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COMMON COUPLED FIXED POINT THEOREMS VIA ψ-CONTRACTIONS IN GENERALIZED G-FUZZY METRIC SPACES

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Abstract. In this paper, we introduce a new class of contractive conditions using ψ-contractions in the setting of symmetric G-fuzzy metric spaces. We establish common coupled fixed point theorems for mappings satisfying the ψ-contractive condition under suitable compatibility assumptions. Our results generalize several known fixed point theorems in the literature and are supported by illustrative examples.

Keywords: G-fuzzy metric spaces; ψ-contractions; fixed point.

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1. Introduction

Mustafa and Sims [10, 11, 14] and Naidu et al. [17] critically examined the structure of the D-metric space originally introduced by Dhage [5, 6, 7, 8] and showed that many of the associated topological claims and fixed point theorems are fundamentally flawed. In response, Mustafa and Sims proposed the notion of a *G-metric space*, a framework which rectifies several deficiencies of D-metrics and allows for a more robust fixed point theory. Subsequently, a number of fixed point results have been developed in G-metric spaces (see [1, 4, 12, 13, 14, 15, 16, 18]), as well as in the context of generalized and fuzzy extensions [19].

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In this paper, we aim to extend these developments by proving two unique common coupled fixed point theorems for multivalued mappings of Jungck-type and for three self-mappings S, T, and R in the setting of symmetric G-fuzzy metric spaces. Our approach utilizes ψ-contractive conditions adapted for fuzzy multivalued frameworks, enhancing the existing theory with more generalized contractive mappings.

In this section, we recall some essential definitions and properties related to G-metric and G-fuzzy metric spaces that form the foundation of our subsequent results.

Definition 1.1. [11] Let X be a nonempty set. A function $G : X \times X \times X \to [0,\infty)$ is called a *G-metric* on X if the following conditions hold for all $x,y,z,a \in X$:

- **(G1)** $G(x,y,z) = 0$ if and only if $x = y = z$,
- **(G2)** $G(x,x,y) > 0$ whenever $x \neq y$,
- **(G3)** $G(x,x,y) \leq G(x,y,z)$ whenever $y \neq z$,
- **(G4)** $G(x,y,z)$ is symmetric in all three variables,
- **(G5)** $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$.

The pair (X,G) is then called a *G-metric space*.

Definition 1.2. [11] A G-metric space (X,G) is said to be *symmetric* if

$$G(x,x,y) = G(x,y,y) \quad \text{for all } x,y \in X.$$

Definition 1.3. [11] Let (X,G) be a G-metric space and $\{x_n\}$ be a sequence in X. A point $x \in X$ is called the *G-limit* of $\{x_n\}$ if

$$\lim_{n,m\to\infty} G(x,x_n,x_m) = 0.$$

In this case, the sequence $\{x_n\}$ is said to be *G-convergent* to x.

Definition 1.4. [11] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ in X is said to be *G-Cauchy* if

$$\lim_{l,m,n\to\infty} G(x_l,x_m,x_n) = 0.$$

The space (X,G) is called *G-complete* if every G-Cauchy sequence in X is G-convergent to some point in X.

Proposition 1.5. [11] Let (X,G) be a G-metric space. The following are equivalent for a sequence $\{x_n\} \subset X$:

(i) $\{x_n\}$ is G-Cauchy,

(ii) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$G(x_n, x_m, x_m) < \varepsilon \quad \text{for all } n,m \geq N.$$

Proposition 1.6. [11] Let (X,G) be a G-metric space. Then G is jointly continuous in all three variables.

Proposition 1.7. [11] In a G-metric space (X,G), the following properties hold for all $x,y,z,a \in X$:

(i) $G(x,y,z) = 0 \Rightarrow x = y = z$,

(ii) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,

(iii) $G(x,y,y) \leq 2G(x,x,y)$,

(iv) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$,

(v) $G(x,y,z) \leq \frac{2}{3}[G(x,a,a) + G(y,a,a) + G(z,a,a)]$.

Proposition 1.8. [11] Let (X,G) be a G-metric space. For a sequence $\{x_n\} \subset X$ and $x \in X$, the following are equivalent:

(i) $x_n \to x$ (G-convergence),

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ as $m,n \to \infty$.

In a recent contribution, Sun and Yang [19] introduced the notion of G-fuzzy metric spaces and established two common fixed point theorems involving four self-mappings.

Definition 1.9. [19] A triple $(X,G,*)$ is called a *G-fuzzy metric space* if X is a nonempty set, $*$ is a continuous t-norm, and $G : X^3 \times (0,\infty) \to [0,1]$ satisfies the following for all $x,y,z,a \in X$ and $t,s > 0$:

(i) $G(x,x,y,t) > 0$ for $x \neq y$,

(ii) $G(x,x,y,t) \geq G(x,y,z,t)$ whenever $y \neq z$,

(iii) $G(x,y,z,t) = 1$ if and only if $x = y = z$,

(iv) G is invariant under any permutation of $\{x,y,z\}$,

(v) $G(x,y,z,t+s) \geq G(a,y,z,t) * G(x,a,a,s)$,

(vi) The mapping $t \mapsto G(x,y,z,t)$ is continuous.

Definition 1.10. [19] A G-fuzzy metric space $(X,G,*)$ is called *symmetric* if

$$G(x,x,y,t) = G(x,y,y,t) \quad \text{for all } x,y \in X \text{ and } t > 0.$$

Example 1.11. Let (X,G) be a G-metric space, and let $*$ be the product t-norm defined by $a * b = ab$ for all $a,b \in [0,1]$. Define

$$G(x,y,z,t) = \frac{t}{t + G(x,y,z)}.$$

Then $(X,G,*)$ is a G-fuzzy metric space.

For $t > 0$, $0 < r < 1$, define the open ball centered at $x \in X$ as:

$$B_G(x,r,t) = \{y \in X : G(x,y,y,t) > 1 - r\}.$$

A subset $A \subseteq X$ is said to be *open* if for every $x \in A$, there exist $t > 0$ and $r \in (0,1)$ such that $B_G(x,r,t) \subseteq A$.

A sequence $\{x_n\}$ in X is:

- *G-convergent* to $x \in X$ if $G(x_n,x_n,x,t) \to 1$ for each $t > 0$,
- *G-Cauchy* if $G(x_n,x_n,x_m,t) \to 1$ as $n,m \to \infty$ for each $t > 0$.

The space is *G-complete* if every G-Cauchy sequence is G-convergent.

Lemma 1.12. [19] *Let $(X,G,*)$ be a G-fuzzy metric space. Then for all $x,y,z \in X$, the function $t \mapsto G(x,y,z,t)$ is non-decreasing.*

Lemma 1.13. [19] *In a G-fuzzy metric space $(X,G,*)$, we have:*

$$\lim_{t \to \infty} G(x,y,z,t) = 1 \quad \text{for all } x,y,z \in X.$$

This is referred to as Property (P).

Definition 1.14. [9] A mapping $\psi : [0,1) \to [0,1)$ is called a *ψ-function* if:

- ψ is continuous and strictly increasing,
- $\psi(t) < t$ for all $t \in (0,1)$,
- $\lim_{t\to 0^+} \psi(t) = 0$.

Lemma 1.15. *Let $(X,G,*)$ be a G-fuzzy metric space. Suppose there exists a ψ-function ψ : $[0,1) \to [0,1)$ and a constant $k \in (0,1)$ such that*

$$\min\{G(x,y,z,kt),\ G(u,v,w,kt)\} \geq \psi(\min\{G(x,y,z,t),\ G(u,v,w,t)\})$$

for all $x,y,z,u,v,w \in X$ and for all $t > 0$. Then it must be that

$$x = y = z \quad \textit{and} \quad u = v = w.$$

Definition 1.16. Let $S, f : X \to X$ be two mappings. They are said to satisfy a *ψ-contractive condition* if for all $x,y \in X$ and $t > 0$,

$$G(Sx,Sy,Sy,t) \geq \psi(G(fx,fy,fy,t)).$$

Definition 1.17. [3] Let $F : X \times X \to X$. An element $(x,y) \in X \times X$ is called a *coupled fixed point* of F if

$$x = F(x,y) \quad \text{and} \quad y = F(y,x).$$

Definition 1.18. [2] Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then:

(i) A pair $(x,y) \in X \times X$ is a *coupled coincidence point* if

$$gx = F(x,y), \quad gy = F(y,x).$$

(ii) A pair $(x,y) \in X \times X$ is a *common coupled fixed point* if

$$x = gx = F(x,y), \quad y = gy = F(y,x).$$

Definition 1.19. [2] Mappings $F : X \times X \to X$ and $g : X \to X$ are said to be *W-compatible* if

$$g(F(x,y)) = F(gx,gy), \quad g(F(y,x)) = F(gy,gx)$$

whenever $gx = F(x,y)$ and $gy = F(y,x)$ for some $(x,y) \in X \times X$.

2. Main Result

Theorem 2.1. *Let $(X,G,*)$ be a symmetric G-fuzzy metric space with $* = \min$. Let $S : X\times X \to X$ and $f : X \to X$ be mappings such that:*

(1) For all $x,y,u,v \in X$ and $t > 0$,

$$G(S(x,y),S(u,v),S(u,v),kt) \geq \psi\big(\min\{G(fx,fu,fu,t),G(fy,fv,fv,t)\}\big),$$

(2) $S(X\times X) \subset f(X)$,

(3) $f(X)$ is G-complete,

(4) The pair (f,S) is W-compatible.

Then S and f have a unique common coupled fixed point (α,α) in $X\times X$.

Proof. Let $x_0,y_0 \in X$, and define sequences $\{x_n\},\{y_n\}$ in X recursively by

$$z_n := S(x_n,y_n) = f(x_{n+1}), \quad p_n := S(y_n,x_n) = f(y_{n+1}) \quad \text{for } n \in \mathbb{N}.$$

Define the following fuzzy distances:

$$d_n(t) := G(z_n,z_{n+1},z_{n+1},t), \quad e_n(t) := G(p_n,p_{n+1},p_{n+1},t).$$

From condition (i) and symmetry of G, we have:

$$\begin{aligned} d_{n+1}(kt) &= G(z_{n+1},z_{n+2},z_{n+2},kt) \\ &= G(S(x_{n+1},y_{n+1}),S(x_{n+2},y_{n+2}),S(x_{n+2},y_{n+2}),kt) \\ &\geq \psi(\min\{G(fx_{n+1},fx_{n+2},fx_{n+2},t),G(fy_{n+1},fy_{n+2},fy_{n+2},t)\}) \\ &= \psi(\min\{d_n(t),e_n(t)\}). \end{aligned} \tag{2.1}$$

Since $\psi(r) > r$ for $r \in [0,1)$, this implies that $\min\{d_n(t),e_n(t)\}$ is a non-decreasing sequence in $[0,1]$ bounded above by 1, hence convergent. Similarly, for $e_{n+1}(kt)$:

$$e_{n+1}(kt) = G(p_{n+1},p_{n+2},p_{n+2},kt) \geq \psi(\min\{e_n(t),d_n(t)\}). \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$\min\{d_{n+1}(kt),e_{n+1}(kt)\} \geq \min\{d_n(t),e_n(t)\}.$$

By repeating this inequality recursively, we get:

$$\begin{aligned}\min\{d_n(t), e_n(t)\} &\geq \min\left\{d_{n-1}\left(\frac{t}{k}\right), e_{n-1}\left(\frac{t}{k}\right)\right\}\\ &\vdots\\ &\geq \min\left\{d_0\left(\frac{t}{k^n}\right), e_0\left(\frac{t}{k^n}\right)\right\}.\end{aligned}$$

That is,

$$\min\{d_n(t), e_n(t)\} \geq \min\left\{G(z_0, z_1, z_1, \tfrac{t}{k^n}), G(p_0, p_1, p_1, \tfrac{t}{k^n})\right\}. \tag{2.3}$$

Now for any $n, p \in \mathbb{N}$, by property (G5) and definition of G-fuzzy metric, we have:

$$\begin{aligned}G(z_n, z_{n+p}, z_{n+p}, t) &\geq G(z_{n+p-1}, z_{n+p}, z_{n+p}, t/p) * G(z_{n+p-2}, z_{n+p-1}, z_{n+p-1}, t/p)\\ &\quad * \cdots * G(z_n, z_{n+1}, z_{n+1}, t/p)\\ &\geq \prod_{i=0}^{p-1} \min\left\{G(z_0, z_1, z_1, \tfrac{t}{pk^{n+i}}), G(p_0, p_1, p_1, \tfrac{t}{pk^{n+i}})\right\}. \quad (2.4)\end{aligned}$$

Letting $n \to \infty$ and using the property (P) of the G-fuzzy metric (i.e., $\lim_{t\to 0^+}$ $G(x,y,z,t) = 1$ for all $x, y, z \in X$), we obtain:

$$\lim_{n\to\infty} G(z_n, z_{n+p}, z_{n+p}, t) = 1 \quad \text{for all } p \in \mathbb{N}. \tag{2.5}$$

This implies that $\{z_n\}$ is a Cauchy sequence in the G-fuzzy metric space $(X, G, *)$. Similarly, we can show that $\{p_n\}$ is also a Cauchy sequence in $(X, G, *)$.

Hence, both sequences $\{z_n\}$ and $\{p_n\}$ are G-Cauchy in $f(X)$. Since $f(X)$ is G-complete by assumption (iii), there exist $\alpha, \beta \in f(X)$ such that

$$\lim_{n\to\infty} G(z_n, \alpha, \alpha, t) = \lim_{n\to\infty} G(p_n, \beta, \beta, t) = 1.$$

This implies that $z_n \to \alpha$ and $p_n \to \beta$ in the fuzzy G-metric space. Since $\alpha, \beta \in f(X)$, there exist $x, y \in X$ such that

$$\alpha = f(x), \quad \beta = f(y).$$

Now we show that $S(x,y) = f(x)$ and $S(y,x) = f(y)$. Observe that:

$$\begin{aligned}G(z_n, S(x,y), S(x,y), kt) &= G(S(x_n, y_n), S(x,y), S(x,y), kt)\\ &\geq \psi\left(\min\{G(fx_n, fx, fx, t), G(fy_n, fy, fy, t)\}\right).\end{aligned}$$

Taking $n \to \infty$ and using continuity and convergence, we get

$$G(fx, S(x,y), S(x,y), kt) = 1 \Rightarrow S(x,y) = f(x).$$

Similarly, we show that $S(y,x) = f(y)$.

Since the pair (f, S) is W-compatible, we have

$$f(\alpha) = f(fx) = f(S(x,y)) = S(fx, fy) = S(\alpha, \beta),$$

$$f(\beta) = f(fy) = f(S(y,x)) = S(fy, fx) = S(\beta, \alpha),$$

which implies:

$$\alpha = f(\alpha) = S(\alpha, \beta), \quad \beta = f(\beta) = S(\beta, \alpha).$$

Hence, (α, β) is a common coupled fixed point of S and f.

To prove uniqueness, suppose (α_1, β_1) is another such point. Then

$$G(\alpha, \alpha_1, \alpha_1, kt) = G(S(\alpha, \beta), S(\alpha_1, \beta_1), S(\alpha_1, \beta_1), kt)$$

$$\geq \psi\left(\min\{G(\alpha, \alpha_1, \alpha_1, t), G(\beta, \beta_1, \beta_1, t)\}\right),$$

$$G(\beta, \beta_1, \beta_1, kt) = G(S(\beta, \alpha), S(\beta_1, \alpha_1), S(\beta_1, \alpha_1), kt)$$

$$\geq \psi\left(\min\{G(\alpha, \alpha_1, \alpha_1, t), G(\beta, \beta_1, \beta_1, t)\}\right).$$

Hence, applying the ψ condition and letting $n \to \infty$ gives:

$$\min\{G(\alpha, \alpha_1, \alpha_1, t), G(\beta, \beta_1, \beta_1, t)\} = 1 \Rightarrow \alpha = \alpha_1, \quad \beta = \beta_1.$$

Thus, the common coupled fixed point is unique.

Now we show $\alpha = \beta$. Observe:

$$G(\alpha, \alpha, \beta, kt) = G(S(\alpha, \beta), S(\alpha, \beta), S(\beta, \alpha), kt)$$

$$\geq \psi\left(\min\{G(\alpha, \alpha, \beta, t), G(\beta, \beta, \alpha, t)\}\right),$$

$$G(\alpha, \beta, \beta, kt) = G(S(\alpha, \beta), S(\beta, \alpha), S(\beta, \alpha), kt)$$

$$\geq \psi\left(\min\{G(\alpha, \beta, \beta, t), G(\beta, \alpha, \alpha, t)\}\right).$$

Taking limits implies:

$$\alpha = \beta.$$

Therefore, (α, α) is the unique common coupled fixed point.

Suppose α_1 is another such point. Then:

$$G(\alpha_1,\alpha,\alpha,t) = G(S(\alpha_1,\alpha_1),S(\alpha,\alpha),S(\alpha,\alpha),t) \geq \psi(G(\alpha_1,\alpha,\alpha,t)).$$

By $\psi(t) < t$, this implies $G(\alpha_1,\alpha,\alpha,t) = 1$, i.e., $\alpha_1 = \alpha$.

Hence, S and f have a unique common coupled fixed point of the form (α,α). □

Example 2.2. Let $X = [0,1]$ with the usual metric $d(x,y) = |x-y|$. Define the mappings and functions:

$$G(x,y) = \frac{|x-y|}{1+|x-y|},$$
$$\psi(t) = \frac{t}{2},$$
$$S(x,y) = \frac{x+y}{3},$$
$$f(x) = \frac{x}{2}.$$

We show that all the conditions of Theorem 2.1 are satisfied:

- The space (X,d) is a complete metric space.
- For all $x,y \in X$, we compute:

  $$G(fx,fy) = \frac{|f(x)-f(y)|}{1+|f(x)-f(y)|} = \frac{\frac{1}{2}|x-y|}{1+\frac{1}{2}|x-y|}.$$

  Also, since

  $$S(x,y) = \frac{x+y}{3}, \quad S(y,x) = \frac{y+x}{3} = S(x,y),$$

  we have

  $$G(Sx,Sy) = \frac{|S(x,y)-S(y,x)|}{1+|S(x,y)-S(y,x)|} = 0.$$

  Therefore,

  $$G(fx,fy) = \frac{\frac{1}{2}|x-y|}{1+\frac{1}{2}|x-y|} \leq \psi\left(G(Sx,Sy)\right) = \psi(0) = 0.$$

  So the contractive condition holds for all $x \neq y$.
- Also, observe that $f(S(x,y)) = f\left(\frac{x+y}{3}\right) = \frac{x+y}{6}$ and $S(fx,fy) = \frac{\frac{x}{2}+\frac{y}{2}}{3} = \frac{x+y}{6}$, so

  $$f(S(x,y)) = S(fx,fy),$$

  which shows that f and S are W-compatible.

- Finally, the point $(0,0)$ satisfies

$$f(0) = 0 = S(0,0),$$

so it is a common fixed point.

Thus, all the conditions of Theorem 2.1 are satisfied, and $(0,0)$ is the unique common fixed point.

Theorem 2.3. *Let $(X,G,*)$ be a symmetric G-complete fuzzy metric space, where $*(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$, and let $\psi : [0,1] \to [0,1]$ be a continuous function satisfying:*

$$\psi(t) < t \quad \textit{for all } t \in (0,1], \quad \textit{and } \psi(0) = 0.$$

Suppose that the mappings $S,T,R : X \times X \to X$ satisfy the following ψ-contractive condition:

$$\begin{aligned} G(S(x,y),T(u,v),R(p,q),kt) \geq \psi\big(&\max\{G(x,u,p,t),\ G(y,v,q,t),\ G(x,x,S(x,y),t),\\ &G(u,u,T(u,v),t),\ G(p,p,R(p,q),t)\}\big) \end{aligned} \tag{2.6}$$

for all $x,y,u,v,p,q \in X$ and some $k \in [0,1)$.

Then there exists a pair $(x,y) \in X \times X$ such that

$$x = S(x,y) = T(x,y) = R(x,y), \quad y = S(y,x) = T(y,x) = R(y,x). \tag{2.7}$$

Moreover, if the common coupled fixed point is of the form (x,x), then it is unique.

Proof. Let $x_0, y_0 \in X$. Define sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$\begin{aligned} x_{3n+1} &= S(x_{3n}, y_{3n}), & y_{3n+1} &= S(y_{3n}, x_{3n}),\\ x_{3n+2} &= T(x_{3n+1}, y_{3n+1}), & y_{3n+2} &= T(y_{3n+1}, x_{3n+1}),\\ x_{3n+3} &= R(x_{3n+2}, y_{3n+2}), & y_{3n+3} &= R(y_{3n+2}, x_{3n+2}), \end{aligned}$$

for all $n = 0,1,2,\ldots$.

Suppose that $x_{3n+1} = x_{3n}$ for some n. Then $S(x,y) = x$, where $x = x_{3n}$ and $y = y_{3n}$. If $T(x,y) \neq R(x,y)$, then by the contractive condition, we have:

$$G(x,T(x,y),R(x,y),kt) = G(S(x,y),T(x,y),R(x,y),kt)$$
$$\geq \psi(\max\{1,1,1,G(x,x,T(x,y),t),G(x,x,R(x,y),t)\})$$
$$\geq G(x,T(x,y),R(x,y),t),$$

a contradiction. Hence, $T(x,y) = R(x,y)$. Since the space is symmetric, we get

$$G(x,T(x,y),T(x,y),kt) \geq G(x,x,T(x,y),t) = G(x,T(x,y),T(x,y),t),$$

and by Lemma 1.15, this implies $T(x,y) = x$. Thus, $S(x,y) = T(x,y) = R(x,y) = x$. Similarly, if $x_{3n+1} = x_{3n+2}$ or $x_{3n+2} = x_{3n+3}$, then again $S(x,y) = T(x,y) = R(x,y) = x$.

Likewise, if $y_{3n} = y_{3n+1}$ or $y_{3n+1} = y_{3n+2}$ or $y_{3n+2} = y_{3n+3}$, then:

$$S(y,x) = T(y,x) = R(y,x) = y,$$

for some $(x,y) \in X \times X$.

Now assume $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n. Define:

$$d_n(t) = G(x_n,x_{n+1},x_{n+2},t), \quad e_n(t) = G(y_n,y_{n+1},y_{n+2},t).$$

From the construction:

$$d_{3n}(kt) = G(x_{3n},x_{3n+1},x_{3n+2},kt)$$
$$= G(S(x_{3n},y_{3n}),T(x_{3n+1},y_{3n+1}),R(x_{3n-1},y_{3n-1}),kt)$$
$$\geq \psi\Big[\max\Big\{d_{3n-1}(t),\, e_{3n-1}(t),\, G(x_{3n},x_{3n},x_{3n+1},t),$$
$$G(x_{3n+1},x_{3n+1},x_{3n+2},t),\, G(x_{3n-1},x_{3n-1},x_{3n},t)\Big\}\Big]$$
$$\geq \psi(\max\{d_{3n-1}(t),\, e_{3n-1}(t),\, d_{3n}(t)\}).$$

Since $\psi(t) < t$ for all $t > 0$, this implies:

$$d_{3n}(kt) \geq \psi(\max\{d_{3n-1}(t), e_{3n-1}(t)\}).$$

Similarly:

$$e_{3n}(kt) \geq \psi(\max\{d_{3n-1}(t), e_{3n-1}(t)\}),$$

and continuing:

$$\min\{d_{n+1}(kt), e_{n+1}(kt)\} \geq \psi\left(\min\{d_n(t), e_n(t)\}\right).$$

Since $\psi(t) < t$, this gives:

$$\begin{aligned}
\min\{d_n(t), e_n(t)\} &\geq \min\left\{d_n\left(\frac{t}{k}\right), e_n\left(\frac{t}{k}\right)\right\} \\
&\geq \min\left\{d_n\left(\frac{t}{k^2}\right), e_n\left(\frac{t}{k^2}\right)\right\} \\
&\vdots \\
&\geq \min\left\{d_0\left(\frac{t}{k^n}\right), e_0\left(\frac{t}{k^n}\right)\right\}. \\
&= \min\left\{G(x_0, x_1, x_2, t/k^n),\ G(y_0, y_1, y_2, t/k^n)\right\}.
\end{aligned}$$

Hence,

$$\min\{d_n(t), e_n(t)\} \geq \min\left\{G(x_0, x_1, x_2, t/k^n),\ G(y_0, y_1, y_2, t/k^n)\right\}. \tag{32}$$

So,

$$G(x_n, x_{n+1}, x_{n+2}, t) \geq \min\left\{G(x_0, x_1, x_2, t/k^n),\ G(y_0, y_1, y_2, t/k^n)\right\}. \tag{33}$$

By property (G3), we also have:

$$\begin{aligned}
G(x_n, x_n, x_{n+1}, t) &\geq G(x_n, x_{n+1}, x_{n+2}, t) \\
&\geq \min\left\{G(x_0, x_1, x_2, t/k^n),\ G(y_0, y_1, y_2, t/k^n)\right\}. \qquad (34)
\end{aligned}$$

Iterating this and using the monotonicity of ψ, we get:

$$\min\{d_n(t), e_n(t)\} \geq \psi^n\left(\min\{d_0(t/k^n), e_0(t/k^n)\}\right) \to 1 \quad \text{as } n \to \infty.$$

So by properties of G, we conclude:

$$G(x_n, x_{n+1}, x_{n+2}, t) \to 1, \quad G(x_n, x_n, x_{n+1}, t) \to 1,$$

and similarly for y_n. Thus, $\{x_n\}$ and $\{y_n\}$ are G-Cauchy sequences in X, and since X is G-complete, there exist $x, y \in X$ such that:

$$x_n \to x, \quad y_n \to y.$$

Now consider the limit:

$$\begin{aligned}G(S(x,y),x_{3n+2},x_{3n+3},kt) &= G(S(x,y),T(x_{3n+1},y_{3n+1}),R(x_{3n+2},y_{3n+2}),kt)\\ &\geq \psi\left(\max\{G(x,x_{3n+1},x_{3n+2},t),G(y,y_{3n+1},y_{3n+2},t),\right.\\ &\quad G(x,x,S(x,y),t),G(x_{3n+1},x_{3n+1},x_{3n+2},t),\\ &\quad \left. G(x_{3n+2},x_{3n+2},x_{3n+3},t)\}\right).\end{aligned}$$

Letting $n\to\infty$, and using the continuity of G and ψ, we get:

$$G(S(x,y),x,x,kt)\geq \psi(G(x,x,S(x,y),t)).$$

By Lemma 1.15, this implies $S(x,y)=x$. As in earlier steps, this leads to:

$$S(x,y)=T(x,y)=R(x,y)=x,\quad S(y,x)=T(y,x)=R(y,x)=y,$$

so (x,y) is a common coupled fixed point.

Now, suppose (x_1,y_1) is another common coupled fixed point. Then:

$$\begin{aligned}G(x,x,x_1,kt) &= G(S(x,y),T(x,y),R(x_1,y_1),kt)\\ &\geq \psi\left(\max\{G(x,x,x_1,t),G(y,y,y_1,t),1,1,1\}\right),\end{aligned}$$

which implies:

$$G(x,x,x_1,kt)\geq \psi(G(x,x,x_1,t)).$$

Similarly:

$$G(y,y,y_1,kt)\geq \psi(G(y,y,y_1,t)).$$

From Lemma 1.15, this gives $x=x_1$, $y=y_1$, so the common coupled fixed point is unique.

Finally, to show $x=y$, consider:

$$\begin{aligned}G(x,x,y,kt) &= G(S(x,y),T(x,y),R(y,x),kt)\\ &\geq \psi\left(\max\{G(x,x,y,t),G(y,y,x,t),1,1,1\}\right)\\ &= \psi(G(x,x,y,t)).\end{aligned}$$

Thus, by Lemma 1.15 again, $x=y$.

Therefore, S,T,R have a unique common coupled fixed point of the form (x,x). □

3. Numerical Example

Example 3.1. Let $X=[0,1]$ and define the function $G:X^3\times(0,\infty)\to[0,1]$ by

$$G(x,y,z,t)=\frac{t}{t+\max\{|x-y|,|y-z|,|z-x|\}}.$$

Then $(X,G,*)$ is a G-fuzzy metric space under the standard triangular norm $a*b=ab$.

Define mappings $f,S,T,R:X\to X$ as follows:

$$f(x)=\frac{x}{4},\quad S(x,y)=\min\{f(x),f(y)\},\quad T(x)=\frac{x}{5},\quad R(x)=\frac{x}{6}.$$

Let $\psi:[0,1]\to[0,1]$ be defined by $\psi(t)=\frac{t}{2}$. Clearly, $\psi\in\Psi$, and $\psi(t)<t$ for all $t>0$.

We claim that the mappings f,S,T,R satisfy all the assumptions of Theorem 2.1 and Theorem 2.3.

(1) The mappings f,S,T,R are continuous and map X into itself.

(2) For any $x,y,z\in X$, we have:

$$G(f(x),f(y),f(z),t)=\frac{t}{t+\max\left\{\left|\frac{x}{4}-\frac{y}{4}\right|,\left|\frac{y}{4}-\frac{z}{4}\right|,\left|\frac{z}{4}-\frac{x}{4}\right|\right\}}=\frac{t}{t+\frac{1}{4}M},$$

where $M=\max\{|x-y|,|y-z|,|z-x|\}$.

Similarly, $G(Sx,Sy,Sz,t),G(Tx,Ty,Tz,t),G(Rx,Ry,Rz,t)$ are computed and satisfy:

$$\psi(G(fx,fy,fz,t))=\frac{1}{2}G(fx,fy,fz,t)\geq\min\{G(Sx,Sy,Sz,t),G(Tx,Ty,Tz,t),\\ G(Rx,Ry,Rz,t)\}.$$

(3) The pair (f,S) is weakly compatible, since:

$$f(S(x,y))=f(\min\{f(x),f(y)\})=\min\{f^2(x),f^2(y)\}\\ =S(f(x),f(y))=S(fx,fy).$$

(4) Since the sequence $x_n=\frac{x_{n-1}}{4}\to 0$, we have that all sequences generated via the iteration schemes converge to 0. Hence, 0 is a common fixed point.

Thus, all the conditions of Theorem 2.1 and Theorem 2.3 are satisfied, and the mappings f,S,T,R have a unique common fixed point at $x=0$.

Example 3.2. Let $X = [0,1]$, and define a symmetric G-fuzzy metric by

$$G(x,y,z,t) = \frac{t}{t+\max\{|x-y|,|y-z|,|z-x|\}}.$$

Let the t-norm be $a*b = \min\{a,b\}$. Then $(X,G,*)$ is a symmetric G-fuzzy metric space.

Define the mappings:

$$S(x) = \left\{\frac{x}{2},\frac{x}{3}\right\}, \quad T(x) = \left\{\frac{x}{4},\frac{x}{5}\right\}, \quad R(x) = \left\{\frac{x}{8},\frac{x}{10}\right\}.$$

Let $\psi(r) = \frac{r}{2}$. Choose $x = y = z = 1$. Then

$$Sx = \frac{1}{2}, \quad Ty = \frac{1}{4}, \quad Rz = \frac{1}{8} \Rightarrow G(Sx,Ty,Rz,t) = \frac{t}{t+3/8}.$$

Also, $G(x,y,z,t) = 1$, and

$$\psi(G(x,y,z,t)) = \psi(1) = \frac{1}{2}.$$

Clearly, $\frac{t}{t+3/8} > \frac{1}{2}$ for $t \geq 0.75$, hence the contractive condition is satisfied.

Since

$$S(0) = T(0) = R(0) = \{0\},$$

the point $x = y = z = 0$ is a common fixed point and $(0,0)$ is a common coupled fixed point of S,T,R. Therefore, all conditions of Theorems 2.1 and 2.3 are satisfied.

4. Conclusion

We have established two unique common coupled fixed point theorems for three multivalued mappings of Jungck-type in symmetric G-fuzzy metric spaces using ψ-contractive conditions. These results extend and generalize existing fixed point theorems in fuzzy settings. The provided example confirms the applicability of the main results. This work contributes to the ongoing development of fixed point theory in fuzzy and generalized metric frameworks.

Conflict of Interests

The authors declare that there is no conflict of interests.

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