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CONVERGENCE AND STABILITY ANALYSIS OF A MODIFIED HYBRID ITERATIVE PROCESS WITH SOME APPLICATIONS

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Abstract. This paper presents a novel modification of the Picard-Noor hybrid iterative process, designed to enhance convergence performance in solving nonlinear equations. The newly proposed scheme is rigorously analyzed and shown to have a superior convergence rate to that of the Picard-S iterative scheme, in Berinde's sense. We not only prove the strong convergence of the proposed scheme but also establish its stability and data dependency, ensuring the method's resilience to slight perturbations in the operator or initial approximation. In comparison to classical methods such as Picard, Mann, Ishikawa, Noor, and their hybrid variants, the process achieves improved performance. A detailed theoretical framework supports the convergence claims, and numerical examples are provided to demonstrate its practical efficiency and accelerated convergence behavior. The analysis confirms that this approach offers a reliable and faster alternative in nonlinear functional analysis and related computational applications. In this study, we also demonstrate some applications in differential equation, machine learning and optimization algorithms. Overall, the findings suggest that the modified Picard-Noor hybrid process not only improves convergence speed but also contributes significantly to the development of stable, data-sensitive iterative algorithms. This advancement, coupled with its application in various fields opens up new directions for research in the development of optimized fixed-point iterative schemes tailored for complex nonlinear systems.

Keywords: Hybrid iteration; convergence; data dependency; stability; modified Picard-Noor iteration.

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1. INTRODUCTION

The Banach contraction theorem is frequently used in developing fixed-point iterative processes. Some renowned iterative processes are the Picard [31], Mann [25], Ishikawa [12], Noor [26], Abbas [1], Agarwal [2], S* [13], SP [30], Normal-S [32] and CR [7] iterative processes. Various hybrid iterative processes, like the Picard-Mann [14], Picard-Ishikawa [27], Picard-Noor [8] and Picard-S [11] iterative processes were introduced. It was found that these hybrid processes converged faster than several iterative processes. We shall establish that our proposed modified Picard-Noor iteration is superior to the Picard-S iteration in terms of convergence rate.

2. PRELIMINARIES

Let X be a Banach space. Throughout we are going to denote C to be a non-void convex and closed subset of X. A point $p \in C$ is a fixed point of $T : C \to C$ if $Tp = p$. Let $Fix(T) = \{p \in C : Tp = p\}$.
A mapping $T : C \to C$ is a contraction if $\exists \theta \in (0,1)$, such that

$$\|Tx - Ty\| \leq \theta \|x - y\| \text{ for all } x, y \in C \tag{2.1}$$

The following are some well-known iterative processes:

Picard iteration

It is defined by $\{r_n\}_{n=1}^{\infty}$ as

$$r_{n+1} = Tr_n \tag{2.2}$$

Mann iteration

It is defined by $\{s_n\}_{n=1}^{\infty}$ as

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n T s_n \tag{2.3}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$.

Ishikawa iteration

It is defined by $\{p_n\}_{n=1}^{\infty}$ as

$$p_{n+1} = (1 - \alpha_n)p_n + \alpha_n T q_n$$

$$q_n = (1-\beta_n)p_n + \beta_n T p_n \tag{2.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0,1)$.

Noor iteration

It is defined by $\{t_n\}_{n=1}^{\infty}$ as

$$t_{n+1} = (1-\alpha_n)t_n + \alpha_n T u_n$$

$$u_n = (1-\beta_n)t_n + \beta_n T v_n$$

$$v_n = (1-\gamma_n)t_n + \gamma_n T t_n \tag{2.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $(0,1)$.

Picard-Mann hybrid iteration

It is defined by $\{p_n\}_{n=1}^{\infty}$ as

$$p_{n+1} = T q_n$$

$$q_n = (1-\alpha_n)p_n + \alpha_n T p_n \tag{2.6}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$.

Picard-Ishikawa hybrid iteration

It is defined by $\{p_n\}_{n=1}^{\infty}$ as

$$p_{n+1} = T q_n$$

$$q_n = (1-\alpha_n)p_n + \alpha_n T r_n$$

$$r_n = (1-\beta_n)p_n + \beta_n T p_n \tag{2.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0,1)$.

Picard-Noor hybrid iteration

Chyne and Kumar [8] introduced this process in 2023. For fixed p_1 in C, this process is defined by $\{p_n\}_{n=1}^{\infty}$ as

$$p_{n+1} = T q_n$$

$$q_n = (1-\alpha_n)p_n + \alpha_n T r_n$$

$$r_n = (1-\beta_n)p_n + \beta_n T s_n$$

$$s_n = (1-\beta_n)p_n + \beta_n T p_n \tag{2.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $(0,1)$.

Picard-S hybrid iteration

Gursoy and Karakaya [11] introduced this process in 2014. For fixed t_1 in C, it is defined by $\{t_n\}_{n=1}^{\infty}$ as

$$t_{n+1} = Tu_n$$

$$u_n = (1-\alpha_n)Tt_n + \alpha_n T v_n$$

$$v_n = (1-\beta_n)t_n + \beta_n T t_n \tag{2.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0,1)$.

The above results inspired us to introduce a Modified Picard-Noor (MPN) which is defined for any fixed x_1 in C, and for $i \in \mathbb{N}$, by the sequence $\{x_n\}_{n=1}^{\infty}$ as

$$x_{n+1} = T^i y_n$$

$$y_n = (1-\alpha_n)Tx_n + \alpha_n T z_n$$

$$z_n = (1-\beta_n)Tx_n + \beta_n T w_n$$

$$w_n = (1-\gamma_n)x_n + \gamma_n T x_n \tag{2.10}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $(0,1)$.

Definition 2.1. [3] *If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are real sequences with $x_n \to x$ and $y_n \to y$, and that $\lim_{n\to\infty} \frac{|x_n - x|}{|y_n - y|} = 0$, then $\{x_n\}_{n=1}^{\infty}$ converges to x faster than $\{y_n\}_{n=1}^{\infty}$ does to y.*

Definition 2.2. [3] *Let the iterations $\{x_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be such that $x_n \to p$ and $t_n \to p$. Suppose $\|x_n - p\| \le a_n$ and $\|t_n - p\| \le b_n$ for all $n \in \mathbb{N}$, where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two positive sequences with $a_n \to 0, b_n \to 0$. If $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$, then $\{x_n\}_{n=1}^{\infty}$ converges faster than $\{t_n\}_{n=1}^{\infty}$ to p.*

Lemma 2.1. *Let a real sequence* $\{k_n\}_{n=1}^{\infty}$ *(*$k_n \geq 0$*) satisfy* $k_{n+1} \leq (1-\mu_n)k_n$*. If* $\{\mu_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $\sum_{n=1}^{\infty} \mu_n = \infty$*, then* $\lim_{n\to\infty} k_n = 0$*.*

Lemma 2.2. *Let the real sequences* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *(*$a_n \geq 0, b_n \geq 0$*) satisfy* $a_{n+1} \leq (1-\lambda_n)a_n + b_n$*, where* $\lambda_n \in (0,1)$ $\forall n \in \mathbb{N}$*,* $\sum_{n=1}^{\infty} \lambda_n = \infty$ *and* $\frac{b_n}{\lambda_n} \to 0$ *as* $n \to \infty$*. Then* $\lim_{n\to\infty} a_n = 0$*.*

Lemma 2.3. *Let* θ *be such that* $0 \leq \theta \leq 1$*, and let* $\{\varepsilon_n\}_{n=1}^{\infty}$ *(*$\varepsilon_n > 0$*) be a sequence, such that* $\lim_{n\to\infty} \varepsilon_n = 0$*. Then for any sequence* $\{\rho_n\}_{n=1}^{\infty}$*, satisfying* $\rho_{n+1} \leq \theta\rho_n + \varepsilon_n$*,* $n = 1,2,3,\ldots$*, we have* $\lim_{n\to\infty} \rho_n = 0$*.*

Lemma 2.4. *Let* $\{\xi_n\}_{n=1}^{\infty}$ *be a real sequence with* $\xi_n \geq 0 \forall n$*. Suppose* $\exists\, n_1 \in \mathbb{N}$*, such that* $\forall n \geq n_1$*, the inequality* $\xi_{n+1} \leq (1-\zeta_n)\xi_n + \zeta_n\lambda_n$ *is satisfied, where* $\zeta_n \in (0,1)$*,* $\forall n \in \mathbb{N}$*,* $\sum_{n=1}^{\infty} \zeta_n = \infty$ *and* $\lambda_n \geq 0$*,* $\forall n \in \mathbb{N}$*. Then the inequality below holds*

$$0 \leq \limsup_{n\to\infty} \xi_n \leq \limsup_{n\to\infty} \lambda_n \tag{2.11}$$

3. Analysis of Convergence and Stability

Theorem 3.1. *Let* $T : C \to C$ *satisfy (2.1). If* $\{x_n\}_{n=1}^{\infty}$ *generated by (2.10), with* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *in* $(0,1)$ *satisfying* $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty$*, then* $\{x_n\}_{n=1}^{\infty}$ *converges to a unique* $p \in Fix(T)$*.*

Proof. Clearly, a unique $p \in Fix(T)$ exists.

We now prove that $x_n \to p$ as $n \to \infty$.

From (2.10) we have

$$\begin{aligned}
\|w_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|T x_n - T p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\theta\|x_n - p\| \\
&= (1-\gamma_n(1-\theta))\|x_n - p\| \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
\|z_n - p\| &= \|(1-\beta_n)x_n + \beta_n T w_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|T w_n - T p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\theta\|w_n - p\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|x_n - p\| + \beta_n \theta \big(1-\gamma_n(1-\theta)\big)\|x_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n \big(1-\gamma_n(1-\theta)\big)\|x_n - p\| \\
&= [1-\beta_n\gamma_n(1-\theta)]\|x_n - p\| \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
\|y_n - p\| &= \|(1-\alpha_n)x_n + \alpha_n T z_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|T z_n - T p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \theta \|z_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \theta \big[1-\beta_n\gamma_n(1-\theta)\|x_n - p\|\big] \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \big(1-\beta_n\gamma_n(1-\theta)\big)\|x_n - p\| \\
&= [1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n - p\| \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|T^i y_n - T p\| \\
&= \|T(T^{i-1} y_n) - T p\| \\
&\leq \theta \|T^{i-1} y_n - p\| \\
&= \theta \|T^{i-1} y_n - T p\| \\
&\leq \theta^2 \|T^{i-2} y_n - p\| \\
&\ldots \\
&\leq \theta^i \|y_n - p\| \\
&\leq \theta^i [1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n - p\| \qquad (3.4)
\end{aligned}
$$

Repeating the above process, we get

$$
(3.5) \qquad
\begin{cases}
\|x_{n+1} - p\| \leq \theta^i [1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n - p\| \\
\|x_n - p\| \leq \theta^i [1-\alpha_{n-1}\beta_{n-1}\gamma_{n-1}(1-\theta)]\|x_{n-1} - p\| \\
\|x_{n-1} - p\| \leq \theta^i [1-\alpha_{n-2}\beta_{n-2}\gamma_{n-2}(1-\theta)]\|x_{n-2} - p\| \\
\qquad \ldots \\
\|x_2 - p\| \leq \theta^i [1-\alpha_1\beta_1\gamma_1(1-\theta)]\|x_1 - p\|
\end{cases}
$$

From (3.5) we get

$$\|x_{n+1}-p\| \leq \|x_1-p\|\theta^{in}\prod_{k=1}^{n}[1-\alpha_k\beta_k\gamma_k(1-\theta)] \tag{3.6}$$

Since $\theta, \alpha_n, \beta_n, \gamma_n \in (0,1)$, we have

$$1-\alpha_n\beta_n\gamma_n(1-\theta) < 1 \tag{3.7}$$

We know that $\forall x \in (0,1)$, $1-x < e^{-x}$. Using these facts and (3.6), we get

$$\|x_{n+1}-p\| \leq \|x_1-p\|\theta^{in}e^{-(1-\theta)\sum_{n=1}^{\infty}\alpha_k\beta_k\gamma_k} \tag{3.8}$$

Taking limit as $n \to \infty$ on both sides of (3.8), we get $\lim_{n\to\infty}\|x_n-p\| = 0$.
Therefore, $\{x_n\}_{n=1}^{\infty}$ converges to a unique $p \in Fix(T)$. □

Theorem 3.2. *Let $T : C \to C$ satisfy (2.1). If $\{x_n\}_{n=1}^{\infty}$ generated by (2.10), with $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ in $(0,1)$ satisfying $\sum_{n=1}^{\infty}\alpha_n\beta_n\gamma_n = \infty$,, then (2.10) is T-stable.*

Proof. Consider a sequence $\{t_n\}_{n=1}^{\infty}$ in C. Let (2.10) generate the sequence $x_{n+1} = F(T,x_n)$. By Theorem 3.1, this sequence converges to a unique $x^* \in Fix(T)$.
Let $\varepsilon_n = \|t_{n+1}-F(T,x_n)\|$. We will show that $\lim_{n\to\infty}\varepsilon_n = 0 \iff \lim_{n\to\infty}t_n = x^*$.
Let $\lim_{n\to\infty}\varepsilon_n = 0$. We have

$$\begin{aligned}
\|t_{n+1}-x^*\| &\leq \|t_{n+1}-F(T,x_n)\| + \|F(T,x_n)-x^*\| \\
&\leq \varepsilon_n + \theta^i\big(1-\alpha_n\beta_n\gamma_n(1-\theta)\big)\|t_n-x^*\| \\
&\leq \varepsilon_n + \big(1-\alpha_n\beta_n\gamma_n(1-\theta)\big)\|t_n-x^*\|
\end{aligned}$$

Now, $\theta \in (0,1)$, $\alpha_n, \beta_n, \gamma_n \in (0,1)$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty}\varepsilon_n = 0$.
Using Lemma (2.2), we get $\lim_{n\to\infty}t_n = x^*$.
Conversely, let $\lim_{n\to\infty}t_n = x^*$.

$$\begin{aligned}
\varepsilon_n &= \|t_{n+1}-F(T,x_n)\| \\
&\leq \|t_{n+1}-x^*\| + \|F(T,x_n)-x^*\| \\
&\leq \|t_{n+1}-x^*\| + \theta^i\big(1-\alpha_n\beta_n\gamma_n(1-\theta)\big)\|t_n-x^*\| \\
&\leq \|t_{n+1}-x^*\| + \big(1-\alpha_n\beta_n\gamma_n(1-\theta)\big)\|t_n-x^*\|
\end{aligned}$$

This implies that $\lim_{n\to\infty} \varepsilon_n = 0$.

Hence, (2.10) is T-stable. □

Theorem 3.3. *Let a contraction $T : C \to C$ satisfy (2.1) with a unique $p \in Fix(T)$. For $t_1 = x_1 \in C$, let (2.9) and (2.10) generate the iterative sequences $\{t_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ respectively, with $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ being real sequences in $(0,1)$ satisfying*

(i) $\alpha \le \alpha_n < 1, \beta \le \beta_n < 1, \gamma \le \gamma_n < 1$ for some $\alpha, \beta, \gamma > 0$ and $\forall n \in \mathbb{N}$.

Then for $i > 2$, $\{x_n\}_{n=1}^{\infty}$ converges to p faster than $\{t_n\}_{n=1}^{\infty}$.

Proof. From (3.6) of Theorem (3.1), we have

$$\|x_{n+1} - p\| \le \|x_1 - p\| \theta^{in} \prod_{k=1}^{n} [1 - \alpha_k \beta_k \gamma_k (1-\theta)] \tag{3.9}$$

Using assumption (i) we can replace $\alpha_k, \beta_k, \gamma_k$ by α, β, γ respectively in (3.9), to get

$$\|x_{n+1} - p\| \le \theta^{in} [1 - \alpha\beta\gamma(1-\theta)]^n \|x_1 - p\| \tag{3.10}$$

Let

$$a_n = \theta^{in} [1 - \alpha\beta\gamma(1-\theta)]^n \|x_1 - p\| \tag{3.11}$$

Using (2.9), we have

$$\begin{aligned}
\|v_n - p\| &= \|(1-\beta_n)t_n + \beta_n T t_n - (1 - \beta_n + \beta_n)p\| \\
&\le (1-\beta_n)\|t_n - p\| + \beta_n \|T t_n - T p\| \\
&\le (1-\beta_n)\|t_n - p\| + \beta_n \theta \|t_n - p\| \\
&= [1 - \beta_n(1-\theta)] \|t_n - p\| \qquad (3.12)
\end{aligned}$$

$$\begin{aligned}
\|u_n - p\| &= \|(1-\alpha_n) T t_n + \alpha_n T v_n - p\| \\
&\le (1-\alpha_n)\|T t_n - T p\| + \alpha_n \|T v_n - T p\| \\
&\le (1-\alpha_n)\theta \|t_n - p\| + \alpha_n \theta \|v_n - p\| \\
&\le (1-\alpha_n)\theta \|t_n - p\| + \alpha_n \theta \big(1 - \beta_n(1-\theta)\big) \|t_n - p\| \\
&= \theta [1 - \alpha_n \beta_n (1-\theta)] \|t_n - p\| \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}\|t_{n+1}-p\| &= \|Tu_n - Tp\| \\ &\leq \theta\|u_n - p\| \\ &= \theta^2[1-\alpha_n\beta_n\gamma_n(1-\theta)]\|t_n - p\| \end{aligned} \tag{3.14}$$

Repeating the above process, we get

$$\begin{cases} \|t_{n+1}-p\| \leq \theta^2[1-\alpha_n\beta_n(1-\theta)]\|t_n-p\| \\ \|t_n-p\| \leq \theta^2[1-\alpha_{n-1}\beta_{n-1}(1-\theta)]\|t_{n-1}-p\| \\ \|t_{n-1}-p\| \leq \theta^2[1-\alpha_{n-2}\beta_{n-2}(1-\theta)]\|t_{n-2}-p\| \\ \dots \\ \|t_2-p\| \leq \theta^2[1-\alpha_1\beta_1(1-\theta)]\|t_1-p\| \end{cases} \tag{3.15}$$

From (3.15) we get

$$\|t_{n+1}-p\| \leq \|t_1-p\|\theta^{2n}\prod_{k=1}^{n}[1-\alpha_k\beta_k(1-\theta)] \tag{3.16}$$

Using assumption (i), we can replace α_k, β_k by α, β respectively in (3.16), to get

$$\begin{aligned}\|t_{n+1}-p\| &\leq \theta^{2n}[1-\alpha\beta(1-\theta)]^n\|t_1-p\| \\ &\leq \theta^{2n}[1-\alpha\beta\gamma(1-\theta)]^n\|t_1-p\| \end{aligned} \tag{3.17}$$

Let

$$b_n = \theta^{2n}[1-\alpha\beta\gamma(1-\theta)]^n\|t_1-p\| \tag{3.18}$$

$$\frac{a_n}{b_n} = \frac{\theta^{in}[1-\alpha\beta\gamma(1-\theta)]^n\|x_1-p\|}{\theta^{2n}[1-\alpha\beta\gamma(1-\theta)]^n\|t_1-p\|} = \frac{\theta^{(i-2)n}\|x_1-p\|}{\|t_1-p\|} \to 0 \text{ as } n\to\infty \text{ (for } i>2).$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ converges faster than $\{t_n\}_{n=1}^{\infty}$. That is, (2.10) converges faster than (2.9).

□

We now compare our iterative process with several processes by some numerical illustrations.

Example 3.1. *Let $C = [1,7] \subseteq X = \mathbb{R}$ and $T : C \to C$ be defined in C by $Tx = \sqrt[3]{x+6}$ for all $x \in C$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$ for each $n \in \mathbb{N}$, and initialize $x_1 = 3$. Let $i = 4$. Clearly, T is a contraction and $Fix(T) = \{2\}$.*

For this example, a comparison of the convergence rate of the modified Picard-Noor with those of Picard-S, Picard-Mann, Picard-Ishikawa and Picard-Noor iterations is shown in table 1 and figure 1.

Step	Modified Picard-Noor ($i = 4$)	Picard-S	Picard-Noor	Picard-Ishikawa	Picard-Mann
1	3.000000000000	3.000000000000	3.000000000000	3.000000000000	3.000000000000
2	2.000024565052	2.005157978169	2.042530334242	2.042587833998	2.044027184316
3	2.000000000618	2.000027605187	2.001847485464	2.001852787043	2.001985087658
4	2.000000000000	2.000000147771	2.000080327413	2.000080680737	2.000089600067
5	2.000000000000	2.000000000791	2.000003492721	2.000003513434	2.000004044438
6	2.000000000000	2.000000000004	2.000000151868	2.000000153001	2.000000182561
7	2.000000000000	2.000000000000	2.000000006603	2.000000006663	2.000000008241
8	2.000000000000	2.000000000000	2.000000000287	2.000000000290	2.000000000372
9	2.000000000000	2.000000000000	2.000000000012	2.000000000013	2.000000000017
10	2.000000000000	2.000000000000	2.000000000001	2.000000000001	2.000000000001
11	2.000000000000	2.000000000000	2.000000000000	2.000000000000	2.000000000000
12	2.000000000000	2.000000000000	2.000000000000	2.000000000000	2.000000000000

TABLE 1.

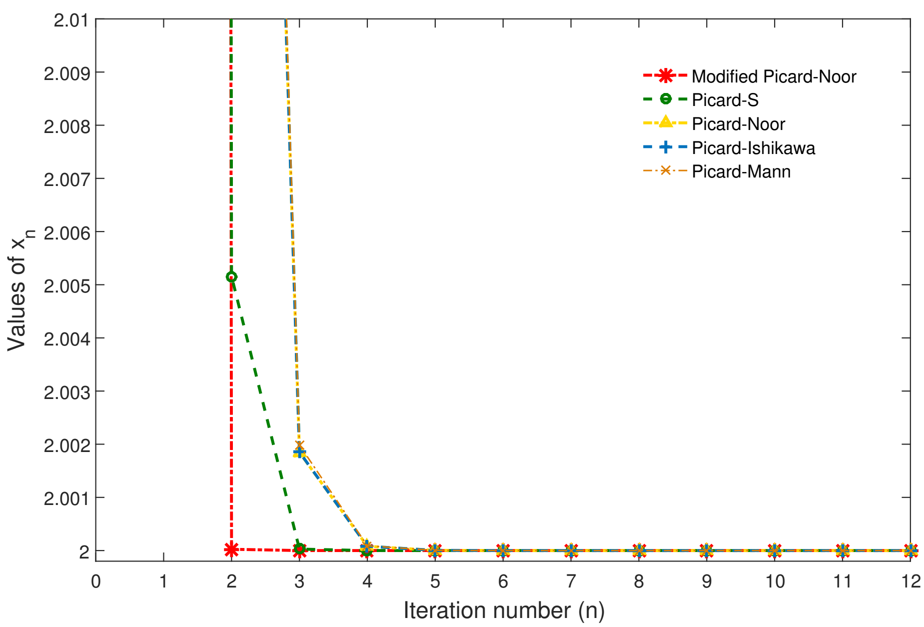

FIGURE 1.

Example 3.2. *Let* $C = [1,6] \subseteq X = \mathbb{R}$ *and* $T : C \to C$ *be defined in* C *by* $Tx = \sqrt[3]{5x+12}$ *for all* $x \in C$*. Choose* $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$ *for each* $n \in \mathbb{N}$*, and initialize* $x_1 = 5$*. Let* $i = 3$*. Clearly,* T *is a contraction and* $Fix(T) = \{3\}$*.*

For this example, a comparison of the convergence rate of the modified Picard-Noor with those of Picard-S, Picard-Mann, Picard-Ishikawa and Picard-Noor iterations is shown in table 2 and figure 2.

Step	Modified Picard-Noor ($i = 3$)	Picard-S	Picard-Noor	Picard-Ishikawa	Picard-Mann
1	5.000000000000	5.000000000000	5.000000000000	5.000000000000	5.000000000000
2	3.006445382140	3.048674228129	3.190549809563	3.191418657751	3.202034989107
3	3.000022562625	3.001324998840	3.019318526797	3.019524682304	3.021967613446
4	3.000000079005	3.000036179723	3.001971120264	3.002004713599	3.002408268640
5	3.000000000277	3.000000987987	3.000201249884	3.000205975600	3.000264252467
6	3.000000000001	3.000000026980	3.000020548830	3.000021164575	3.000028998545
7	3.000000000000	3.000000000737	3.000002098174	3.000002174735	3.000003182278
8	3.000000000000	3.000000000020	3.000000214238	3.000000223462	3.000000349221
9	3.000000000000	3.000000000001	3.000000021875	3.000000022962	3.000000038323
10	3.000000000000	3.000000000000	3.000000002234	3.000000002359	3.000000004206
11	3.000000000000	3.000000000000	3.000000000228	3.000000000242	3.000000000462
12	3.000000000000	3.000000000000	3.000000000023	3.000000000025	3.000000000051

TABLE 2.

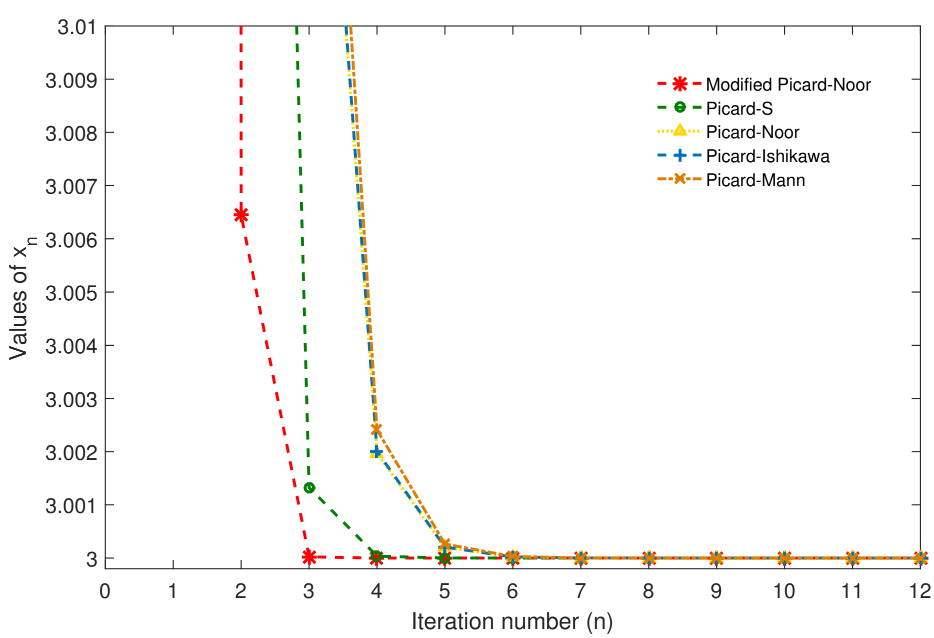

FIGURE 2.

4. Data Dependency

Theorem 4.1. *Let T be a contraction map with approximate operator $\tilde{T}$. Let the iterative sequence $\{x_n\}_{n=1}^{\infty}$ for T be generated by (2.10). We define $\{\tilde{x}_n\}_{n=1}^{\infty}$ as follows:*

$$\begin{aligned} \tilde{x}_{n+1} &= \tilde{T}^i \tilde{y}_n \\ \tilde{y}_n &= (1-\alpha_n)\tilde{x}_n + \alpha_n \tilde{T}\tilde{z}_n \\ \tilde{z}_n &= (1-\beta_n)\tilde{x}_n + \beta_n \tilde{T}\tilde{w}_n \\ \tilde{w}_n &= (1-\gamma_n)\tilde{x}_n + \gamma_n \tilde{T}\tilde{x}_n \end{aligned} \tag{4.1}$$

where α_n, β_n, γ_n in $(0,1)$ satisfying

(i) $\frac{1}{2} \leq \alpha_n(1-\theta(1-\beta_n(1-\theta(1-\gamma_n(1-\theta))))), \forall n \in \mathbb{N}$

(ii) $\sum_{n=1}^{\infty} \alpha_n(1-\theta(1-\beta_n(1-\theta(1-\gamma_n(1-\theta))))) = \infty$

If $Tp = p$ and $\tilde{T}\tilde{p} = \tilde{p}$ be such that $\lim_{n\to\infty} \tilde{x} = \tilde{p}$, then $\|p-\tilde{p}\| \leq \frac{7\varepsilon}{1-\theta}$, where $\varepsilon > 0$ is fixed.

Proof.

$$\begin{aligned} \|x_{n+1} - \tilde{x}_{n+1}\| &= \|T^i y_n - \tilde{T}^i \tilde{y}_n\| \\ &= \|T^i y_n - T^i \tilde{y}_n + T^i \tilde{y}_n - \tilde{T}^i \tilde{y}_n\| \\ &\leq \|T^i y_n - T^i \tilde{y}_n\| + \|T^i \tilde{y}_n - \tilde{T}^i \tilde{y}_n\| \\ &\leq \theta \|T^{i-1} y_n - T^{i-1} \tilde{y}_n\| + \frac{\varepsilon}{i} \\ &\leq \theta^2 \|T^{i-2} y_n - T^{i-2} \tilde{y}_n\| + \frac{2\varepsilon}{i} \\ &\cdots \\ &\leq \theta^i \|y_n - \tilde{y}_n\| + \varepsilon \end{aligned} \tag{4.2}$$

$$\begin{aligned} \|w_n - \tilde{w}_n\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - (1-\gamma_n)\tilde{x}_n - \gamma_n \tilde{T}\tilde{x}_n\| \\ &\leq (1-\gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n \|T x_n - \tilde{T}\tilde{x}_n\| \\ &\leq (1-\gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n \left\{ \|T x_n - T\tilde{x}_n\| + \|T\tilde{x}_n - \tilde{T}\tilde{x}_n\| \right\} \\ &\leq (1-\gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n \theta \|x_n - \tilde{x}_n\| + \gamma_n \varepsilon \end{aligned}$$

$$= \big(1-\gamma_n(1-\theta)\big)\|x_n-\tilde{x}_n\|+\gamma_n\varepsilon \tag{4.3}$$

$$\begin{aligned}
\|z_n-\tilde{z}_n\| &= \|(1-\beta_n)x_n+\beta_n Tw_n-(1-\beta_n)\tilde{x}_n-\beta_n\tilde{T}\tilde{w}_n\| \\
&\leq (1-\beta_n)\|x_n-\tilde{x}_n\|+\beta_n\|Tw_n-\tilde{T}\tilde{w}_n\| \\
&\leq (1-\beta_n)\|x_n-\tilde{x}_n\|+\beta_n\left\{\|Tw_n-T\tilde{w}_n\|+\|T\tilde{w}_n-\tilde{T}\tilde{w}_n\|\right\} \\
&\leq (1-\beta_n)\|x_n-\tilde{x}_n\|+\beta_n\theta\|w_n-\tilde{w}_n\|+\beta_n\varepsilon \\
&\leq (1-\beta_n)\|x_n-\tilde{x}_n\|+\beta_n\theta\left\{\big(1-\gamma_n(1-\theta)\big)\|x_n-\tilde{x}_n\|+\gamma_n\varepsilon\right\}+\beta_n\varepsilon
\end{aligned}$$

$$= [1-\beta_n(1-\theta(1-\gamma_n(1-\theta)))]\|x_n-\tilde{x}_n\|+\beta_n\varepsilon(1+\gamma_n\theta) \tag{4.4}$$

$$\begin{aligned}
\|y_n-\tilde{y}_n\| &= \|(1-\alpha_n)x_n+\alpha_n Tz_n-(1-\alpha_n)\tilde{x}_n-\alpha_n\tilde{T}\tilde{z}_n\| \\
&\leq (1-\alpha_n)\|x_n-\tilde{x}_n\|+\alpha_n\|Tz_n-\tilde{T}\tilde{z}_n\| \\
&\leq (1-\alpha_n)\|x_n-\tilde{x}_n\|+\alpha_n\left\{\|Tz_n-T\tilde{z}_n\|+\|T\tilde{z}_n-\tilde{T}\tilde{z}_n\|\right\} \\
&= (1-\alpha_n)\|x_n-\tilde{x}_n\|+\alpha_n\theta\|z_n-\tilde{z}_n\|+\alpha_n\varepsilon \\
&= (1-\alpha_n)\|x_n-\tilde{x}_n\|+\alpha_n\theta\left\{\begin{aligned}&[1-\beta_n(1-\theta(1-\gamma_n(1-\theta)))]\|x_n-\tilde{x}_n\| \\ &+\beta_n\varepsilon(1+\gamma_n\theta)\end{aligned}\right\}+\alpha_n\varepsilon
\end{aligned}$$

$$= [1-\alpha_n(1-\theta(1-\beta_n(1-\theta(1-\gamma_n(1-\theta)))))]\|x_n-\tilde{x}_n\|+\alpha_n\varepsilon(1+\beta_n\theta(1+\gamma_n\theta)) \tag{4.5}$$

From (4.2) and (4.5), we have

$$\begin{aligned}
\|x_{n+1}-\tilde{x}_{n+1}\| &\leq \theta^i[1-\alpha_n(1-\theta(1-\beta_n(1-\theta(1-\gamma_n(1-\theta)))))]\|x_n-\tilde{x}_n\| \\
&\quad +\alpha_n\varepsilon(1+\beta_n\theta(1+\gamma_n\theta)) \\
&\leq [1-\alpha_n(1-\theta(1-\beta_n(1-\theta(1-\gamma_n(1-\theta)))))]\|x_n-\tilde{x}_n\| \\
&\quad +\alpha_n\theta^i\varepsilon(1+\beta_n\theta(1+\gamma_n\theta))+\varepsilon \\
&= [1-\alpha_n(1-\theta(1-\beta_n(1-\theta(1-\gamma_n(1-\theta)))))]\|x_n-\tilde{x}_n\|+\alpha_n\theta^i\varepsilon \\
&\quad +\alpha_n\beta_n\theta^{i+1}\varepsilon+\alpha_n\beta_n\gamma_n\theta^{i+2}\varepsilon+\varepsilon
\end{aligned}$$

$$\leq [1-\alpha_n(1-\theta(1-\beta_n(1-\theta(1-\gamma_n(1-\theta)))))]\|x_n-\tilde{x}_n\|+\alpha_n\beta_n\gamma_n\varepsilon+3\varepsilon \tag{4.6}$$

From assumption (i) we have

$$\begin{aligned}\|x_{n+1}-\tilde{x}_{n+1}\| &\leq [1-\alpha_n(1-\theta(1-\beta_n(1-\theta(1-\gamma_n(1-\theta)))))]\|x_n-\tilde{x}_n\|+\alpha_n\beta_n\gamma_n\varepsilon+\\ &\quad +3(1-\alpha_n\beta_n\gamma_n+\alpha_n\beta_n\gamma_n)\varepsilon\\ &\leq [1-\alpha_n(1-\theta(1-\beta_n(1-\theta(1-\gamma_n(1-\theta)))))]\|x_n-\tilde{x}_n\|\\ &\quad +\alpha_n\beta_n\gamma_n(1-\theta)\frac{7\varepsilon}{1-\theta} \end{aligned} \tag{4.7}$$

Let $\xi := \|x_n-\tilde{x}_n\|$, $\zeta_n := \alpha_n(1-\theta(1-\beta_n(1-\theta(1-\gamma_n(1-\theta))))) \in (0,1)$, $\lambda_n := \dfrac{7\varepsilon}{1-\theta}$.
By Lemma (2.4), we have

$$0 \leq \limsup_{n\to\infty}\|x_n-\tilde{x}_n\| \leq \limsup_{n\to\infty}\frac{7\varepsilon}{1-\theta} \tag{4.8}$$

We know that $\lim_{n\to\infty} x_n = p$ (by Theorem 3.1). Also by assumption, $\lim_{n\to\infty}\tilde{x}_n = \tilde{p}$.
Therefore, we have $\|p-\tilde{p}\| \leq \dfrac{7\varepsilon}{1-\theta}$.
This completes the proof. □

5. Application in Differential Equation

Consider the norm $\|x-y\|_\infty = \max_{t\in[a,b]}|x(t)-y(t)|$ for $C([a,b])$, which is the space of real-valued continuous functions on $[a,b]$. Clearly, $\big(C([a,b]),\|\cdot\|_\infty\big)$ is a Banach space.
Consider the following delay differential equation:

$$x'(t) = f(t,x(t),x(t-\tau)),\ t\in[t_0,b] \tag{5.1}$$

with initialization condition

$$x(t) = \varphi(t),\ t\in[t_0-\tau,t_0] \tag{5.2}$$

We assume the following conditions are satisfied:

- (C_1) $t_0,b\in\mathbb{R}$, $\tau>0$;
- (C_2) $f\in C([t_0,b]\times\mathbb{R}^2,\mathbb{R})$;
- (C_3) $\varphi\in C([t_0-\tau,b],\mathbb{R})$;
- (C_4) there exists $L_f>0$ such that $\forall u_i,v_i\in\mathbb{R}$, $i=1,2$, $t\in[t_0,b]$

$$\big|f(t,u_1,u_2)-f(t,v_1,v_2)\big| \leq L_f\sum_{i=1}^{2}|u_i-v_i| \tag{5.3}$$

(C_5) $2L_f(b-t_0)<1$

A solution x of (5.1) – (5.2) is a function $x \in C\big([t_0-\tau,b],\mathbb{R}\big)\bigcap C^1\big([t_0,b],\mathbb{R}\big)$.

We reformulate (5.1) – (5.2) as an integral equation as follows:

$$x(t)=\begin{cases}\varphi(t), & t\in[t_0-\tau,t_0],\\ \varphi(t_0)+\int_{t_0}^{t} f(s,x(s),x(s-\tau))ds, & t\in[t_0,b]\end{cases} \tag{5.4}$$

Firstly, the following result was given by Coman et al. [9], which is required for our main result.

Theorem 5.1. *If (C_1) – (C_5) are satisfied, then a unique solution, say x^*, to (5.1) – (5.2) exists in $C\big([t_0-\tau,b],\mathbb{R}\big)\bigcap C^1\big([t_0,b],\mathbb{R}\big)$ and*

$$x^*=\lim_{n\to\infty} T^n(x) \text{ for any } x\in C\big([t_0-\tau,b],\mathbb{R}\big). \tag{5.5}$$

We now prove the following main result.

Theorem 5.2. *If (C_1) – (C_5) are satisfied, then a unique solution, say x^*, to (5.1) – (5.2) exists in $C\big([t_0-\tau,b],\mathbb{R}\big)\bigcap C^1\big([t_0,b],\mathbb{R}\big)$ and the Modified Picard-Noor iterative process (2.10) with real sequences $\alpha_n,\beta_n,\gamma_n$ in $(0,1)$ such that $\sum_{n=1}^{\infty}\alpha_n\beta_n\gamma_n=\infty$, converges to x^*.*

Proof. Let (2.10) generate the iterative sequence $\{x_n\}_{n=1}^{\infty}$ for the operator

$$x(t)=\begin{cases}\varphi(t), & t\in[t_0-\tau,t_0],\\ \varphi(t_0)+\int_{t_0}^{t} f(s,x(s),x(s-\tau))ds, & t\in[t_0,b]\end{cases} \tag{5.6}$$

Let $x^*\in Fix(T)$. We will prove that $x_n\to x^*$ as $n\to\infty$.
We can see that $x_n\to x^*$ for each $t\in[t_0-\tau,t_0]$.
Now, for each $t\in[t_0,b]$ we have

$$\begin{aligned}\|w_n-x^*\|_\infty &= \|(1-\gamma_n)x_n+\gamma_nTx_n-x^*\|_\infty\\ &\le (1-\gamma_n)\|x_n-x^*\|_\infty+\gamma_n\|Tx_n-Tx^*\|_\infty\\ &\le (1-\gamma_n)\|x_n-x^*\|_\infty+\gamma_n\max_{t\in[t_0-\tau,b]}\big|Tx_n(t)-Tx^*(t)\big|\end{aligned}$$

$$
\begin{aligned}
&= (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-\tau,b]} \left| \begin{matrix} \varphi(t_0) + \int_{t_0}^{t} f(s, x_n(s), x_n(s-\tau))ds \\ -\varphi(t_0) - \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))ds \end{matrix} \right| \\
&= (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-\tau,b]} \left| \begin{matrix} \int_{t_0}^{t} f(s, x_n(s), x_n(s-\tau))ds \\ - \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))ds \end{matrix} \right| \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} \left| \begin{matrix} f(s, x_n(s), x_n(s-\tau))ds \\ - f(s, x^*(s), x^*(s-\tau))ds \end{matrix} \right| \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f\big(|x_n(s) - x^*(s)| + |x_n(s-\tau) - x^*(s-\tau)|\big)ds \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \int_{t_0}^{t} L_f \left(\begin{matrix} \max_{t\in[t_0-\tau,b]} |x_n(s) - x^*(s)| \\ + \max_{t\in[t_0-\tau,b]} |x_n(s-\tau) - x^*(s-\tau)| \end{matrix} \right) ds \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \int_{t_0}^{t} L_f\big(\|x_n - x^*\|_\infty + \|x_n - x^*\|_\infty\big)ds \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + 2\gamma_n L_f (t - t_0)\|x_n - x^*\|_\infty \\
&\leq [1 - \gamma_n\big(1 - 2L_f(b - t_0)\big)]\|x_n - x^*\|_\infty \qquad (5.7)
\end{aligned}
$$

$$
\begin{aligned}
\|z_n - x^*\|_\infty &= \|(1-\beta_n)x_n + \beta_n T w_n - x^*\|_\infty \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \|T w_n - T x^*\|_\infty \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-\tau,b]} \big|T w_n(t) - T x^*(t)\big| \\
&= (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-\tau,b]} \left| \begin{matrix} \varphi(t_0) + \int_{t_0}^{t} f(s, w_n(s), w_n(s-\tau))ds \\ -\varphi(t_0) - \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))ds \end{matrix} \right| \\
&= (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-\tau,b]} \left| \begin{matrix} \int_{t_0}^{t} f(s, w_n(s), w_n(s-\tau))ds \\ - \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))ds \end{matrix} \right| \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} \left| \begin{matrix} f(s, w_n(s), w_n(s-\tau))ds \\ - f(s, x^*(s), x^*(s-\tau))ds \end{matrix} \right| \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f\big(|w_n(s) - x^*(s)| + |w_n(s-\tau) - x^*(s-\tau)|\big)ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \int_{t_0}^{t} L_f \begin{pmatrix} \max\limits_{t\in[t_0-\tau,b]} |w_n(s) - x^*(s)| \\ + \max\limits_{t\in[t_0-\tau,b]} |w_n(s-\tau) - x^*(s-\tau)| \end{pmatrix} ds \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \int_{t_0}^{t} L_f \big(\|w_n - x^*\|_\infty + \|w_n - x^*\|_\infty\big) ds \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + 2\beta_n L_f (t-t_0)\|w_n - x^*\|_\infty \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + 2\beta_n L_f (b-t_0)[1-\gamma_n\big(1-2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty
\end{aligned} \tag{5.8}
$$

Using condition (C_5), that is $2L_f(b-t_0) < 1$ in (5.8), we have

$$
\begin{aligned}
\|z_n - x^*\|_\infty &\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n[1-\gamma_n\big(1-2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \\
&= [1-\beta_n\gamma_n\big(1-2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
\|y_n - x^*\|_\infty &= \|(1-\alpha_n)x_n + \alpha_n T z_n - x^*\|_\infty \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \|T z_n - T x^*\|_\infty \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\tau,b]} \big|T z_n(t) - T x^*(t)\big| \\
&= (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\tau,b]} \left| \begin{matrix} \varphi(t_0) + \int_{t_0}^{t} f(s, z_n(s), z_n(s-\tau)) ds \\ -\varphi(t_0) - \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau)) ds \end{matrix} \right| \\
&= (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\tau,b]} \left| \begin{matrix} \int_{t_0}^{t} f(s, z_n(s), z_n(s-\tau)) ds \\ -\int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau)) ds \end{matrix} \right| \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} \left| \begin{matrix} f(s, z_n(s), z_n(s-\tau)) ds \\ -f(s, x^*(s), x^*(s-\tau)) ds \end{matrix} \right| \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f \big(|z_n(s) - x^*(s)| + |z_n(s-\tau) - x^*(s-\tau)|\big) ds \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \int_{t_0}^{t} L_f \begin{pmatrix} \max\limits_{t\in[t_0-\tau,b]} |z_n(s) - x^*(s)| \\ + \max\limits_{t\in[t_0-\tau,b]} |z_n(s-\tau) - x^*(s-\tau)| \end{pmatrix} ds \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \int_{t_0}^{t} L_f \big(\|z_n - x^*\|_\infty + \|z_n - x^*\|_\infty\big) ds \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + 2\alpha_n L_f (t-t_0)\|z_n - x^*\|_\infty
\end{aligned}
$$

$$\leq (1-\alpha_n)\|x_n - x^*\|_\infty + 2\alpha_n L_f(b-t_0)[1-\beta_n\gamma_n\big(1-2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \tag{5.10}$$

Using condition (C_5), that is $2L_f(b-t_0) < 1$ in (5.10), we have

$$\begin{aligned}\|y_n - x^*\|_\infty &\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n[1-\beta_n\gamma_n\big(1-2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \\ &= [1-\alpha_n\beta_n\gamma_n\big(1-2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \end{aligned} \tag{5.11}$$

$$\begin{aligned}\|x_{n+1} - x^*\|_\infty &= \|T^i y_n - Tx^*\|_\infty \\
&= \|T(T^{i-1}y_n) - Tx^*\|_\infty \\
&= \max_{t\in[t_0-\tau,b]} \left| \int_{t_0}^t f(s, T^{i-1}y_n(s), T^{i-1}y_n(s-\tau))ds - \int_{t_0}^t f(s, x^*(s), x^*(s-\tau))ds \right| \\
&\leq \max_{t\in[t_0-\tau,b]} \int_{t_0}^t \big| f(s, T^{i-1}y_n(s), T^{i-1}y_n(s-\tau))ds - f(s, x^*(s), x^*(s-\tau)) \big| ds \\
&\leq \max_{t\in[t_0-\tau,b]} \int_{t_0}^t L_f\big(|T^{i-1}y_n(s) - x^*(s)| + |T^{i-1}y_n(s-\tau) - x^*(s-\tau)|\big) ds \\
&\leq 2L_f(b-t_0)\|T^{i-1}y_n - x^*\|_\infty \\
&\leq \|T^{i-1}y_n - x^*\|_\infty \qquad \text{(using condition } C_1) \\
&\leq \|T^{i-2}y_n - x^*\|_\infty \\
&\cdots \\
&\leq \|y_n - x^*\|_\infty \\
&\leq 2L_f(b-t_0)\big[1-\alpha_n\beta_n\gamma_n\big(1-2L_f(b-t_0)\big)\big]\|x_n - x^*\|_\infty \end{aligned} \tag{5.12}$$

Now, take $\mu_n = \alpha_n\beta_n\gamma_n\big(1-2L_f(b-t_0)\big) < 1$ and $k_n = \|x_n - x^*\|_\infty$.
By Lemma 2.1, we get $\lim_{n\to\infty} \|x_n - x^*\|_\infty = 0$.
Hence the proof. □

6. Application in Machine Learning and Optimization Algorithms

Machine learning (ML) optimization algorithms can often be framed as fixed-point problems. Convergence of these algorithms to a solution is guaranteed by the existence of fixed points.

6.1. Gradient descent in Metric Spaces.

The gradient descent update rule is:

$$x_{n+1} = x_n - \eta \bigtriangledown f(x_n) \tag{6.1}$$

with learning rate η, and gradient $\bigtriangledown f(x_n)$ of the objective function at x_n.
(6.1) is an iterative algorithm for optimization, mainly used in ML models such as support vector machine and neural networks [10] [33]. It converges to a point x^*, where:

$$\bigtriangledown f(x^*) = 0 \tag{6.2}$$

We can introduce a regularization function Ψ, like the l_1-norm or l_2-norm to respectively promote sparsity or ensure smoothness as follows:

$$x_{n+1} = x_n - \eta \bigtriangledown f(x_n) - \lambda \Psi(x_n) \tag{6.3}$$

Here, λ is a regularization parameter.

6.2. Reinforcement Learning and Value Iteration. For policy optimization, we iteratively solve the Bellman's equation via value iteration method [5]. For this, we frame a fixed point problem as follows:

$$V_{n+1} = T(V_n),\ T(V)(s) = \max_a \left[R(s,a) + \gamma \sum_{s'} P(s'|s,a)V(s') \right] \tag{6.4}$$

Here, T is the Bellman operator, which is a contraction mapping [4], s is a state in space S, a is an action from the action space A, $R(s,a)$ is a reward received when taking the action a in state s, $V(s')$ is the value in the next state s', $\max_a$ is the greedy policy improvement step, which chooses the best action, $\gamma \in (0,1)$ is the discount factor, which gives less weight to future rewards, and $P(s'|s,a)$ is the probability of transitioning to state s' after taking action a in state s. Under sup-norm, with T being a γ-contraction, and thus, it converges to a unique fixed point (by Banach fixed point theorem). This can be applied in Deep Q-learning, dynamic programming and policy evaluation.

6.3. Proximal Point Methods. Optimization problems can be solved by proximal point method using the following iteration [6]:

$$x_{n+1} = \text{prox}_{\lambda f}(x_n) := \arg\min_x \left\{ f(x) + \frac{1}{2\lambda} \|x - x_n\|^2 \right\} \tag{6.5}$$

6.4. Neural Equilibrium Models (DEQ). Neural equilibrium models usually define hidden states by solving:

$$z^* = f_\theta(z^*, x) \tag{6.6}$$

and train directly via fixed point iteration:

$$z_{n+1} = f_\theta(z_n, x)$$

Here, convergence depends on ensuring that f_θ is contractive or nonexpansive. This can be applied to transformer variants (e.g., deep equilibrium transformers) and recurrent-like architectures.

6.5. Illustrative Examples.

Example 6.1. *Consider the following objective function:*

$$f(x) = \frac{1}{2}(x-3)^2 + \lambda|x| \tag{6.7}$$

with λ being the regularization parameter for sparsity, $\Psi(x) = |x|$ representing the l_1-norm. We are going to use the following update rule:

$$x_{n+1} = x_n - \eta \bigtriangledown f(x_n) - \lambda\Psi(x_n) \tag{6.8}$$

Here, $\bigtriangledown f(x_n) = x-3$. So we define $T : \mathbb{R} \to \mathbb{R}$ by

$$T(x) = x - \eta(x-3) - \lambda.sign(x) \tag{6.9}$$

We are going to use our modified Picard-Noor process (2.10) with iterate $i = 3$. Let $\lambda = 0.4$, $\eta = 0.1$, $\alpha_n = \beta_n = \gamma_n = 0.7$ (fixed for simplicity).

Step 1: Initialize $x_1 = 0$.

Step 2: Compute $T(x_1)$

$$T(x_1) = 0 - 0.1(0-3) - 0.4 \times sign(0)$$

Step 3: Compute w_1

$$w_1 = (1-0.7)\times 0 + 0.7 \times 0.3 = 0.21$$

Step 4: Compute $T(w_1)$

$$\begin{aligned} T(0.21) &= 0.21 - 0.1(0.21-3) - 0.4 \times sign(0.21) \\ &= 0.21 - 0.1(0.21-3) - 0.4 \times 1 \\ &= 0.089 \end{aligned}$$

Step 5: Compute z_1

$$z_1 = (1-0.7) \times 0 + 0.7 \times 0.089 = 0.0623$$

Step 6: Compute $T(z_1)$

$$\begin{aligned} T(0.0623) &= 0.0623 - 0.1(0.0623-3) - 0.4 \times sign(0.0623) \\ &= 0.0623 - 0.1(0.0623-3) - 0.4 \times 1 = -0.04393 \end{aligned}$$

Step 7: Compute y_1

$$y_1 = (1-0.7) \times 0 + 0.7 \times (-0.04393) = -0.03075$$

Step 8: Compute $T(y_1), T^2(y_1), T^3(y_1)$

$$\begin{aligned} T(y_1) &= T(-0.03075) \\ &= -0.03075 - 0.1(-0.03075-3) - 0.4 \times sign(-0.03075) \\ &= -0.03075 - 0.1(-0.03075-3) - 0.4(-1) \\ &= 0.6723 \end{aligned}$$

Similarly,

$T^2(y_1) = 0.5051$ *and* $T^3(y_1) = 0.3546$

The following table gives the iterated values up to x_{10} *(corrected to 6 decimal places):*

x_1	*0.000000*	x_6	*1.338451*
x_2	*0.354600*	x_7	*1.501381*
x_3	*0.660896*	x_8	*1.640651*
x_4	*0.923263*	x_9	*1.760398*
x_5	*1.147363*	x_{10}	*1.864432*

Example 6.2. *Consider a Markov Decision Process having the following:*

- ***States:*** $S = \{s_1, s_2\}$
- ***Action:*** a_1 *(only one for simplicity)*
- ***Rewards:*** $R(s_1, a_1) = 5$ *and* $R(s_2, a_2) = 10$
- ***Transition probabilities:***

$$P(s_1|s_1, a_1) = 0.5, \quad P(s_2|s_1, a_1) = 0.5$$

$$P(s_1|s_2, a_1) = 0.2, \quad P(s_2|s_2, a_1) = 0.8$$

- ***Discount factor:*** $\gamma = 0.9$

The value function update using the Bellman operator is:

$$T(V)(s) = R(s, a_1) + \gamma \sum_{s'} P(s'|s, a_1) V(s')$$

We apply the modified Picard-Noor iteration scheme (2.10).

$$\text{Let } x_n = [V_n(s_1), V_n(s_2)], \quad \alpha_n = \beta_n = \gamma_n = 0.5$$

Step 1: Initialize $x_1 = [0, 0]$.

Step 2: Compute $T(x_1)$

$$T(x_1)(s_1) = 5 + 0.9\big[0.5 \times 0 + 0.5 \times 0\big] = 5$$

$$T(x_1)(s_2) = 10 + 0.9\big[0.2 \times 0 + 0.8 \times 0\big] = 10$$

So,

$$T(x_1) = [5, 10]$$

Step 3: Compute w_1

$$w_1 = 0.5[0, 0] + 0.5[5, 10] = [2.5, 5]$$

Step 4: Compute $T(w_1)$

$$T(w_1)(s_1) = 5 + 0.9\big[0.5 \times 2.5 + 0.5 \times 5\big] = 8.375$$

$$T(w_1)(s_2) = 10 + 0.9\big[0.2 \times 2.5 + 0.8 \times 5\big] = 14.05$$

$$\text{So, } T(w_1) = [8.375, 14.05]$$

Step 5: Compute z_1

$$z_1 = 0.5[0,0] + 0.5[8.375, 14.05] = [4.1875,\ 7.025]$$

Step 6: Compute $T(z_1)$

$$T(z_1)(s_1) = 5 + 0.9\big[0.5 \times 4.1875 + 0.5 \times 7.025\big] = 10.0456$$

$$T(z_1)(s_2) = 10 + 0.9\big[0.2 \times 4.1875 + 0.8 \times 7.025\big] = 15.8118$$

So,

$$T(z_1) = [10.0456,\ 15.8118]$$

Step 7: Compute y_1

$$y_1 = 0.5[0,0] + 0.5[10.0456, 15.81175] = [5.0228,\ 7.9059]$$

Step 8: Compute $T(y_1), T^2(y_1), T^3(y_1)$

$$T(y_1)(s_1) = 5 + 0.9\big[0.5 \times 5.0228 + 0.5 \times 7.9059\big] = 10.8089$$

$$T(y_1)(s_2) = 10 + 0.9\big[0.2 \times 5.0228 + 0.8 \times 7.9059\big] = 16.5928$$

So,

$$T(y_1) = [10.8089,\ 16.5928]$$

Similarly,

$$T^2(y_1) = [17.3308,\ 23.8924]$$

$$T^3(y_1) = [23.5504,\ 30.3221]$$

The following table gives the iterated values up to x_{10} *(corrected to 5 decimal places):*

Iteration number	$V(s_1)$	$V(s_2)$
0	*0*	*0*
1	*23.55040*	*30.32210*
2	*42.48697*	*49.33540*
3	*55.18438*	*62.03368*
4	*63.67639*	*70.52570*
5	*69.35556*	*76.20488*
6	*73.15361*	*80.00292*
7	*75.69361*	*82.54292*
8	*77.39228*	*84.24159*
9	*78.52829*	*85.37760*
10	*79.28802*	*86.13733*

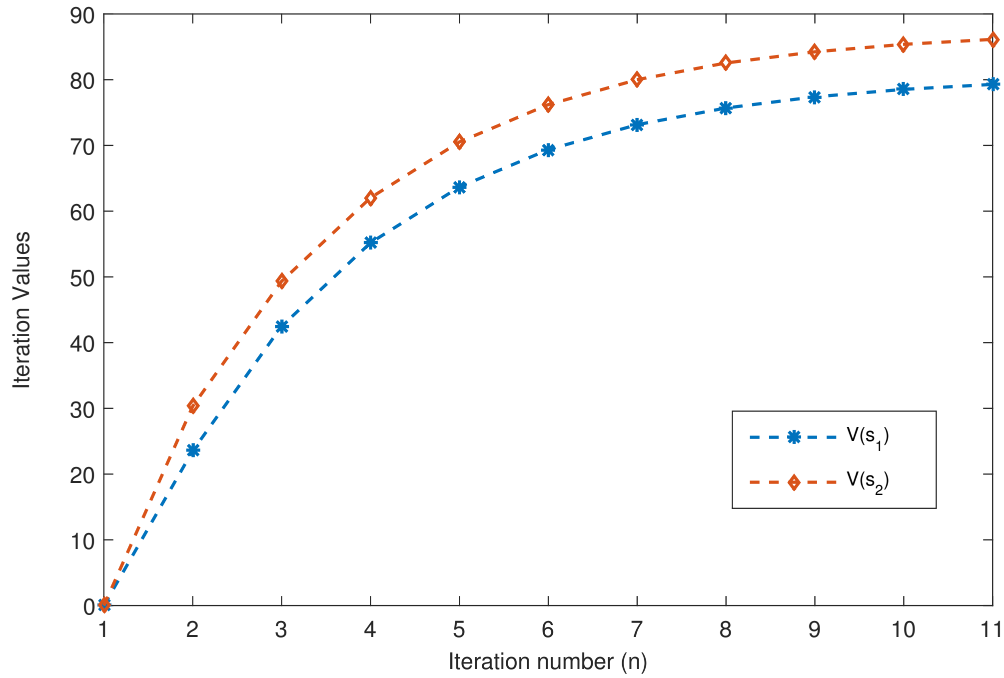

FIGURE 3.

7. CONCLUSION

The above results show that the Modified Picard-Noor iterative process is faster the Picard-S iterative process. This study suggests that by modifying the known iterative processes, the convergence rate becomes better. In particular, if we replace the contraction mapping in the known iterative processes by a suitable i^{th} iterate, we see an improvement in the fastness of the processes. With the demonstration of applications to in differential equations, machine learning and optimization algorithm, we strengthen the need for more research and advancement in the development of hybrid fixed-point iterative processes.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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