfying, for all $\mathfrak{y}_1, æ_2 \in \mathscr{S}$, $\mathfrak{y}_2, æ_1 \in \mathscr{T}$, the inequality

$$\begin{aligned}
&\varphi\left(\varsigma_\rho\left(\mathscr{A}(\mathfrak{y}_1,\mathfrak{y}_2),\mathscr{A}(æ_1,æ_2)\right)\right)\\
(1)\qquad &\leq \mathscr{F}_G\Big(\ \varphi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)),\wp(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)),\varpi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))\ \Big)
\end{aligned}$$

where $\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2) = \max\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{f}\mathfrak{y}_1,\mathfrak{f}æ_1),\varsigma_\rho(\mathfrak{f}\mathfrak{y}_2,\mathfrak{f}æ_2)\\ \frac{\varsigma_\rho(\mathfrak{f}\mathfrak{y}_1,\mathscr{A}(\mathfrak{y}_2,\mathfrak{y}_1)).\varsigma_\rho(\mathfrak{f}æ_1,\mathscr{A}(æ_2,æ_1))}{2\mathfrak{v}^2\left[1+\varsigma_\rho(\mathfrak{f}\mathfrak{y}_1,\mathfrak{f}æ_1)\right]},\frac{\varsigma_\rho(\mathfrak{f}\mathfrak{y}_2,\mathscr{A}(\mathfrak{y}_1,\mathfrak{y}_2)).\varsigma_\rho(\mathfrak{f}æ_2,\mathscr{A}(æ_1,æ_2))}{2\mathfrak{v}^2\left[1+\varsigma_\rho(\mathfrak{f}\mathfrak{y}_2,\mathfrak{f}æ_2)\right]}\end{array}\right\}$, $\varphi \in \mathfrak{A}$, $\wp \in \Delta$, $\varpi \in \mathfrak{K}$, $\mathscr{F}_G \in \mathscr{C}_G$.

Theorem 3.3: $\mathscr{S}$ and $\mathscr{T}$ be two nonempty closed subsets of a complete PbMS $(£,\varsigma_\rho)$ with coefficient $\mathfrak{v} \geq 1$. Let $\mathscr{A} : £^2 \to £$ and $\mathfrak{f} : £ \to £$ are $\mathscr{F}_G(\varphi,\wp,\varpi)$-cyclic coupled rational type contraction with respect to $\mathscr{S}$ and $\mathscr{T}$ and $\mathscr{S} \cap \mathscr{T} \neq \emptyset$ and with regard to a $\mathscr{C}_G$-class functions $\mathscr{F}_G$ with $\varphi(\mathfrak{e}) > \wp(\mathfrak{e})$ for all $\mathfrak{e} > 0$. Assume

(3.3.1) $\mathscr{A}(£^2) \subseteq \mathfrak{f}(£)$ and $\mathfrak{f}(£)$ is complete subspace of $£$ w.r.t $\mathscr{S}$ and $\mathscr{T}$

(3.3.2) $(\mathscr{A},\mathfrak{f})$ is weakly compatible pair.

Then $\mathscr{A}$ and $\mathfrak{f}$ have a USCCFP in $£$.

Proof Let $\mathfrak{y}_0 \in \mathscr{S}, æ_0 \in \mathscr{T}$ be any two elements and from (3.3.1) the sequences $\{\mathfrak{y}_z\},\{ß_z\} \subseteq \mathscr{S}$ and $\{æ_z\},\{œ_z\} \subseteq \mathscr{T}$ be defined as

$$\mathscr{A}(\mathfrak{y}_z,æ_z) = \mathfrak{f}\mathfrak{y}_{z+1} = œ_z, \quad \mathscr{A}(æ_z,\mathfrak{y}_z) = \mathfrak{f}æ_{z+1} = ß_z \ \forall\, z \geq 0.$$

Case(i): If for some z, we have $œ_z = ß_{z+1}$ and $ß_z = œ_{z+1}$ then we have

$$\begin{aligned}
\varphi\left(\varsigma_\rho(ß_z,œ_z)\right) &= \varphi\left(\varsigma_\rho(œ_{z+1},ß_{z+1})\right) = \varphi\left(\varsigma_\rho\left(\mathscr{A}(\mathfrak{y}_{z+1},æ_{z+1}),\mathscr{A}(æ_{z+1},\mathfrak{y}_{z+1})\right)\right)\\
&\leq \mathscr{F}_G\left(\begin{array}{c}\varphi(\mathbb{M}(\mathfrak{y}_{z+1},æ_{z+1},æ_{z+1},\mathfrak{y}_{z+1})),\wp(\mathbb{M}(\mathfrak{y}_{z+1},æ_{z+1},æ_{z+1},\mathfrak{y}_{z+1})),\\ \varpi(\mathbb{M}(\mathfrak{y}_{z+1},æ_{z+1},æ_{z+1},\mathfrak{y}_{z+1}))\end{array}\right)\\
&\leq \max\Big\{\ \varphi(\mathbb{M}(\mathfrak{y}_{z+1},æ_{z+1},æ_{z+1},\mathfrak{y}_{z+1})),\wp(\mathbb{M}(\mathfrak{y}_{z+1},æ_{z+1},æ_{z+1},\mathfrak{y}_{z+1}))\ \Big\}
\end{aligned}$$

where

$$\mathbb{M}(\mathfrak{y}_{z+1},\text{æ}_{z+1},\text{æ}_{z+1},\mathfrak{y}_{z+1}) = \max\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{f}\mathfrak{y}_{z+1},\mathfrak{f}\text{æ}_{z+1}),\varsigma_\rho(\mathfrak{f}\text{æ}_{z+1},\mathfrak{f}\mathfrak{y}_{z+1})\\ \frac{\varsigma_\rho(\mathfrak{f}\mathfrak{y}_{z+1},\mathscr{A}(\text{æ}_{z+1},\mathfrak{y}_{z+1})).\varsigma_\rho(\mathfrak{f}\text{æ}_{z+1},\mathscr{A}(\mathfrak{y}_{z+1},\text{æ}_{z+1}))}{2\mathfrak{v}^2[1+\varsigma_\rho(\mathfrak{f}\mathfrak{y}_{z+1},\mathfrak{f}\text{æ}_{z+1})]},\\ \frac{\varsigma_\rho(\mathfrak{f}\text{æ}_{z+1},\mathscr{A}(\mathfrak{y}_{z+1},\text{æ}_{z+1})).\varsigma_\rho(\mathfrak{f}\mathfrak{y}_{z+1},\mathscr{A}(\text{æ}_{z+1},\mathfrak{y}_{z+1}))}{2\mathfrak{v}^2[1+\varsigma_\rho(\mathfrak{f}\text{æ}_{z+1},\mathfrak{f}\mathfrak{y}_{z+1})]}\end{array}\right\}$$

$$= \max\left\{\begin{array}{c}\varsigma_\rho(\text{œ}_z,\text{ß}_z),\varsigma_\rho(\text{ß}_z,\text{œ}_z)\\ \frac{\varsigma_\rho(\text{œ}_z,\text{ß}_{z+1}).\varsigma_\rho(\text{ß}_z,\text{œ}_{z+1})}{2\mathfrak{v}^2[1+\varsigma_\rho(\text{œ}_z,\text{ß}_z)]},\frac{\varsigma_\rho(\text{ß}_z,\text{œ}_{z+1}).\varsigma_\rho(\text{œ}_z,\text{ß}_{z+1})}{2\mathfrak{v}^2[1+\varsigma_\rho(\text{ß}_z,\text{œ}_z)]}\end{array}\right\}$$

$$= \varsigma_\rho(\text{œ}_z,\text{ß}_z).$$

Accordingly, we conclude that

$$\varphi\left(\varsigma_\rho(\text{œ}_z,\text{ß}_z)\right) \leq \mathscr{F}_G\Big(\ \varphi(\varsigma_\rho(\text{œ}_z,\text{ß}_z)),\wp(\varsigma_\rho(\text{œ}_z,\text{ß}_z)),\varpi(\varsigma_\rho(\text{œ}_z,\text{ß}_z))\ \Big) < \varphi(\varsigma_\rho(\text{œ}_z,\text{ß}_z)).$$

Therefore, $\mathscr{F}_G\Big(\ \varphi(\varsigma_\rho(\text{œ}_z,\text{ß}_z)),\wp(\varsigma_\rho(\text{œ}_z,\text{ß}_z)),\varpi(\varsigma_\rho(\text{œ}_z,\text{ß}_z))\ \Big) = \varphi\left(\varsigma_\rho(\text{œ}_z,\text{ß}_z)\right)$.
By the condition (iii) of Definition (2.8), it can be deduced that either
$\varphi(\varsigma_\rho(\text{œ}_z,\text{ß}_z)) = 0$ or $\wp(\varsigma_\rho(\text{œ}_z,\text{ß}_z)) = 0$ implies $\varsigma_\rho(\text{œ}_z,\text{ß}_z) = 0$. Thus $\text{œ}_z = \text{ß}_z$ so that $\mathscr{S}\cap\mathscr{T}\neq\emptyset$ and $(\text{œ}_z,\text{œ}_z)$ is SCCFP of $\mathscr{A}$ and $\mathfrak{f}$.
Case (ii): Suppose that $\text{œ}_z \neq \text{ß}_{z+1}$ and $\text{ß}_z \neq \text{œ}_{z+1}$ for all $z \geq 0$.
Then for all $z \geq 0$, $\text{ß}_z \in \mathscr{S}$ and $\text{œ}_z \in \mathscr{T}$. By (1), we have

$$\varphi\left(\varsigma_\rho(\text{ß}_z,\text{œ}_{z+1})\right) = \varphi\left(\varsigma_\rho\left(\mathscr{A}(\text{æ}_z,\mathfrak{y}_z),\mathscr{A}(\mathfrak{y}_{z+1},\text{æ}_{z+1})\right)\right)$$

$$\leq \mathscr{F}_G\Big(\ \varphi(\mathbb{M}(\text{æ}_z,\mathfrak{y}_z,\mathfrak{y}_{z+1},\text{æ}_{z+1})),\wp(\mathbb{M}(\text{æ}_z,\mathfrak{y}_z,\mathfrak{y}_{z+1},\text{æ}_{z+1})),\varpi(\mathbb{M}(\text{æ}_z,\mathfrak{y}_z,\mathfrak{y}_{z+1},\text{æ}_{z+1}))\ \Big)$$

$$(2) \leq \max\Big\{\ \varphi(\mathbb{M}(\text{æ}_z,\mathfrak{y}_z,\mathfrak{y}_{z+1},\text{æ}_{z+1})),\wp(\mathbb{M}(\text{æ}_z,\mathfrak{y}_z,\mathfrak{y}_{z+1},\text{æ}_{z+1}))\ \Big\}$$

where

$$\mathbb{M}(\text{æ}_z,\mathfrak{y}_z,\mathfrak{y}_{z+1},\text{æ}_{z+1}) = \max\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{f}\text{æ}_z,\mathfrak{f}\mathfrak{y}_{z+1}),\varsigma_\rho(\mathfrak{f}\mathfrak{y}_z,\mathfrak{f}\text{æ}_{z+1})\\ \frac{\varsigma_\rho(\mathfrak{f}\text{æ}_z,\mathscr{A}(\mathfrak{y}_z,\text{æ}_z)).\varsigma_\rho(\mathfrak{f}\mathfrak{y}_{z+1},\mathscr{A}(\text{æ}_{z+1},\mathfrak{y}_{z+1}))}{2\mathfrak{v}^2[1+\varsigma_\rho(\mathfrak{f}\text{æ}_z,\mathfrak{f}\mathfrak{y}_{z+1})]},\\ \frac{\varsigma_\rho(\mathfrak{f}\mathfrak{y}_z,\mathscr{A}(\text{æ}_z,\mathfrak{y}_z)).\varsigma_\rho(\mathfrak{f}\text{æ}_{z+1},\mathscr{A}(\mathfrak{y}_{z+1},\text{æ}_{z+1}))}{2\mathfrak{v}^2[1+\varsigma_\rho(\mathfrak{f}\mathfrak{y}_z,\mathfrak{f}\text{æ}_{z+1})]}\end{array}\right\}$$

$$= \max\left\{\begin{array}{c}\varsigma_\rho(\text{ß}_{z-1},\text{œ}_z),\varsigma_\rho(\text{œ}_{z-1},\text{ß}_z)\\ \frac{\varsigma_\rho(\text{ß}_{z-1},\text{œ}_z).\varsigma_\rho(\text{œ}_z,\text{ß}_{z+1})}{2\mathfrak{v}^2[1+\varsigma_\rho(\text{ß}_{z-1},\text{œ}_z)]},\frac{\varsigma_\rho(\text{œ}_{z-1},\text{ß}_z).\varsigma_\rho(\text{ß}_z,\text{œ}_{z+1})}{2\mathfrak{v}^2[1+\varsigma_\rho(\text{œ}_{z-1},\text{ß}_z)]}\end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c}\varsigma_\rho(ß_{z-1},œ_z),\varsigma_\rho(œ_{z-1},ß_z)\\ \varsigma_\rho(ß_z,œ_{z+1}),\varsigma_\rho(œ_z,ß_{z+1})\end{array}\right\}$$

Hence, from Eq.(2) we have

$$\varphi\left(\varsigma_\rho\left(ß_z,œ_{z+1}\right)\right)\leq \max\left\{\begin{array}{c}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(ß_{z-1},œ_z),\varsigma_\rho(œ_{z-1},ß_z)\\ \varsigma_\rho(ß_z,œ_{z+1}),\varsigma_\rho(œ_z,ß_{z+1})\end{array}\right\}\right),\\ \wp\left(\max\left\{\begin{array}{c}\varsigma_\rho(ß_{z-1},œ_z),\varsigma_\rho(œ_{z-1},ß_z)\\ \varsigma_\rho(ß_z,œ_{z+1}),\varsigma_\rho(œ_z,ß_{z+1})\end{array}\right\}\right)\end{array}\right\}. \tag{3}$$

Suppose, $\varsigma_\rho(ß_{z-1},œ_z)<\varsigma_\rho(ß_z,œ_{z+1})$ and $\varsigma_\rho(œ_{z-1},ß_z)<\varsigma_\rho(œ_z,ß_{z+1})$. Hence, from Eq. (3), we have

$$\begin{aligned}\varphi\left(\varsigma_\rho\left(ß_z,œ_{z+1}\right)\right)&\leq \max\left\{\begin{array}{c}\varphi\left(\max\left\{\varsigma_\rho(ß_z,œ_{z+1}),\varsigma_\rho(œ_z,ß_{z+1})\right\}\right),\\ \wp\left(\max\left\{\varsigma_\rho(ß_z,œ_{z+1}),\varsigma_\rho(œ_z,ß_{z+1})\right\}\right)\end{array}\right\}\\ &<\varphi\left(\max\left\{\varsigma_\rho(ß_z,œ_{z+1}),\varsigma_\rho(œ_z,ß_{z+1})\right\}\right). \qquad (4)\end{aligned}$$

Similarly, we can prove

$$\varphi\left(\varsigma_\rho\left(œ_z,ß_{z+1}\right)\right)<\varphi\left(\max\left\{\varsigma_\rho(ß_z,œ_{z+1}),\varsigma_\rho(œ_z,ß_{z+1})\right\}\right) \tag{5}$$

Using property of φ and combining (4), (5), we get

$$\begin{aligned}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho\left(ß_z,œ_{z+1}\right),\\ \varsigma_\rho\left(œ_z,ß_{z+1}\right)\end{array}\right\}\right)&=\max\left\{\begin{array}{c}\varphi\left(\varsigma_\rho\left(ß_z,œ_{z+1}\right)\right),\\ \varphi\left(\varsigma_\rho\left(œ_z,ß_{z+1}\right)\right)\end{array}\right\}\\ &<\varphi\left(\max\left\{\varsigma_\rho(ß_z,œ_{z+1}),\varsigma_\rho(œ_z,ß_{z+1})\right\}\right).\end{aligned}$$

From non-decreasing property of φ, this implies that
$\max\left\{\begin{array}{c}\varsigma_\rho\left(ß_z,œ_{z+1}\right),\\ \varsigma_\rho\left(œ_z,ß_{z+1}\right)\end{array}\right\}<\max\left\{\begin{array}{c}\varsigma_\rho(ß_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,ß_{z+1})\end{array}\right\}$ which is a contradiction.
Accordingly, we conclude that $\max\left\{\begin{array}{c}\varsigma_\rho\left(ß_z,œ_{z+1}\right),\\ \varsigma_\rho\left(œ_z,ß_{z+1}\right)\end{array}\right\}<\max\left\{\begin{array}{c}\varsigma_\rho(ß_{z-1},œ_z),\\ \varsigma_\rho(œ_{z-1},ß_z)\end{array}\right\}$.
For simplification we denote $\Delta_z=\max\left\{\begin{array}{c}\varsigma_\rho\left(ß_z,œ_{z+1}\right),\\ \varsigma_\rho\left(œ_z,ß_{z+1}\right)\end{array}\right\}$, then we have

$$\Delta_z<\Delta_{z-1}\ \forall\ z\geq 1 \tag{6}$$

Therefore, $\{\Delta_z\}$ is a decreasing sequence and converges to $\delta \geq 0$. Now we claim that $\delta = 0$ and letting $z \to \infty$ in (2), we have that

$$\begin{aligned}\varphi(\delta) &\leq \mathscr{F}_G\Big(\varphi(\delta), \wp(\delta), \varpi(\delta) \Big) \\ &\leq \max\Big\{ \varphi(\delta), \wp(\delta) \Big\} = \varphi(\delta).\end{aligned}$$

Therefore, $\mathscr{F}_G\Big(\varphi(\delta), \wp(\delta), \varpi(\delta) \Big) = \varphi(\delta)$. By the condition (iii) of Definition (2.8), it can be deduced that either $\varphi(\delta) = 0$ or $\wp(\delta) = 0$. Hence, $\delta = 0$. Suppose $\delta > 0$ and letting $z \to \infty$ in (6), we have that $\delta < \delta$, is a contradiction . Hence $\delta = 0$. Thus $\lim_{z\to\infty} \Delta_z = 0$. It follows that

$$\lim_{z\to\infty} \varsigma_\rho(ß_z, œ_{z+1}) = \lim_{z\to\infty} \varsigma_\rho(œ_z, ß_{z+1}) = 0. \tag{7}$$

From (7) and (ς_{ρ_2}), we have that

$$\lim_{z\to\infty} \varsigma_\rho(ß_z, œ_z) = \lim_{z\to\infty} \varsigma_\rho(œ_z, ß_z) = 0. \tag{8}$$

From (ς_{ρ_4}), (7) and (8), we have

$$\frac{1}{\mathfrak{v}}\varsigma_\rho(ß_z, ß_{z+1}) + \varsigma_\rho(œ_z, œ_z) \quad \leq \varsigma_\rho(ß_z, œ_z) + \varsigma_\rho(œ_z, ß_{z+1}) \to 0 \text{ as } z \to \infty$$

and

$$\frac{1}{\mathfrak{v}}\varsigma_\rho(œ_z, œ_{z+1}) + \varsigma_\rho(ß_z, ß_z) \quad \leq \varsigma_\rho(œ_z, ß_z) + \varsigma_\rho(ß_z, œ_{z+1}) \to 0 \text{ as } z \to \infty.$$

It follows that

$$\lim_{z\to\infty} \varsigma_\rho(ß_z, ß_{z+1}) = \lim_{z\to\infty} \varsigma_\rho(œ_z, œ_z) = \lim_{z\to\infty} \varsigma_\rho(œ_z, œ_{z+1}) = \lim_{z\to\infty} \varsigma_\rho(ß_z, ß_z) = 0. \tag{9}$$

From definition of d_{ς_ρ}, (9), we have that

$$\lim_{z\to\infty} d_{\varsigma_\rho}(ß_z, œ_{z+1}) = \lim_{z\to\infty} d_{\varsigma_\rho}(œ_z, ß_{z+1}) = 0. \tag{10}$$

To demonstrate the Cauchy nature of $\{œ_z\}$ and $\{ß_z\}$ in $(£, \varsigma_\rho)$. If, in b-metric space $(£, d_{\varsigma_\rho})$, it is sufficient to demonstrate that $\{œ_z\}$ and $\{ß_z\}$ are Cauchy sequences, then let's assume that either $\{œ_z\}$ or $\{ß_z\}$ is not. This gives that $d_{\varsigma_\rho}(œ_z, œ_w) \nrightarrow 0$ and $d_{\varsigma_\rho}(ß_z, ß_w) \nrightarrow 0$ as $z, w \to \infty$. Consequently, $\max\{d_{\varsigma_\rho}(œ_z, œ_w), d_{\varsigma_\rho}(ß_z, ß_w)\} \nrightarrow 0$ as $z, w \to \infty$.

Then, there are sequences of natural numbers that increase monotonically and have a $\varepsilon > 0$. $\forall\, z \geq 0$ $\varkappa(z) > \wp(z) > z$, $\{\varkappa(z)\}_{z\geq 0}$, $\{\wp(z)\}_{z\geq 0}$,

$$\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)}, ß_{\wp(z)})\right\} \geq \varepsilon \tag{11}$$

and

$$\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)-1}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)-1}, ß_{\wp(z)})\right\} < \varepsilon. \tag{12}$$

From (11) and (12), we have that

$$\begin{aligned}
\varepsilon &\leq \max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)}, ß_{\wp(z)})\right\} \\
&\leq \mathfrak{v}.\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)}, œ_{\varkappa(z)-1}), d_{\varsigma\rho}(ß_{\varkappa(z)}, ß_{\varkappa(z)-1})\right\} \\
&\quad +\mathfrak{v}.\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)-1}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)-1}, ß_{\wp(z)})\right\} \\
&< \mathfrak{v}.\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)}, œ_{\varkappa(z)-1}), d_{\varsigma\rho}(ß_{\varkappa(z)}, ß_{\varkappa(z)-1})\right\} + \mathfrak{v}.\varepsilon
\end{aligned}$$

Using $z \to \infty$ as the upper limit and (9), we obtain that

$$\varepsilon \leq \lim_{z\to\infty} \sup \max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)}, ß_{\wp(z)})\right\} \leq \mathfrak{v}.\varepsilon. \tag{13}$$

Also

$$\begin{aligned}
\varepsilon &\leq \max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)}, ß_{\wp(z)})\right\} \\
&\leq \mathfrak{v}.\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)}, œ_{\varkappa(z)-1}), d_{\varsigma\rho}(ß_{\varkappa(z)}, ß_{\varkappa(z)-1})\right\} \\
&\quad +\mathfrak{v}.\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)-1}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)-1}, ß_{\wp(z)})\right\}
\end{aligned}$$

Assuming $z \to \infty$ as the upper limit, we can infer from (9) that

$$\frac{\varepsilon}{\mathfrak{v}} \leq \lim_{k\to\infty} \sup \max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)-1}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)-1}, ß_{\wp(z)})\right\}. \tag{14}$$

On other hand

$$\begin{aligned}
&\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)-1}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)-1}, ß_{\wp(z)})\right\} \\
\leq\ &\mathfrak{v}\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)-1}, œ_{\varkappa(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)-1}, ß_{\varkappa(z)})\right\} + \mathfrak{v}\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)}, ß_{\wp(z)})\right\}.
\end{aligned}$$

With $z \to \infty$ as the upper limit, and from (13) and (14), we obtain

$$\frac{\varepsilon}{\mathfrak{v}} \leq \lim_{k\to\infty} \sup \max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)-1}, œ_{\wp(z)}), d_{\varsigma\rho}(ß_{\varkappa(z)-1}, ß_{\wp(z)})\right\} \leq \varepsilon\mathfrak{v}^2. \tag{15}$$

also, from (11), we have that

$$\begin{aligned}\varepsilon &\leq \max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)},œ_{\wp(z)}),d_{\varsigma\rho}(ß_{\varkappa(z)},ß_{\wp(z)})\right\}\\ &\leq \mathfrak{v}\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)},œ_{\wp(z)+1}),d_{\varsigma\rho}(ß_{\varkappa(z)},ß_{\wp(z)+1})\right\}\\ &\quad+\mathfrak{v}\max\left\{d_{\varsigma\rho}(œ_{\wp(z)+1},œ_{\wp(z)}),d_{\varsigma\rho}(ß_{\wp(z)+1},ß_{\wp(z)})\right\}\end{aligned}$$

Using $z\to\infty$ as the upper limit and (9), we obtain

$$\frac{\varepsilon}{\mathfrak{v}}\leq \lim_{z\to\infty}\sup\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)},œ_{\wp(z)+1}),d_{\varsigma\rho}(ß_{\varkappa(z)},ß_{\wp(z)+1})\right\}. \tag{16}$$

On other hand

$$\begin{aligned}&\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)},œ_{\wp(z)+1}),d_{\varsigma\rho}(ß_{\varkappa(z)},ß_{\wp(z)+1})\right\}\\ \leq\ &\mathfrak{v}\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)},œ_{\wp(z)}),d_{\varsigma\rho}(ß_{\varkappa(z)},ß_{\wp(z)})\right\}\\ &+\mathfrak{v}.\max\left\{d_{\varsigma\rho}(œ_{\wp(z)},œ_{\wp(z)+1}),d_{\varsigma\rho}(ß_{\wp(z)},ß_{\wp(z)+1})\right\}\end{aligned}$$

Using $z\to\infty$ as the upper limit, we can obtain from (9), (13),

$$\frac{\varepsilon}{\mathfrak{v}}\leq \lim_{z\to\infty}\sup\max\left\{d_{\varsigma\rho}(œ_{\varkappa(z)},œ_{\wp(z)+1}),d_{\varsigma\rho}(ß_{\varkappa(z)},ß_{\wp(z)+1})\right\}\leq\varepsilon\mathfrak{v}^2. \tag{17}$$

Using the Eq. (1), we have

$$\begin{aligned}\varphi\left(\varsigma\rho(œ_{\varkappa(z)},œ_{\wp(z)+1})\right) &= \varphi\left(\varsigma\rho(\mathscr{A}\left(\mathfrak{y}_{\varkappa(z)},æ_{\varkappa(z)}\right),\mathscr{A}\left(\mathfrak{y}_{\wp(z)+1},æ_{\wp(z)+1}\right))\right)\\ &\leq \mathscr{F}_G\begin{pmatrix}\varphi(\mathbb{M}(\mathfrak{y}_{\varkappa(z)},æ_{\varkappa(z)},\mathfrak{y}_{\wp(z)+1},æ_{\wp(z)+1})),\\ \wp(\mathbb{M}(\mathfrak{y}_{\varkappa(z)},æ_{\varkappa(z)},\mathfrak{y}_{\wp(z)+1},æ_{\wp(z)+1})),\\ \varpi(\mathbb{M}(\mathfrak{y}_{\varkappa(z)},æ_{\varkappa(z)},\mathfrak{y}_{\wp(z)+1},æ_{\wp(z)+1}))\end{pmatrix}\\ &\leq \max\begin{Bmatrix}\varphi(\mathbb{M}(\mathfrak{y}_{\varkappa(z)},æ_{\varkappa(z)},\mathfrak{y}_{\wp(z)+1},æ_{\wp(z)+1})),\\ \wp(\mathbb{M}(\mathfrak{y}_{\varkappa(z)},æ_{\varkappa(z)},\mathfrak{y}_{\wp(z)+1},æ_{\wp(z)+1}))\end{Bmatrix}\\ &< \varphi(\mathbb{M}(\mathfrak{y}_{\varkappa(z)},æ_{\varkappa(z)},\mathfrak{y}_{\wp(z)+1},æ_{\wp(z)+1}))\\ &= \varphi\left(\max\begin{Bmatrix}\varsigma\rho(œ_{\varkappa(z)-1},œ_{\wp(z)}),\varsigma\rho(ß_{\varkappa(z)-1},ß_{\wp(z)})\\ \frac{\varsigma\rho(œ_{\varkappa(z)-1},ß_{\varkappa(z)})\cdot\varsigma\rho(œ_{\wp(z)},ß_{\wp(z)+1})}{2\mathfrak{v}^2\left[1+\varsigma\rho(œ_{\varkappa(z)-1},œ_{\wp(z)})\right]},\\ \frac{\varsigma\rho(ß_{\varkappa(z)-1},œ_{\varkappa(z)})\cdot\varsigma\rho(ß_{\wp(z)},œ_{\wp(z)+1})}{2\mathfrak{v}^2\left[1+\varsigma\rho(ß_{\varkappa(z)-1},ß_{\wp(z)})\right]}\end{Bmatrix}\right).\end{aligned}$$

Because of

$$\mathbb{M}(\mathfrak{y}_{\varkappa(z)},\text{æ}_{\varkappa(z)},\mathfrak{y}_{\wp(z)+1},\text{æ}_{\wp(z)+1}) = \max\left\{\begin{array}{c} \varsigma_\rho(\mathfrak{f}\mathfrak{y}_{\varkappa(z)},\mathfrak{f}\mathfrak{y}_{\wp(z)+1}),\varsigma_\rho(\mathfrak{f}\text{æ}_{\varkappa(z)},\mathfrak{f}\text{æ}_{\wp(z)+1}) \\ \frac{\varsigma_\rho(\mathfrak{f}\mathfrak{y}_{\varkappa(z)},\mathscr{A}(\text{æ}_{\varkappa(z)},\mathfrak{y}_{\varkappa(z)}))\cdot\varsigma_\rho(\mathfrak{f}\mathfrak{y}_{\wp(z)+1},\mathscr{A}(\text{æ}_{\wp(z)+1},\mathfrak{y}_{\wp(z)+1}))}{2\mathfrak{v}^2[1+\varsigma_\rho(\mathfrak{f}\mathfrak{y}_{\varkappa(z)},\mathfrak{f}\mathfrak{y}_{\wp(z)+1})]}, \\ \frac{\varsigma_\rho(\mathfrak{f}\text{æ}_{\varkappa(z)},\mathscr{A}(\mathfrak{y}_{\varkappa(z)},\text{æ}_{\varkappa(z)}))\cdot\varsigma_\rho(\mathfrak{f}\text{æ}_{\wp(z)+1},\mathscr{A}(\mathfrak{y}_{\wp(z)+1},\text{æ}_{\wp(z)+1}))}{2\mathfrak{v}^2[1+\varsigma_\rho(\mathfrak{f}\text{æ}_{\varkappa(z)},\mathfrak{f}\text{æ}_{\wp(z)+1})]} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} \varsigma_\rho(\text{œ}_{\varkappa(z)-1},\text{œ}_{\wp(z)}),\varsigma_\rho(\text{ß}_{\varkappa(z)-1},\text{ß}_{\wp(z)}) \\ \frac{\varsigma_\rho(\text{œ}_{\varkappa(z)-1},\text{ß}_{\varkappa(z)})\cdot\varsigma_\rho(\text{œ}_{\wp(z)},\text{ß}_{\wp(z)+1})}{2\mathfrak{v}^2[1+\varsigma_\rho(\text{œ}_{\varkappa(z)-1},\text{œ}_{\wp(z)})]}, \\ \frac{\varsigma_\rho(\text{ß}_{\varkappa(z)-1},\text{œ}_{\varkappa(z)})\cdot\varsigma_\rho(\text{ß}_{\wp(z)},\text{œ}_{\wp(z)+1})}{2\mathfrak{v}^2[1+\varsigma_\rho(\text{ß}_{\varkappa(z)-1},\text{ß}_{\wp(z)})]} \end{array}\right\}.$$

Similarly,

$$\varphi\left(\varsigma_\rho(\text{ß}_{\varkappa(z)},\text{ß}_{\wp(z)+1})\right) < \varphi\left(\max\left\{\begin{array}{c} \varsigma_\rho(\text{œ}_{\varkappa(z)-1},\text{œ}_{\wp(z)}),\varsigma_\rho(\text{ß}_{\varkappa(z)-1},\text{ß}_{\wp(z)}) \\ \frac{\varsigma_\rho(\text{œ}_{\varkappa(z)-1},\text{ß}_{\varkappa(z)})\cdot\varsigma_\rho(\text{œ}_{\wp(z)},\text{ß}_{\wp(z)+1})}{2\mathfrak{v}^2[1+\varsigma_\rho(\text{œ}_{\varkappa(z)-1},\text{œ}_{\wp(z)})]}, \\ \frac{\varsigma_\rho(\text{ß}_{\varkappa(z)-1},\text{œ}_{\varkappa(z)})\cdot\varsigma_\rho(\text{ß}_{\wp(z)},\text{œ}_{\wp(z)+1})}{2\mathfrak{v}^2[1+\varsigma_\rho(\text{ß}_{\varkappa(z)-1},\text{ß}_{\wp(z)})]} \end{array}\right\}\right).$$

Therefore,

$$\varphi\left(\max\left\{\begin{array}{c} \varsigma_\rho(\text{œ}_{\varkappa(z)},\text{œ}_{\wp(z)+1}), \\ \varsigma_\rho(\text{ß}_{\varkappa(z)},\text{ß}_{\wp(z)+1}) \end{array}\right\}\right) = \max\left\{\begin{array}{c} \varphi\left(\varsigma_\rho(\text{œ}_{\varkappa(z)},\text{œ}_{\wp(z)+1})\right), \\ \varphi\left(\varsigma_\rho(\text{ß}_{\varkappa(z)},\text{ß}_{\wp(z)+1})\right) \end{array}\right\}$$

$$< \varphi\left(\max\left\{\begin{array}{c} \varsigma_\rho(\text{œ}_{\varkappa(z)-1},\text{œ}_{\wp(z)}),\varsigma_\rho(\text{ß}_{\varkappa(z)-1},\text{ß}_{\wp(z)}) \\ \frac{\varsigma_\rho(\text{œ}_{\varkappa(z)-1},\text{ß}_{\varkappa(z)})\cdot\varsigma_\rho(\text{œ}_{\wp(z)},\text{ß}_{\wp(z)+1})}{2\mathfrak{v}^2[1+\varsigma_\rho(\text{œ}_{\varkappa(z)-1},\text{œ}_{\wp(z)})]}, \\ \frac{\varsigma_\rho(\text{ß}_{\varkappa(z)-1},\text{œ}_{\varkappa(z)})\cdot\varsigma_\rho(\text{ß}_{\wp(z)},\text{œ}_{\wp(z)+1})}{2\mathfrak{v}^2[1+\varsigma_\rho(\text{ß}_{\varkappa(z)-1},\text{ß}_{\wp(z)})]} \end{array}\right\}\right).$$

Using $z \to \infty$ as the upper limit and (9), (15), (17), we obtain that

$$\varphi(\varepsilon.\mathfrak{v}^2) \le \mathscr{F}_G\left(\varphi(\varepsilon.\mathfrak{v}^2),\wp(\varepsilon.\mathfrak{v}^2),\varpi(\varepsilon.\mathfrak{v}^2) \right) < \varphi(\varepsilon\mathfrak{v}^2)$$

Therefore, $\mathscr{F}_G\left(\varphi(\varepsilon.\mathfrak{v}^2),\wp(\varepsilon.\mathfrak{v}^2),\varpi(\varepsilon.\mathfrak{v}^2) \right) = \varphi\left(\varepsilon.\mathfrak{v}^2\right)$. By the condition (*iii*) of Definition (2.8), it can be deduced that either $\varphi(\varepsilon.\mathfrak{v}^2) = 0$ or $\wp(\varepsilon.\mathfrak{v}^2) = 0$. Hence, $\varepsilon.\mathfrak{v}^2 = 0$ implies $\varepsilon = 0$ and $\mathfrak{v} = 0$ is contradiction.

Thus, in $(\pounds, d_{\varsigma_\rho})$, $\{\text{œ}_z\}$ and $\{\text{ß}_z\}$ are Cauchy sequences.Assume that the complete subspace of $\pounds$ is $\mathfrak{f}(\pounds)$ w.r.t closed subsets $\mathscr{S}$ and $\mathscr{T}$. Then $\{\text{œ}_z\} \subseteq \mathscr{S}$ and $\{\text{ß}_z\} \subseteq \mathscr{T}$ converge to $\eth \in \mathscr{S}$, $\mho \in \mathscr{T}$ in $(\mathfrak{f}(\pounds), d_{\varsigma_\rho})$, so there exist $\mathfrak{y}, \text{æ} \in \mathfrak{f}(\pounds)$ such that $\lim_{z\to\infty} \text{œ}_z = \eth = \mathfrak{f}\mathfrak{y}$ and $\lim_{z\to\infty} \text{ß}_z = \mho = \mathfrak{f}\text{æ}$.

That is $\lim\limits_{z\to\infty} d_{\varsigma_\rho}(\mathfrak{f}\mathfrak{y}_{z+1},\eth)=0, \lim\limits_{z\to\infty} d_{\varsigma_\rho}(\mathfrak{f}\ae_{z+1},\mho)=0$ for some $\eth=\mathfrak{f}\mathfrak{y}$, $\mho=\mathfrak{f}\ae$,
we have that

$$\varsigma_\rho(\eth,\eth)=\lim_{z,w\to\infty}\varsigma_\rho(\mathfrak{f}\mathfrak{y}_z,\mathfrak{f}\mathfrak{y}_w)=\lim_{z\to\infty}\varsigma_\rho(\mathfrak{f}\mathfrak{y}_z,\eth)=\lim_{z\to\infty}\varsigma_\rho(\mathfrak{f}\mathfrak{y}_{z+1},\eth)=0.$$

and

$$\varsigma_\rho(\mho,\mho)=\lim_{z,w\to\infty}\varsigma_\rho(\mathfrak{f}\ae_z,\mathfrak{f}\ae_w)=\lim_{z\to\infty}\varsigma_\rho(\mathfrak{f}\ae_z,\mho)=\lim_{z\to\infty}\varsigma_\rho(\mathfrak{f}\ae_{z+1},\mho)=0$$

also we have

$$\varsigma_\rho(\eth,\mho)=\lim_{z\to\infty}\varsigma_\rho(\mathfrak{f}\mathfrak{y}_{z+1},\mathfrak{f}\ae_{z+1})=\lim_{z\to\infty}\varsigma_\rho(\oe_z,\ss_z)=0. \tag{18}$$

Therefore, $\eth=\mho$, Since $\mathscr{S}\cap\mathscr{T}\neq\emptyset$ then from the above it follows that $\eth\in\mathscr{S}\cap\mathscr{T}$. We now assert that $\mathscr{A}(\mathfrak{y},\ae)=\eth$. From (1), we have

$$\begin{aligned}
\varphi\left(\varsigma_\rho(\mathscr{A}(\mathfrak{y},\ae),\oe_z)\right) &= \mathscr{F}\left(\varsigma_\rho(\mathscr{A}(\mathfrak{y},\ae),\mathscr{A}(\mathfrak{y}_z,\ae_z))\right)\\
&\le \mathscr{F}_G\begin{pmatrix}\varphi(\mathbb{M}(\mathfrak{y},\ae,\mathfrak{y}_z,\ae_z)),\wp(\mathbb{M}(\mathfrak{y},\ae,\mathfrak{y}_z,\ae_z)),\\ \varpi(\mathbb{M}(\mathfrak{y},\ae,\mathfrak{y}_z,\ae_z))\end{pmatrix}\\
&\le \max\left\{\ \varphi(\mathbb{M}(\mathfrak{y},\ae,\mathfrak{y}_z,\ae_z)),\wp(\mathbb{M}(\mathfrak{y},\ae,\mathfrak{y}_z,\ae_z))\ \right\}\\
&< \varphi(\mathbb{M}(\mathfrak{y},\ae,\mathfrak{y}_z,\ae_z))\\
&= \varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}_z),\varsigma_\rho(\mathfrak{f}\ae,\mathfrak{f}\ae_z)\\ \frac{\varsigma_\rho(\mathfrak{f}\mathfrak{y},\mathscr{A}(\ae,\mathfrak{y})).\varsigma_\rho(\mathfrak{f}\mathfrak{y}_z,\mathscr{A}(\ae_z,\mathfrak{y}_z))}{2\mathfrak{v}^2\left[1+\varsigma_\rho(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}_z)\right]},\\ \frac{\varsigma_\rho(\mathfrak{f}\ae,\mathscr{A}(\mathfrak{y},\ae)).\varsigma_\rho(\mathfrak{f}\ae_z,\mathscr{A}(\mathfrak{y}_z,\ae_z))}{2\mathfrak{v}^2\left[1+\varsigma_\rho(\mathfrak{f}\ae,\mathfrak{f}\ae_z)\right]}\end{array}\right\}\right).
\end{aligned}$$

Because of

$$\mathbb{M}(\mathfrak{y},\ae,\mathfrak{y}_z,\ae_z)\ =\max\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}_z),\varsigma_\rho(\mathfrak{f}\ae,\mathfrak{f}\ae_z)\\ \frac{\varsigma_\rho(\mathfrak{f}\mathfrak{y},\mathscr{A}(\ae,\mathfrak{y})).\varsigma_\rho(\mathfrak{f}\mathfrak{y}_z,\mathscr{A}(\ae_z,\mathfrak{y}_z))}{2\mathfrak{v}^2\left[1+\varsigma_\rho(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}_z)\right]},\\ \frac{\varsigma_\rho(\mathfrak{f}\ae,\mathscr{A}(\mathfrak{y},\ae)).\varsigma_\rho(\mathfrak{f}\ae_z,\mathscr{A}(\mathfrak{y}_z,\ae_z))}{2\mathfrak{v}^2\left[1+\varsigma_\rho(\mathfrak{f}\ae,\mathfrak{f}\ae_z)\right]}\end{array}\right\}.$$

In the above inequality, if we let $z\to\infty$, we obtain that $\varphi\left(\varsigma_\rho(\mathscr{A}(\mathfrak{y},\ae),\eth)\right)=0$ implies that $\varsigma_\rho(\mathscr{A}(\mathfrak{y},\ae),\eth)=0$ and hence $\mathscr{A}(\mathfrak{y},\ae)=\eth=\mathfrak{f}\mathfrak{y}$. Since, $(\mathscr{A},\mathfrak{f})$ as a weakly compatible pair, we obtain

$$\mathfrak{f}\eth=\mathfrak{f}^2\mathfrak{y}=\mathfrak{f}(\mathscr{A}(\mathfrak{y},\ae))=\mathscr{A}(\mathfrak{f}\mathfrak{y},\mathfrak{f}\ae)=\mathscr{A}(\eth,\mho)$$

We now establish that $\mathfrak{f}\partial = \partial$. We can see from (1) that

$$
\begin{aligned}
\varphi\left(\varsigma_\rho\left(\mathfrak{f}\partial, æ_z\right)\right) &= \varphi\left(\varsigma_\rho\left(\mathscr{A}(\partial,\mho), \mathscr{A}(\mathfrak{y}_z, æ_z)\right)\right) \\
&\leq \mathscr{F}_G\Big(\ \varphi(\mathbb{M}(\partial,\mho,\mathfrak{y}_z,æ_z)), \wp(\mathbb{M}(\partial,\mho,\mathfrak{y}_z,æ_z)), \varpi(\mathbb{M}(\partial,\mho,\mathfrak{y}_z,æ_z))\ \Big) \\
&\leq \max\Big\{\ \varphi(\mathbb{M}(\partial,\mho,\mathfrak{y}_z,æ_z)), \wp(\mathbb{M}(\partial,\mho,\mathfrak{y}_z,æ_z))\ \Big\} \\
&< \varphi(\mathbb{M}(\partial,\mho,\mathfrak{y}_z,æ_z)) \\
&= \varphi\left(\max\left\{\begin{array}{c} \varsigma_\rho\left(\mathfrak{f}\partial, \mathfrak{f}\mathfrak{y}_z\right), \varsigma_\rho\left(\mathfrak{f}\mho, \mathfrak{f}æ_z\right) \\ \frac{\varsigma_\rho(\mathfrak{f}\partial,\mathscr{A}(\mho,\partial)).\varsigma_\rho(\mathfrak{f}\mathfrak{y}_z,\mathscr{A}(æ_z,\mathfrak{y}_z))}{2\mathfrak{v}^2\left[1+\varsigma_\rho(\mathfrak{f}\partial,\mathfrak{f}\mathfrak{y}_z)\right]}, \\ \frac{\varsigma_\rho(\mathfrak{f}\mho,\mathscr{A}(\partial,\mho)).\varsigma_\rho(\mathfrak{f}æ_z,\mathscr{A}(\mathfrak{y}_z,æ_z))}{2\mathfrak{v}^2\left[1+\varsigma_\rho(\mathfrak{f}\mho,\mathfrak{f}æ_z)\right]} \end{array}\right\}\right).
\end{aligned}
$$

If we let $z \to \infty$ in the inequality above, we get that

$$
\varphi\left(\varsigma_\rho\left(\mathfrak{f}\partial,\partial\right)\right) \leq \mathscr{F}_G\Big(\ \varphi(\varsigma_\rho(\mathfrak{f}\partial,\partial)), \wp(\varsigma_\rho(\mathfrak{f}\partial,\partial)), \varpi(\varsigma_\rho(\mathfrak{f}\partial,\partial))\ \Big) < \varphi\left(\varsigma_\rho\left(\mathfrak{f}\partial,\partial\right)\right).
$$

Therefore, $\mathscr{F}_G\Big(\ \varphi(\varsigma_\rho(\mathfrak{f}\partial,\partial)), \wp(\varsigma_\rho(\mathfrak{f}\partial,\partial)), \varpi(\varsigma_\rho(\mathfrak{f}\partial,\partial))\ \Big) = \varphi\left(\varsigma_\rho\left(\mathfrak{f}\partial,\partial\right)\right)$. By the condition (iii) of Definition (2.8), it can be deduced that $\varphi\left(\varsigma_\rho\left(\mathfrak{f}\partial,\partial\right)\right) = 0$ implies $\varsigma_\rho\left(\mathfrak{f}\partial,\partial\right) = 0$. It follows that $\mathscr{A}(\partial,\mho) = \partial = \mathfrak{f}\partial$. Again, in view of (18), we conclude that $\mathscr{A}(\partial,\partial) = \partial = \mathfrak{f}\partial$; that is, we have a strong common coupled fixed point of $\mathscr{A}$ and $\mathfrak{f}$.

For uniqueness let us suppose (∂^*,∂^*) be another SCCFP of $\mathscr{A}$ and $\mathfrak{f}$. Then from (1), we have

$$
\begin{aligned}
\varphi\left(\varsigma_\rho(\partial,\partial^*)\right) &= \mathscr{F}\left(\varsigma_\rho\left(\mathscr{A}(\partial,\partial), \mathscr{A}(\partial^*,\partial^*)\right)\right) \\
&\leq \mathscr{F}_G\Big(\ \varphi(\mathbb{M}(\partial,\partial,\partial^*,\partial^*)), \wp(\mathbb{M}(\partial,\partial,\partial^*,\partial^*)), \varpi(\mathbb{M}(\partial,\partial,\partial^*,\partial^*))\ \Big) \\
&\leq \max\Big\{\ \varphi(\mathbb{M}(\partial,\partial,\partial^*,\partial^*)), \wp(\mathbb{M}(\partial,\partial,\partial^*,\partial^*))\ \Big\} \\
&< \varphi(\mathbb{M}(\partial,\partial,\partial^*,\partial^*)) \\
&= \varphi\left(\max\left\{\begin{array}{c} \varsigma_\rho\left(\mathfrak{f}\partial, \mathfrak{f}\partial^*\right), \varsigma_\rho\left(\mathfrak{f}\partial, \mathfrak{f}\partial^*\right) \\ \frac{\varsigma_\rho(\mathfrak{f}\partial,\mathscr{A}(\partial,\partial)).\varsigma_\rho(\mathfrak{f}\partial^*,\mathscr{A}(\partial^*,\partial^*))}{2\mathfrak{v}^2\left[1+\varsigma_\rho(\mathfrak{f}\partial,\mathfrak{f}\partial^*)\right]}, \\ \frac{\varsigma_\rho(\mathfrak{f}\partial,\mathscr{A}(\partial,\partial)).\varsigma_\rho(\mathfrak{f}\partial^*,\mathscr{A}(\partial^*,\partial^*))}{2\mathfrak{v}^2\left[1+\varsigma_\rho(\mathfrak{f}\partial,\mathfrak{f}\partial^*)\right]} \end{array}\right\}\right)
\end{aligned}
$$

Accordingly, we conclude that

$$
\varphi\left(\varsigma_\rho(\partial,\partial^*)\right) \leq \mathscr{F}_G\Big(\ \varphi(\varsigma_\rho(\partial,\partial^*)), \wp(\varsigma_\rho(\partial,\partial^*)), \varpi(\varsigma_\rho(\partial,\partial^*))\ \Big) < \varphi\left(\varsigma_\rho(\partial,\partial^*)\right).
$$

Therefore, $\mathscr{F}_G\left(\ \varphi(\varsigma_\rho(\eth,\eth^*)),\wp(\varsigma_\rho(\eth,\eth^*)),\varpi(\varsigma_\rho(\eth,\eth^*))\ \right)=\varphi\left(\varsigma_\rho(\eth,\eth^*)\right)$. By the condition (iii) of Definition (2.8), it can be deduced that $\varphi\left(\varsigma_\rho(\eth,\eth^*)\right)=0$ implies $\varsigma_\rho(\eth,\eth^*)=0$ and hence the USCCFP of $\mathscr{A}$ and $\mathfrak{f}$ is $(\eth,\eth)$. This completes the proof of the theorem.

Corollary 3.4: $\mathscr{S}$ and $\mathscr{T}$ be two nonempty closed subsets of a complete PbMS $(\pounds,\varsigma_\rho)$ with coefficient $\mathfrak{v}\geq 1$. Let $\mathscr{A}:\pounds^2\to\pounds$ be a $\mathscr{F}_G(\varphi,\wp,\varpi)$-cyclic coupled contraction with respect to $\mathscr{S}$ and $\mathscr{T}$ and $\mathscr{S}\cap\mathscr{T}\neq\emptyset$ and with regard to a $\mathscr{C}_G$-class functions $\mathscr{F}_G$ with $\varphi(\mathfrak{e})>\wp(\mathfrak{e})$ for all $\mathfrak{e}>0$, $\mathfrak{y}_1,\text{æ}_2\in\mathscr{S}$, $\mathfrak{y}_2,\text{æ}_1\in\mathscr{T}$, the inequality

$$\varphi\left(\varsigma_\rho\left(\mathscr{A}(\mathfrak{y}_1,\mathfrak{y}_2),\mathscr{A}(\text{æ}_1,\text{æ}_2)\right)\right)\leq\mathscr{F}_G\left(\begin{array}{c}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{y}_1,\text{æ}_1),\\ \varsigma_\rho(\mathfrak{y}_2,\text{æ}_2)\end{array}\right\}\right),\wp\left(\max\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{y}_1,\text{æ}_1),\\ \varsigma_\rho(\mathfrak{y}_2,\text{æ}_2)\end{array}\right\}\right),\\ \varpi\left(\max\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{y}_1,\text{æ}_1),\\ \varsigma_\rho(\mathfrak{y}_2,\text{æ}_2)\end{array}\right\}\right)\end{array}\right)$$

where $\varphi\in\mathfrak{A}$, $\wp\in\Delta$, $\varpi\in\mathfrak{K}$, $\mathscr{F}_G\in\mathscr{C}_G$. Then in $\mathscr{S}\cap\mathscr{T}$, there is a USCFP for $\mathscr{A}$.

Proof Using the identity map on $\mathscr{S}\cup\mathscr{T}$, $\mathfrak{f}=I_{\mathscr{S}\cup\mathscr{T}}$, we can determine from Theorem (3.3) that $\mathscr{A}$ has a USCFP.

Example 3.5: Let $\pounds=[-1,2]$ together with PbMS $\varsigma_\rho:\pounds^2\to[0,\infty)$ as $\varsigma_\rho(\mathfrak{y};\text{æ})=[\max\{\mathfrak{y},\text{æ}\}]^2$ for all $\mathfrak{y},\text{æ}\in\pounds$ with coefficient $\mathfrak{v}=4>1$. Let $\mathscr{S}=[0,1]$ and $\mathscr{T}=[-1,0]$. Then $\mathscr{S}$ and $\mathscr{T}$ are nonempty closed subset of $\pounds$ and $\varsigma_\rho(\mathscr{S};\mathscr{T})=0$.

Let $\mathscr{A}:\pounds^2\to\pounds$ as $\mathscr{A}(\mathfrak{y};\text{æ})=\begin{cases}0 & \text{if} \quad \mathfrak{y},\text{æ}\in\mathscr{S}\cap\mathscr{T}\\ 1 & \text{otherwise}\end{cases}$

and $\mathfrak{f}:\pounds\to\pounds$ as $\mathfrak{f}(\text{æ})=\begin{cases}0 & \text{if} \quad \text{æ}\in\mathscr{S}\cap\mathscr{T}\\ 1 & \text{if} \quad \text{æ}\in\mathscr{S}\\ 2 & \text{if} \quad \text{æ}\in\mathscr{T}\end{cases}$.

Obviously, $\mathscr{A}(0,0)=0=\mathfrak{f}0$ implies that $(0,0)$ is a CCIP of $\mathscr{A}$ and $\mathfrak{f}$. Moreover, $\mathfrak{f}(\pounds)=\{0,1,2\}$ and $\mathscr{A}(\pounds^2)=\{0,1\}$. Hence, $\mathscr{A}(\pounds^2)\subseteq\mathfrak{f}(\pounds)$ and other hand

$\mathscr{A}(0,0)=\mathscr{A}(\mathfrak{f}0,\mathfrak{f}0)=\mathfrak{f}\mathscr{A}(0,0)=\mathfrak{f}0=0$, then $(\mathscr{A},\mathfrak{f})$ is ω-compatible.

After that, $\mathscr{F}_G:[0,\infty)\times[0,\infty)\times[0,\infty)\to\mathbb{R}$ as $\mathscr{F}_G(\mathfrak{e},\varsigma,\rho)=\varsigma-\frac{\mathfrak{e}}{1+\rho}$ and $\varphi,\wp,\varpi:\mathbb{R}^+\to\mathbb{R}^+$ as $\varphi(\mathfrak{e})=\frac{3\mathfrak{e}}{2}$, $\wp(\mathfrak{e})=\begin{cases}\mathfrak{e} & \text{for} \quad \mathfrak{e}<1\\ \frac{\mathfrak{e}}{2}+\frac{3}{4} & \text{for} \quad \mathfrak{e}\geq 1,\end{cases}$ and $\varpi(\mathfrak{e})=\frac{\mathfrak{e}}{2}$. From above we conclude that the inequality (1) is satisfied for all $\mathfrak{y}_1,\mathfrak{y}_2,\text{æ}_1,\text{æ}_2\in\pounds$, it follows that

Case (i): for all $\mathfrak{y}_1, æ_2 \in \mathscr{S}$, $\mathfrak{y}_2, æ_1 \in \mathscr{T}$, then

$$\varphi\left(\varsigma_\rho\left(\mathscr{A}(\mathfrak{y}_1,\mathfrak{y}_2),\mathscr{A}(æ_1,æ_2)\right)\right)=\varphi\left(\varsigma_\rho(1,1)\right)=\varphi(1)=\frac{3}{2}$$

and

$$\begin{aligned}\mathscr{F}_G\left(\begin{array}{c}\varphi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)),\wp(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)),\\ \varpi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))\end{array}\right) &= \wp(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)) \\ &\quad -\frac{\varphi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))}{1+\varpi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))} \\ &< \wp(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))\end{aligned}$$

where

$$\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)=\max\left\{\begin{array}{c}\varsigma_\rho(1,2),\varsigma_\rho(2,1)\\ \frac{\varsigma_\rho(1,1).\varsigma_\rho(2,1)}{2.4^2\left[1+\varsigma_\rho(1,2)\right]},\frac{\varsigma_\rho(2,1).\varsigma_\rho(1,1)}{2.4^2\left[1+\varsigma_\rho(2,1)\right]}\end{array}\right\}=4.$$

Therefore,

$$\frac{3}{2}=\varphi(1)\le\mathscr{F}_G\left(\;\varphi(4),\wp(4),\varpi(4)\;\right)<\wp(4)<\varphi(4)=6$$

Hence, Eq.(1) holds.

Case (ii): for all $\mathfrak{y}_1, æ_2 \in \mathscr{T}$, $\mathfrak{y}_2, æ_1 \in \mathscr{S}$, then

$$\varphi\left(\varsigma_\rho\left(\mathscr{A}(\mathfrak{y}_1,\mathfrak{y}_2),\mathscr{A}(æ_1,æ_2)\right)\right)=\varphi\left(\varsigma_\rho(1,1)\right)=\varphi(1)=\frac{3}{2}$$

and

$$\begin{aligned}\mathscr{F}_G\left(\begin{array}{c}\varphi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)),\wp(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)),\\ \varpi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))\end{array}\right) &= \wp(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)) \\ &\quad -\frac{\varphi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))}{1+\varpi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))} \\ &< \wp(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))\end{aligned}$$

where

$$\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)=\max\left\{\begin{array}{c}\varsigma_\rho(2,1),\varsigma_\rho(1,2),\\ \frac{\varsigma_\rho(2,1).\varsigma_\rho(1,1)}{2.4^2\left[1+\varsigma_\rho(2,1)\right]},\frac{\varsigma_\rho(1,1).\varsigma_\rho(2,1)}{2.4^2\left[1+\varsigma_\rho(1,2)\right]},\end{array}\right\}=4.$$

Therefore,

$$\frac{3}{2}=\varphi(1)\leq\mathscr{F}_G\left(\ \varphi(4),\wp(4),\varpi(4)\ \right)<\wp(4)<\varphi(4)=6$$

Hence, Eq.(1) holds.

Case (iii): for all $\mathfrak{y}_1,æ_2,\mathfrak{y}_2,æ_1\in\mathscr{S}\cap\mathscr{T}$, then

$$\varphi\left(\varsigma_\rho\left(\mathscr{A}(\mathfrak{y}_1,\mathfrak{y}_2),\mathscr{A}(æ_1,æ_2)\right)\right)=\varphi\left(\varsigma_\rho(0,0)\right)=\varphi(0)=0$$

and

$$\begin{aligned}\mathscr{F}_G\left(\begin{array}{c}\varphi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)),\wp(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)),\\ \varpi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))\end{array}\right)&=\wp(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))\\ &\quad-\frac{\varphi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))}{1+\varpi(\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2))}\\ &=0\end{aligned}$$

where

$$\mathbb{M}(\mathfrak{y}_1,\mathfrak{y}_2,æ_1,æ_2)=\max\left\{\begin{array}{c}\varsigma_\rho(0,0),\varsigma_\rho(0,0),\\ \frac{\varsigma_\rho(0,0).\varsigma_\rho(0,0)}{2.4^2\left[1+\varsigma_\rho(0,0)\right]},\frac{\varsigma_\rho(0,0).\varsigma_\rho(0,0)}{2.4^2\left[1+\varsigma_\rho(0,0)\right]},\end{array}\right\}=0.$$

Therefore,

$$0=\varphi(0)\leq\mathscr{F}_G\left(\ \varphi(0),\wp(0),\varpi(0)\ \right)=0.$$

Hence, Eq.(1) holds. Thus, Eq.(1) holds and all the requirements of Theorem 3.3 are satisfied. According to Theorem 3.3, $\mathscr{A}$ and $\mathfrak{f}$ have a unique strong coupled fixed point, which is $(0,0)$.

3.1. Application to Integral Equations.

In this part, we demonstrate how to prove the existence and uniqueness of a solution to a system of non-linear Fredholm integral equations using the coupled fixed point results that were obtained in our Corollary (3.4).

Let the set of all continuous functions defined on $[a,b]$ be $£=(C[a,b],\mathbb{R})$.

Let the mapping

$$d_{\varsigma_\rho}:£\times£\to[0,\infty)\text{ be defined as }d_{\varsigma_\rho}(\mathfrak{f},\mathfrak{g})=||(\mathfrak{f}-\mathfrak{g})^2||=\sup_{s\in[a,b]}|\mathfrak{f}(s)-\mathfrak{g}(s)|^2\ \forall\ \mathfrak{f},\mathfrak{g}\in£.$$

Then obviously, the pair $(\pounds, d_{\varsigma_\rho})$ is a complete b-metric space with $\mathfrak{v} = 2$.

The system of non-linear Fredholm integral equations that follows is now examined:

$$(19) \qquad \begin{cases} \mathfrak{x}(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \frac{1}{b-a}\int_a^b \mathscr{K}(\mathfrak{v}, \mathfrak{x}(\mathfrak{u}), \mathfrak{y}(\mathfrak{u}))d\mathfrak{u}, \mathfrak{v}, \mathfrak{u} \in [a,b] \\ \mathfrak{y}(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \frac{1}{b-a}\int_a^b \mathscr{K}(\mathfrak{v}, \mathfrak{y}(\mathfrak{u}), \mathfrak{x}(\mathfrak{u}))d\mathfrak{u}, \mathfrak{v}, \mathfrak{u} \in [a,b] \end{cases}$$

where $\mathscr{K} : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ and $\mathfrak{f} : [a,b] \to \mathbb{R}$ are continuous.

Define the operator $\mathscr{A} : \pounds^2 \to \pounds$ by

$$\mathscr{A}(\mathfrak{x}, \mathfrak{y})(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \frac{1}{b-a}\int_a^b \mathscr{K}(\mathfrak{v}, \mathfrak{x}(\mathfrak{u}), \mathfrak{y}(\mathfrak{u}))d\mathfrak{u}.$$

Note that Eq.(19) has a coupled solution iff $\mathscr{A}$ has a CFP.

Theorem 3.1.1: The following condition is satisfied if $\mathscr{K} : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous:
$|\mathscr{K}(\mathfrak{v}, \mathfrak{z}(\mathfrak{u}), \mathfrak{w}(\mathfrak{u})) - \mathscr{K}(\mathfrak{v}, \mathfrak{x}(\mathfrak{u}), \mathfrak{y}(\mathfrak{u}))| \leq \sqrt{\frac{1}{3}\mathbb{M}(\mathfrak{z}, \mathfrak{w}, \mathfrak{x}, \mathfrak{y})}$
where $\mathbb{M}(\mathfrak{z}, \mathfrak{w}, \mathfrak{x}, \mathfrak{y}) = \max\left\{ |\mathfrak{z}(\mathfrak{u}) - \mathfrak{x}(\mathfrak{u})|^2, |\mathfrak{w}(\mathfrak{u}) - \mathfrak{y}(\mathfrak{u})|^2 \right\}$, for all $\mathfrak{z}, \mathfrak{y} \in \mathscr{S}$ and $\mathfrak{w}, \mathfrak{x} \in \mathscr{T}$ with $\mathfrak{x} \neq \mathfrak{y} \neq \mathfrak{z} \neq \mathfrak{w}$ and $\mathfrak{v}, \mathfrak{u} \in [a,b]$. Then the system of integral equations (19) has a unique solution.

Proof This proof will be carried out using Corollary 3.4. Because the function $\mathscr{K}$ is continuous, so is the operator $\mathscr{A} : \pounds^2 \to \pounds$ defined above. Let $\varphi, \wp, \varpi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(a) = a$, $\wp(a) = \frac{3a}{2}$ and $\varpi(a) = \frac{a}{6}$ and $\mathscr{F}_G : [0,\infty) \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ as $\mathscr{F}_G(\mathfrak{e}, \varsigma, \rho) = \varsigma - \mathfrak{e} - \rho$ where $\frac{\varsigma - \mathfrak{e}}{2} > \rho$. To show that the mapping $\mathscr{A}$ creates a $\mathscr{F}_G(\varphi, \wp, \varpi)$-cyclic coupled contraction with respect to $\mathscr{S}$ and $\mathscr{T}$ and $\mathscr{S} \cap \mathscr{T} \neq \emptyset$ and with regard to a $\mathscr{C}_G$-class functions $\mathscr{F}_G$, we proceed as follows:

$$\begin{aligned} |\mathscr{A}(\mathfrak{z}, \mathfrak{w})(\mathfrak{v}) - \mathscr{A}(\mathfrak{x}, \mathfrak{y})(\mathfrak{v})| &= \frac{1}{b-a}\left|\int_a^b (\mathscr{K}(\mathfrak{v}, \mathfrak{z}(\mathfrak{u}), \mathfrak{w}(\mathfrak{u})) - \mathscr{K}(\mathfrak{v}, \mathfrak{x}(\mathfrak{u}), \mathfrak{y}(\mathfrak{u})))d\mathfrak{u}\right| \\ &\leq \frac{1}{b-a}\int_a^b |\mathscr{K}(\mathfrak{v}, \mathfrak{z}(\mathfrak{u}), \mathfrak{w}(\mathfrak{u})) - \mathscr{K}(\mathfrak{v}, \mathfrak{x}(\mathfrak{u}), \mathfrak{y}(\mathfrak{u}))|d\mathfrak{u} \\ &\leq \frac{1}{b-a}\int_a^b \sqrt{\frac{1}{3}\mathbb{M}(\mathfrak{z}, \mathfrak{w}, \mathfrak{x}, \mathfrak{y})}d\mathfrak{u}. \end{aligned}$$

By squaring spermium on both sides of the aforementioned discrepancy, we get

$$\begin{aligned}\sup_{\mathfrak{v}\in[a,b]} |\mathscr{A}(\mathfrak{z},\mathfrak{w})(\mathfrak{v})-\mathscr{A}(\mathfrak{x},\mathfrak{y})(\mathfrak{v})|^2 &\leq \frac{1}{3(b-a)^2}\sup_{\mathfrak{v}\in[a,b]}\mathbb{M}(\mathfrak{z},\mathfrak{w},\mathfrak{x},\mathfrak{y})(\int_a^b 1du)^2.\\ &\leq \frac{1}{3}\max\left\{\begin{array}{c}\sup_{\mathfrak{v}\in[a,b]}|\mathfrak{z}(\mathfrak{u})-\mathfrak{x}(\mathfrak{u})|^2,\\ \sup_{\mathfrak{v}\in[a,b]}|\mathfrak{w}(\mathfrak{u})-\mathfrak{y}(\mathfrak{u})|^2,\end{array}\right\}.\end{aligned}$$

yields that

$$d_{\varsigma_\rho}\left(\mathscr{A}(\mathfrak{z},\mathfrak{w}),\mathscr{A}(\mathfrak{x},\mathfrak{y})\right) \leq \frac{1}{3}\max\left\{\ d_{\varsigma_\rho}\left(\mathfrak{z},\mathfrak{x}\right),d_{\varsigma_\rho}\left(\mathfrak{w},\mathfrak{y}\right)\ \right\}$$

Now for any PbMS ς_ρ on $£$, we can have a b-metric d_{ς_ρ} on $£$ by

$$d_{\varsigma_\rho}(\mathfrak{z},\mathfrak{w})=\begin{cases}\varsigma_\rho(\mathfrak{z},\mathfrak{w}) & \text{if} \quad \mathfrak{z}\neq\mathfrak{w}\\ 0 & \text{if} \quad \mathfrak{z}=\mathfrak{w}.\end{cases}$$

Then the last inequality can be written as:

$$\varsigma_\rho\left(\mathscr{A}(\mathfrak{z},\mathfrak{w}),\mathscr{A}(\mathfrak{x},\mathfrak{y})\right) \leq \frac{1}{3}\max\left\{\ \varsigma_\rho\left(\mathfrak{z},\mathfrak{x}\right),\varsigma_\rho\left(\mathfrak{w},\mathfrak{y}\right)\ \right\}.$$

$$\varphi\left(\varsigma_\rho\left(\mathscr{A}(\mathfrak{z},\mathfrak{w}),\mathscr{A}(\mathfrak{x},\mathfrak{y})\right)\right)\leq \mathscr{F}_G\left(\begin{array}{c}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho\left(\mathfrak{z},\mathfrak{x}\right),\\ \varsigma_\rho\left(\mathfrak{w},\mathfrak{y}\right)\end{array}\right\}\right),\wp\left(\max\left\{\begin{array}{c}\varsigma_\rho\left(\mathfrak{z},\mathfrak{x}\right),\\ \varsigma_\rho\left(\mathfrak{w},\mathfrak{y}\right)\end{array}\right\}\right),\\ \varpi\left(\max\left\{\begin{array}{c}\varsigma_\rho\left(\mathfrak{z},\mathfrak{x}\right),\\ \varsigma_\rho\left(\mathfrak{w},\mathfrak{y}\right)\end{array}\right\}\right)\end{array}\right).$$

This shows that $\mathscr{A}$ is a $\mathscr{F}_G(\varphi,\wp,\varpi)$-cyclic coupled contraction . We thus infer that $\mathscr{A}$ has a unique coupled solution in $\mathscr{S}\cap\mathscr{T}$ to the integral equation (19) since all the requirements of the Corollary (3.4) are met.

3.2. Applications to Homotopy.

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 3.2.1: Let $\mathscr{S}$ and $\mathscr{T}$ be nonempty subsets of complete PbMS $(£,\varsigma_\rho)$ with the coefficient $\mathfrak{v}\geq 1$, $\Xi_1\subseteq\mathscr{S}$, $\Xi_2\subseteq\mathscr{T}$ and $\overline{\Xi}_1\subseteq\mathscr{S}$, $\overline{\Xi}_2\subseteq\mathscr{T}$ be an open and closed subsets such that $\Xi_1\subseteq\overline{\Xi}_1$, $\Xi_2\subseteq\overline{\Xi}_2$. Assume that the operator

$\mathfrak{L}_\varsigma : (\overline{\Xi}_1\times\overline{\Xi}_2)\cup(\overline{\Xi}_2\times\overline{\Xi}_1)\times[0,1]\to £$ satisfies the following requirements:

$\tau_0)$ æ $\neq \mathfrak{A}_\varsigma$(æ, œ, λ), œ $\neq \mathfrak{L}_\varsigma$(œ, æ, λ), for each æ $\in\overline{\Xi}_1$, œ $\in\overline{\Xi}_2$ and $\lambda\in[0,1]$ (here $\partial(\Xi_1\cup\Xi_2)$ is boundary of $\Xi_1\cup\Xi_2$ in $\mathscr{S}\cup\mathscr{T}$);

τ_1) $\forall$ æ, $\mathfrak{x} \in \overline{\Xi}_1$, œ, ß $\in \overline{\Xi}_2$, $\lambda \in [0,1]$, $\varphi \in \mathfrak{A}$, $\wp \in \Delta$, $\varpi \in \mathfrak{K}$, $\mathscr{F}_G \in \mathscr{C}_G$ with $\varphi(\mathfrak{e}) > \wp(\mathfrak{e})$ for all $\mathfrak{e} > 0$, satisfying

$$\varphi\left(\mathfrak{v}\varsigma_\rho\left(\ \mathfrak{L}_\varsigma(æ,œ,\lambda),\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda)\ \right)\right) \leq \quad \mathscr{F}_G\left(\begin{array}{c} \varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho\,(æ,ß)\,,\\ \varsigma_\rho\,(\mathfrak{x},œ)\end{array}\right\}\right), \\ \wp\left(\max\left\{\begin{array}{c}\varsigma_\rho\,(æ,ß)\,,\\ \varsigma_\rho\,(\mathfrak{x},œ)\end{array}\right\}\right), \\ \varpi\left(\max\left\{\begin{array}{c}\varsigma_\rho\,(æ,ß)\,,\\ \varsigma_\rho\,(\mathfrak{x},œ)\end{array}\right\}\right)\end{array}\right)$$

τ_2) $\exists\, M \geq 0 \ni \varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda),\mathfrak{L}_\varsigma(œ,æ,\zeta)) \leq M|\lambda-\zeta|$ for every æ $\in \overline{\Xi}_1$, œ $\in \overline{\Xi}_2$ and $\lambda,\zeta \in [0,1]$.

Then $\mathfrak{L}_\varsigma(.,s)$ has a CFP for some $s \in [0,1] \iff \mathfrak{L}_\varsigma(.,t)$ has a CFP for some $t \in [0,1]$.

Proof Consider the set

$$\mathscr{A} = \{\lambda \in [0,1] : æ = \mathfrak{L}_\varsigma(æ,œ,\lambda), œ = \mathfrak{L}_\varsigma(œ,æ,\lambda) \text{ for some } æ \in \Xi_1, œ \in \Xi_2\}.$$

Since $\mathfrak{L}_\varsigma(.,0)$ has a CFP in $(\Xi_1\times\Xi_2)\cap(\Xi_2\times\Xi_1)$, we know that $(0,0)\in\mathscr{A}^2$. demonstrating that $\mathscr{A}$ is not an empty set. In $[0,1]$, we will show that $\mathscr{A}$ is both open and closed. As a result, the connectedness may provide $\mathscr{A}=[0,1]$. As a result, $\mathfrak{L}_\varsigma(.,1)$ in $(\Xi_1\times\Xi_2)\cap(\Xi_2\times\Xi_1)$ has a CFP. First, we show that $\mathscr{A}$ is closed in $[0,1]$. Let $\{\lambda_z\}_{z=1}^{\infty}\subseteq\mathscr{A}$, where $z\to\infty$ and $\lambda_z\to\lambda\in[0,1]$. Showing that $\lambda\in\mathscr{A}$ is necessary. Considering that $\lambda_z\in\mathscr{A}$ for $z=1,2,3,\cdots$, $\exists$ æ$_z \in \Xi_1$, œ$_z\in\Xi_2$, and that $æ_z=\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda_z), œ_z=\mathfrak{L}_\varsigma(œ_z,æ_z,\lambda_z)$. Think about

$$\begin{aligned}\varsigma_\rho(æ_z,œ_{z+1}) &= \varsigma_\rho(\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda_z),\mathfrak{L}_\varsigma(œ_{z+1},æ_{z+1},\lambda_{z+1}))\\ &\leq \mathfrak{v}\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda_z),\mathfrak{L}_\varsigma(œ_{z+1},æ_{z+1},\lambda_z))\\ +\varsigma_\rho(\mathfrak{L}_\varsigma(œ_{z+1},æ_{z+1},\lambda_z),\mathfrak{L}_\varsigma(œ_{z+1},æ_{z+1},\lambda_{z+1}))\\ -\varsigma_\rho(\mathfrak{L}_\varsigma(œ_{z+1},æ_{z+1},\lambda_z),\mathfrak{L}_\varsigma(œ_{z+1},æ_{z+1},\lambda_z))\end{array}\right\}\\ &\leq \mathfrak{v}\varsigma_\rho(\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda_z),\mathfrak{L}_\varsigma(œ_{z+1},æ_{z+1},\lambda_z))+\mathfrak{v}M|\lambda_z-\lambda_{z+1}|.\end{aligned}$$

Letting $z\to\infty$, we get

$$\lim_{z\to\infty}\varsigma_\rho(æ_z,œ_{z+1}) \leq \lim_{z\to\infty}\mathfrak{v}\varsigma_\rho(\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda_z),\mathfrak{L}_\varsigma(œ_{z+1},æ_{z+1},\lambda_z))+0.$$

From (τ_1) , we obtain

$$\begin{aligned}
&\lim_{z\to\infty}\varphi\left(\varsigma_\rho(æ_z,œ_{z+1})\right)\\
&\leq \lim_{z\to\infty}\mathscr{F}_G\left(\begin{array}{c}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right),\wp\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right),\\ \varpi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right)\end{array}\right)\\
&\leq \lim_{z\to\infty}\max\left\{\begin{array}{c}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right)\\ \wp\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right)\end{array}\right\}\\
&< \lim_{z\to\infty}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right).
\end{aligned}$$

Similarly

$$\lim_{z\to\infty}\varphi\left(\varsigma_\rho(œ_z,æ_{z+1})\right) < \lim_{z\to\infty}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right)$$

Therefore, we have

$$\begin{aligned}
\lim_{z\to\infty}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right) &= \lim_{z\to\infty}\max\left\{\begin{array}{c}\varphi\left(\varsigma_\rho(æ_z,œ_{z+1})\right),\\ \varphi\left(\varsigma_\rho(œ_z,æ_{z+1})\right)\end{array}\right\}\\
&\leq \lim_{z\to\infty}\mathscr{F}_G\left(\begin{array}{c}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right),\\ \wp\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right),\\ \varpi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right)\end{array}\right)\\
&\leq \lim_{z\to\infty}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}\right).
\end{aligned}$$

Put $\delta=\lim_{z\to\infty}\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,œ_{z+1}),\\ \varsigma_\rho(œ_z,æ_{z+1})\end{array}\right\}$ then $\mathscr{F}_G\left(\ \varphi(\delta),\wp(\delta),\varpi(\delta)\ \right)=\varphi(\delta)$. By the condition (iii) of Definition (2.8), it can be deduced that either $\varphi(\delta)=0$ or $\wp(\delta)=0$. Hence, $\delta=0$. It follows that

(20) $$\lim_{z\to\infty}\varsigma_\rho(æ_z,œ_{z+1})=\lim_{z\to\infty}\varsigma_\rho(œ_z,æ_{z+1})=0.$$

From (20) and (ς_{ρ_2}), we have that

(21) $$\lim_{z\to\infty}\varsigma_\rho\,(æ_z,œ_z)=\lim_{z\to\infty}\varsigma_\rho\,(œ_z,æ_z)=0.$$

From (ς_{ρ_4}), (20) and (21), we have

$$\frac{1}{\mathfrak{v}}\varsigma_\rho\,(æ_z,æ_{z+1})+\varsigma_\rho\,(œ_z,œ_z)\quad\leq\varsigma_\rho\,(æ_z,œ_z)+\varsigma_\rho\,(œ_z,æ_{z+1})\to 0 \text{ as } z\to\infty$$

and

$$\frac{1}{\mathfrak{v}}\varsigma_\rho\,(œ_z,œ_{z+1})+\varsigma_\rho\,(æ_z,æ_z)\quad\leq\varsigma_\rho\,(œ_z,æ_z)+\varsigma_\rho\,(æ_z,œ_{z+1})\to 0 \text{ as } z\to\infty.$$

It follows that

$$\lim_{z\to\infty}\varsigma_\rho\,(æ_z,æ_{z+1})=\lim_{z\to\infty}\varsigma_\rho\,(œ_z,œ_z)=\lim_{z\to\infty}\varsigma_\rho\,(œ_z,œ_{z+1})=\lim_{z\to\infty}\varsigma_\rho\,(æ_z,æ_z)=0.$$

(22)

The Cauchy sequences in $(£,\varsigma_\rho)$ are $\{æ_z\}$ and $\{œ_z\}$, as we now show. If, on the other hand, $\{æ_z\}$ and $\{œ_z\}$ are not Cauchy, then. $z_k>w_k>k$ is a monotonic increasing sequence of natural integers $\{w_k\}$ and $\{z_k\}$ with $\varepsilon>0$,

(23) $$\varsigma_\rho(æ_{w_k},æ_{z_k})\geq\varepsilon\quad\ \varsigma_\rho(œ_{w_k},œ_{z_k})\geq\varepsilon$$

and

(24) $$\varsigma_\rho(æ_{w_{k+1}},æ_{z_k})<\varepsilon\quad\ \varsigma_\rho(œ_{w_{k+1}},œ_{z_k})<\varepsilon$$

From (23) and (24), we obtain

$$\begin{aligned}\varepsilon\ &\leq\varsigma_\rho(æ_{w_k},æ_{z_k})\\ &\leq\mathfrak{v}\left(\varsigma_\rho(æ_{w_k},æ_{w_k+1})+\varsigma_\rho(æ_{w_k+1},æ_{z_k})-\varsigma_\rho(æ_{w_k+1},æ_{w_k+1})\right)\end{aligned}$$

Using $k\to\infty$ as the limit, we obtain that

$$\begin{aligned}
\varphi(\varepsilon) &\leq \lim_{k\to\infty}\varphi\left(\mathfrak{v}\varsigma_\rho(æ_{w_k+1},æ_{z_k})\right)=\lim_{k\to\infty}\varphi\left(\mathfrak{v}\varsigma_\rho(\mathfrak{L}_\varsigma(æ_{w_{k+1}},œ_{w_{k+1}},\lambda_{w_{k+1}}),\mathfrak{L}_\varsigma(æ_{z_k},œ_{z_k},\lambda_{z_k}))\right)\\
&\leq \lim_{z\to\infty}\mathscr{F}_G\left(\begin{array}{c}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_{w_{k+1}},æ_{z_k}),\\ \varsigma_\rho(œ_{w_{k+1}},œ_{z_k})\end{array}\right\}\right),\wp\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_{w_{k+1}},æ_{z_k}),\\ \varsigma_\rho(œ_{w_{k+1}},œ_{z_k})\end{array}\right\}\right),\\ \varpi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_{w_{k+1}},æ_{z_k}),\\ \varsigma_\rho(œ_{w_{k+1}},œ_{z_k})\end{array}\right\}\right)\end{array}\right)\\
&\leq \lim_{z\to\infty}\max\left\{\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_{w_{k+1}},æ_{z_k}),\\ \varsigma_\rho(œ_{w_{k+1}},œ_{z_k})\end{array}\right\}\right),\wp\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_{w_{k+1}},æ_{z_k}),\\ \varsigma_\rho(œ_{w_{k+1}},œ_{z_k})\end{array}\right\}\right)\right\}\\
&< \lim_{z\to\infty}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_{w_{k+1}},æ_{z_k}),\\ \varsigma_\rho(œ_{w_{k+1}},œ_{z_k})\end{array}\right\}\right).\\
&< \varphi(\varepsilon).
\end{aligned}$$

Therefore, $\mathscr{F}_G\left(\ \varphi(\varepsilon),\wp(\varepsilon),\varpi(\varepsilon)\ \right)=\varphi(\varepsilon)$. By the condition (iii) of Definition (2.8), it can be deduced that either $\varphi(\varepsilon)=0$ or $\wp(\varepsilon)=0$. Hence, $\varepsilon=0$ is contradiction. Consequently, $\{æ_z\}$ is a Cauchy sequence in $(£,\varsigma_\rho)$, and $\lim_{z,w\to\infty}\varsigma_\rho(æ_z,æ_w)=0$. Likewise, we have $\{œ_z\}$ is a Cauchy sequence in $(£,\varsigma_\rho)$ and $\lim_{z,w\to\infty}\varsigma_\rho(œ_z,œ_w)=0$. $(£,\varsigma_\rho)$ is complete, so $ß\in\Xi_1$, $\mathfrak{x}\in\Xi_2$ exist.

$$\varsigma_\rho(ß,ß)=\lim_{z\to\infty}\varsigma_\rho(æ_z,ß)=\lim_{z\to\infty}\varsigma_\rho(æ_{z+1},ß)=\lim_{z,\ w\to\infty}\varsigma_\rho(æ_z,æ_w)=0.$$

$$\varsigma_\rho(\mathfrak{x},\mathfrak{x})=\lim_{z\to\infty}\varsigma_\rho(œ_z,\mathfrak{x})=\lim_{z\to\infty}\varsigma_\rho(œ_{z+1},\mathfrak{x})=\lim_{z,\ w\to\infty}\varsigma_\rho(œ_z,œ_w)=0$$

also we have

$$\varsigma_\rho(ß,\mathfrak{x})=\varsigma_\rho(\lim_{z\to\infty}æ_z,\lim_{z\to\infty}œ_z)=\lim_{z\to\infty}\varsigma_\rho(æ_z,œ_z)=0.$$

Therefore, $ß=\mathfrak{x}$, Since $\Xi_1\cap\Xi_2\neq\emptyset$ then from the above it follows that $ß\in\Xi_1\cap\Xi_2$. From Lemma 2.6, we get $\lim_{z\to\infty}\varsigma_\rho(æ_z,\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda))=\varsigma_\rho(ß,\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda))$.

Now,

$$\begin{aligned}
\varsigma_\rho(æ_z,\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda)) &=\varsigma_\rho(\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda_z),\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda))\\
&\leq\mathfrak{v}\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda_z),\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda))+\varsigma_\rho(\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda),\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda))\\ -\varsigma_\rho(\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda),\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda))\end{array}\right\}\\
&\leq\mathfrak{v}M|\lambda_z-\lambda|+\mathfrak{v}\varsigma_\rho(\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda),\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda)).
\end{aligned}$$

Letting $z \to \infty$ and using property of φ, we obtain

$$\begin{aligned}
\varphi\left(\varsigma_\rho(ß,\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda))\right) &\leq \lim_{z\to\infty}\varphi\left(\mathfrak{v}\varsigma_\rho(\mathfrak{L}_\varsigma(æ_z,œ_z,\lambda),\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda))\right) \\
&\leq \lim_{z\to\infty}\mathscr{F}_G\left(\begin{array}{c}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,ß),\\ \varsigma_\rho(œ_z,\mathfrak{x})\end{array}\right\}\right),\wp\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,ß),\\ \varsigma_\rho(œ_z,\mathfrak{x})\end{array}\right\}\right),\\ \varpi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ_z,ß),\\ \varsigma_\rho(œ_z,\mathfrak{x})\end{array}\right\}\right)\end{array}\right) \\
&< \varphi(0)=0
\end{aligned}$$

consequently, we conclude that $\varsigma_\rho(ß,\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda))=0$ and $\varsigma_\rho(\mathfrak{x},\mathfrak{L}_\varsigma(\mathfrak{x},ß,\lambda))=0$. In order for $\mathfrak{L}_\varsigma(ß,\mathfrak{x},\lambda)=ß$ and $\mathfrak{L}_\varsigma(\mathfrak{x},ß,\lambda)=\mathfrak{x}$ to be equal.Therefore, $\lambda\in A$. Thus, in [0, 1], $\mathscr{A}$ is closed. Let $\mathscr{A}$ contain λ_0. When $æ_0=\mathfrak{L}_\varsigma(æ_0,œ_0,\lambda_0)$, then $\exists\, æ_0\in\Xi_1, œ_0\cap\Xi_2$ as well as $œ_0=\mathfrak{L}_\varsigma(œ_0,æ_0,\lambda_0)$. $B_{\varsigma_\rho}(æ_0,r)\subseteq\Xi_1$ and $B_{\varsigma_\rho}(œ_0,r)\subseteq\Xi_2$ since $\Xi_1\cup\Xi_2$ is open.
Choose $\lambda\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$ such that $|\lambda-\lambda_0|\leq\frac{1}{M^z}<\varepsilon$.
Then for $æ\in\overline{B_{\varsigma_\rho}(æ_0,r)}=\{æ\in £/\varsigma_\rho(æ,æ_0)\leq r+\varsigma_\rho(æ_0,æ_0)\}$ and
$œ\in\overline{B_{\varsigma_\rho}(œ_0,r)}=\{œ\in £/\varsigma_\rho(œ,œ_0)\leq r+\varsigma_\rho(œ_0,œ_0)\}$. Now we have

$$\begin{aligned}
\varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda),æ_0) &= \varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda),\mathfrak{L}_\varsigma(æ_0,œ_0,\lambda_0)) \\
&\leq \mathfrak{v}\left\{\begin{array}{c}\varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda),\mathfrak{L}_\varsigma(æ,œ,\lambda_0))+\varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda_0),\mathfrak{L}_\varsigma(æ_0,œ_0,\lambda_0))\\ -\varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda_0),\mathfrak{L}_\varsigma(æ,œ,\lambda_0))\end{array}\right\} \\
&\leq \mathfrak{v}M|\lambda-\lambda_0|+\mathfrak{v}\varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda_0),\mathfrak{L}_\varsigma(æ_0,œ_0,\lambda_0)) \\
&\leq \mathfrak{v}\tfrac{1}{M^{z-1}}+\mathfrak{v}\varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda_0),\mathfrak{L}_\varsigma(æ_0,œ_0,\lambda_0)).
\end{aligned}$$

Letting $z\to\infty$ and using the property of φ, we obtain

$$\begin{aligned}
\varphi(\varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda),æ_0)) &\leq \varphi\left(\mathfrak{v}\varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda_0),\mathfrak{L}_\varsigma(æ_0,œ_0,\lambda_0))\right) \\
&\leq \lim_{z\to\infty}\mathscr{F}_G\left(\begin{array}{c}\varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ,æ_0),\\ \varsigma_\rho(œ,œ_0)\end{array}\right\}\right),\wp\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ,æ_0),\\ \varsigma_\rho(œ,œ_0)\end{array}\right\}\right),\\ \varpi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ,æ_0),\\ \varsigma_\rho(œ,œ_0)\end{array}\right\}\right)\end{array}\right) \\
&< \varphi\left(\max\left\{\begin{array}{c}\varsigma_\rho(æ,æ_0),\\ \varsigma_\rho(œ,œ_0)\end{array}\right\}\right)
\end{aligned}$$

Similarly

$$\varphi\left(\varsigma_\rho(\mathfrak{L}_\varsigma(œ,æ,\lambda),œ_0)\right) < \varphi\left(\max\left\{\varsigma_\rho(æ,æ_0),\varsigma_\rho(œ,œ_0)\right\}\right).$$

Thus

$$\begin{aligned}\max\{\varsigma_\rho(\mathfrak{L}_\varsigma(æ,œ,\lambda),æ_0),\varsigma_\rho(\mathfrak{L}_\varsigma(œ,æ,\lambda),œ_0)\} &< \max\{\varsigma_\rho(æ,æ_0),\varsigma_\rho(œ,œ_0)\}\\ &\leq \max\{r+\varsigma_\rho(æ_0,æ_0),r+\varsigma_\rho(œ_0,œ_0)\}\end{aligned}.$$

Hence, for every constant $\lambda \in (\lambda_0-\varepsilon,\lambda_0+\varepsilon)$, this means that
$\mathfrak{L}_\varsigma(.,\lambda) : \overline{B_{\varsigma_\rho}(æ_0,r)} \to \overline{B_{\varsigma_\rho}(æ_0,r)}$ and $\mathfrak{L}_\varsigma(.,\lambda) : \overline{B_{\varsigma_\rho}(œ_0,r)} \to \overline{B_{\varsigma_\rho}(œ_0,r)}$. Theorem 3.2.1 is satisfied in all of its conditions since (τ_1) likewise holds. This leads us to conclude that $\mathfrak{L}_\varsigma(.,\lambda)$ in $(\overline{\Xi}_1\times\overline{\Xi}_2)\cap(\overline{\Xi}_2\times\overline{\Xi}_1)$ has a CFP. But this CFP must be in $(\Xi_1\times\Xi_2)\cap(\Xi_2\times\Xi_1)$ since (τ_0) holds. $\lambda\in\mathscr{A}$ for any $\lambda\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$. $(\lambda_0-\varepsilon,\lambda_0+\varepsilon)\subseteq\mathscr{A}$, thus. As a result, $\mathscr{A}$ is open in [0, 1]. We use the same technique for the opposite inference.

4. Conclusions

This work utilizes several CFP results and relevant examples to elucidate the main conclusions within the framework of PbMS. It employs $\mathscr{F}_G(\varphi,\wp,\varpi)$-cyclic rational contraction via a generalized $\mathscr{C}_G$-class function. Additionally, it demonstrates applications to homotopy and nonlinear integral equations.

Conflict of Interests

The authors declare that there is no conflict of interests.

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