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EXISTENCE OF APPROXIMATE BEST PROXIMITY POINT OF THE TRIPLETS (Q_1, Q_2, Q_3) ON G-METRIC SPACES AND ITS APPLICATIONS

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Abstract. In this paper, we investigate the existence (qualitative results) and the diameter (quantitative results) of the approximate best proximity point of the triplets (Q_1, Q_2, Q_3) on G-metric spaces using various self-maps which includes $G-B$ contraction, G-Bianchini contraction, and so on. Also, a few examples are provided to demonstrate our findings. Finally, we discuss some applications of approximate best proximity point results in the domain of differential equations rigorously.

Keywords: G-metric space; $G-B$ contraction; G-Bianchini contraction; approximate best proximity point.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Because of its wide range of applications in various fields of mathematics, including optimization techniques, numerical analysis, fluid dynamics, and etc. the fixed point theory (FPT) and the best proximity point theory $(BPPT)$ serves as the fruitful roles in non-linear analysis. Originally, the notion of FPT was developed in the early 1900's. The father of FPT, mathematician Brouwer [4], proposed FP results for continuous mappings on finite dimensional spaces. In 1922, Banach [2] established and confirmed the renowned Banach contraction principle (BCP). After that, several authors used the BCP in numerous ways and presented numerous FP results

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[see, [3], [5], [6], [7], [11], [12], [13], [25], [27]]. Naturally, the conditions for $FP's$ existence are very strict. As a result, there is no assurance that $FP's$ will always exists. In the absence of exact FP, the $BPPT$ may be used because the FP methods have overly strict limitations. This is the primary reason for attempting to locate $BPPT$ on metric spaces. When a direct solution is not feasible, especially for non-self mappings, $BPPT$, a generalization of FPT, are essential for locating the best approximation solutions to the equation $T\wp = \wp$, where $T : \chi^{\star} \to \chi^{\divideontimes}$ are disjoint subsets of a metric space. In the similar way, the approximate $FPT(AFPT)$, also was developed and the approximate solutions (only ε-differences) are determined in many complicated situations. In this way, $AFPT$ and $ABPPT$ has been researched in various metric spaces over the decades by several researchers.

$$FPT \Longrightarrow AFPT \Longrightarrow BPPT \Longrightarrow ABPPT$$

Regarding this, R. Theivaraman et al. [see, [28],[29],[30],[31]] proved many $AFPT$ results by using several self mappings such as contraction mappings, convex contraction mappings and rational contraction mappings on various metric spces. In particular, M. Marudai and V. Bright [16], pointed out many $BPPT$ results by using B-contraction operator on metric spaces. Moreover, W. A. Kirk.et.al.[15] showed the fundamental FP results for mappings satisfying cyclical contraction conditions, which long-windedly explains about the notion of cyclic mappings and its theorems. In this regard, several generalized metric spaces have been produced over the decades by various researchers. Particularly, G-metric space, which is the most generalized of all extended metric spaces. The authors, Mustafa and Sims [22], initially formulated the concept of G-metric spaces and proved many FP results in complete G-metric spaces for contraction mappings, expansive mappings, and so on. For more details, one may refer to [[1], [8], [9], [10], [17], [18], [20], [23]]. Later, Obiedat and Mustafa [refer to [19], [21], [24]], studied FP results for Reich-type contraction mapping on G-metric spaces as well as non-symmetric metric spaces.

Definition 1.1. [10][22] *Let χ be a nonempty set and the function $d_G : \chi \times \chi \times \chi \to [0,\infty)$ satisfy the following axioms:*

$(G_1) G(\wp, q, s) = 0$ *if* $\wp = q = s$ *whenever* $\wp, q, s \in \chi$*;*

$(G_2) G(\wp, q, s) > 0$ *whenever* $\wp, q \in \chi$ *with* $\wp \neq q$*;*

$(G_3) G(\wp,\wp,q) \leq G(\wp,q,s)$ *whenever* $\wp,q,s \in \chi$ *with* $q \neq s$;

$(G_4) G(\wp,q,s) = G(\wp,s,q) = G(q,s,\wp) = ...$, *(symmetry in all three variables);*

$(G_5) G(\wp,q,s) \leq [G(\wp,\ell,\ell) + G(\ell,q,s)]$, *for every* $\wp,q,s,\ell \in \chi$.

Then (χ,d_G) *is called a G-metric space.*

Proposition 1.2. [10][22] *Let* (χ,d_G) *be a G-metric space, then for any* $\wp,q,s \in \chi$ *such that* $G(\wp,q,s) = 0$, *we have that* $\wp = q = s$.

Definition 1.3. [10][22] *Let* (χ,d_G) *be a G-metric space. A sequence* $\{\wp_n\}$ *is said to be a G-cauchy sequence if for every* $\varepsilon > 0$, *there exist* $N \in \mathbb{N}$ *such that* $G(\wp_n,\wp_m,\wp_l) < \varepsilon$, *for every* $n,m,l \geq N$, *that is* $G(\wp_n,\wp_m,\wp_l) \longrightarrow 0$ *as* $n,m,l \longrightarrow +\infty$.

Proposition 1.4. [10][22] *In a G-metric space* (χ,d_G), *the following are equivalent.*

(i) *The sequence* $\{\wp_n\} \subseteq \chi$ *is a G-cauchy.*

(ii) *For every* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $G(\wp_n,\wp_m,\wp_m) < \varepsilon$, *for all* $n,m \geq N$.

Proposition 1.5. [10][22] *Every G-metric* (χ,G) *defines a G-metric space* (χ,d_G), *by*

(i) $d_G(\wp,q) = G(\wp,q,q) + G(q,\wp,\wp)$

if (χ,G) *is symmetric in G-metric space, then*

(ii) $d_G(\wp,q) = 2G(\wp,q,s)$.

Inspired and encouraged by the above all results, we go one step forward and study the *ABPPT* concept and the leading results are proved. In particular, this work is a extension of [28] and the consequences of the extensions are discussed.

The article is organized as follows: Section 1 is introductory. In Section 2, we present the preliminary notions, some notations, essential definitions, needed lemmas and theorems from the previous work, which are used throughout the paper. In Section 3, we present *ABPPT* results for contraction mappings such as the $G-B$ contraction, the G-Bianchini contraction , G-Hardy Rogers contraction mappings and their consequences, where we proved both qualitative and quantitative results. In Section 4, we present some applications of *ABPPT* in the domain of differential equations that support the main findings of this paper. Finally, in Section 5, we present some conclusions.

2. Preliminaries

We now give some definitions, lemmas and theorems which we will use in proving our main results.

Definition 2.1. [15] *Let Q_1, Q_2 and Q_3 be three non-empty subsets of a G-metric space (χ, d_G). A mapping $Q : Q_1 \cup Q_2 \cup Q_3 \to Q_1 \cup Q_2 \cup Q_3$ is said to be a cyclic mapping if $Q(Q_1) \subseteq Q_2$, $Q(Q_2) \subseteq Q_3$ and $Q(Q_3) \subseteq Q_1$.*

Definition 2.2. [28][29] *Let Q_1, Q_2 and Q_3 are three nonempty closed subsets of a G-metric space (χ, d_G) and $Q : Q_1 \cup Q_2 \cup Q_3 \to Q_1 \cup Q_2 \cup Q_3$ be a cyclic map. Then $\wp$ is said to be an approximate best proximity point of the triplets (Q_1, Q_2, Q_3) if*

$$[G(\wp, Q\wp, Q\wp) + G(Q\wp, \wp, \wp)] \leq d_G(Q_1, Q_2, Q_3) + \varepsilon, \text{ for all } \varepsilon > 0.$$

Remark 2.3. [28][29] *Let*

$$P_{B\varepsilon}(Q_1, Q_2, Q_3) = \{\wp \in (Q_1 \cup Q_2 \cup Q_3) \big| [G(\wp, Q\wp, Q\wp) + G(Q\wp, \wp, \wp)] < d_G(Q_1, Q_2, Q_3)\varepsilon, \text{ for some } \varepsilon > 0\},$$

be denotes the set of all approximate best proximity point of the triplets (Q_1, Q_2, Q_3) for a given $\varepsilon > 0$. Also, the triplets (Q_1, Q_2, Q_3) is said to be an approximate best proximity point property if

$$[G(\wp, Q\wp, Q\wp) + G(Q\wp, \wp, \wp)] \leq d_G(Q_1, Q_2, Q_3) \neq 0.$$

Theorem 2.4. *Let Q_1, Q_2 and Q_3 are three non-empty closed subsets of a G-metric space (χ, d_G). Suppose that $Q : Q_1 \cup Q_2 \cup Q_3 \to Q_1 \cup Q_2 \cup Q_3$ is a cyclic mapping and*

$$\lim_{n\to\infty} [G(Q^n\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(Q^{n+1}\wp, Q^n\wp, Q^n\wp)] = d_G(Q_1, Q_2, Q_3), \text{ for some } \wp \in Q_1 \cup Q_2 \cup Q_3.$$

Then the triplets (Q_1, Q_2, Q_3) is called an approximate best proximity triplets.

Definition 2.5. [28][29] *Let $Q : Q_1 \cup Q_2 \cup Q_3 \to Q_1 \cup Q_2 \cup Q_3$ be a continuous map such that $Q(Q_1) \subseteq Q_2$, $Q(Q_2) \subseteq Q_3$ and $Q(Q_3) \subseteq Q_1$ and let $\varepsilon > 0$. We define the diameter*

$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3))$, i.e.,

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,q_3)) = sup\{G(\wp,q,q)+G(q,\wp,\wp) : \wp,q \in P_{B\varepsilon}(Q_1,Q_2,Q_3)\}.$$

Lemma 2.6. *Let Q_1,Q_2 and Q_3 are three non-empty closed subsets of a G-metric space (χ,d_G). a mapping $Q : Q_1\cup Q_2\cup Q_3 \to Q_1\cup Q_2\cup Q_3$ be satisfying $Q(Q_1)\subseteq Q_2$, $Q(Q_2)\subseteq Q_3$ and $Q(Q_3)\subseteq Q_1$ is a contraction map and $\varepsilon>0$. Suppose that for every $\varepsilon>0$, the followings hold:*

(i) $P_{B\varepsilon}(Q_1,Q_2,Q_3)\neq\emptyset$; and

(ii) for every $\psi>0$, there exists $\Phi(\psi)>0$ such that

$$[G(\wp,q,q)+G(q,\wp,\wp)]-[G(Q_{\wp},Q_q,Q_q)+G(Q_q,W_{\wp},W_{\wp})]<\psi$$

implies that $G(\wp,q,q)+G(q,\wp,\wp)\leq\Phi(\psi)$, for all $\wp,q\in P_{B\varepsilon}(Q_1,Q_2,Q_3)\neq\emptyset$. Then;

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3))\leq\Phi(2d_G(Q_1,Q_2,Q_3)+\varepsilon).$$

Definition 2.7. [28][29] *Let (χ,d_G) be a G-metric space and Q_1,Q_2,Q_3 are three nonempty closed subsets of χ. A cyclic mapping $Q : Q_1\cup Q_2\cup Q_3\subseteq\chi\to Q_1\cup Q_2\cup Q_3\subseteq\chi$ is said to be a $G-B$ contraction mapping if there exists $\ell_1,\ell_2,\ell_3\in(0,1)$ with $2\ell_1+\ell_2+2\ell_3<1$ and for every $\wp,q\in Q_1\cup Q_2\cup Q_3\subseteq\chi$ such that*

$$\begin{aligned}G(Q\wp,Qq,Qq)+G(Qq,Q\wp,Q\wp) \leq\ & \ell_1[G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)\\ &+G(q,Qq,Qq)+G(Qq,q,q)]\\ &+\ell_2[G(\wp,q,q)+G(q,\wp,\wp)]\\ &+\ell_3[G(\wp,Qq,Qq)+G(Qq,\wp,\wp)\\ &+G(q,Q\wp,Q\wp)+G(Q\wp,q,q)]\end{aligned} \tag{2.1}$$

Definition 2.8. [28][29] *Let (χ,d_G) be a G-metric space and Q_1,Q_2,Q_3 are three nonempty closed subsets of χ. A cyclic mapping $Q : Q_1\cup Q_2\cup Q_3\subseteq\chi\to Q_1\cup Q_2\cup Q_3\subseteq\chi$ is said to be a G-Hardy and Roger's contraction mapping if there exists $\ell_1,\ell_2,\ell_3,\ell_4$ and $\ell_5\in(0,1)$ with*

$\ell_1+\ell_2+\ell_3+\ell_4+\ell_5<1$ *and for every* $\wp, q \in Q_1 \cup Q_2 \cup Q_3 \subseteq \chi$ *such that*

$$\begin{aligned} G(Q\wp,Qq,Qq)+G(Qq,Q\wp,Q\wp) \leq\ & \ell_2[G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)] \\ & +\ell_3[G(q,Qq,Qq)+G(Qq,q,q)] \\ & +\ell_1[G(\wp,q,q)+G(q,\wp,\wp)] \\ & +\ell_4[G(\wp,Qq,Qq)+G(Qq,\wp,\wp)] \\ & +\ell_5[G(q,Q\wp,Q\wp)+G(Q\wp,q,q)] \end{aligned} \tag{2.2}$$

Definition 2.9. [28][29] *Let* (χ, d_G) *be a G-metric space and* Q_1, Q_2, Q_3 *are three nonempty closed subsets of* χ. *A cyclic mapping* $Q: Q_1 \cup Q_2 \cup Q_3 \subseteq \chi \to Q_1 \cup Q_2 \cup Q_3 \subseteq \chi$ *is said to be a G-Bianchini contraction mapping if there exists* $\ell \in (0,1)$ *and for every* $\wp, q \in Q_1 \cup Q_2 \cup Q_3 \subseteq \chi$ *such that*

$$G(Q\wp,Qq,Qq)+G(Qq,Q\wp,Q\wp) \leq \ell B[G(\wp,q,q)+G(q,\wp,\wp)]$$

where,

$$\begin{aligned} B[G(\wp,q,q)+&G(q,\wp,\wp)] \\ &= max\{G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp), G(q,Qq,Qq)+G(Qq,q,q)\}. \end{aligned}$$

Definition 2.10. [26] *A mapping* $\Upsilon: \mathbb{R}_+ \to \mathbb{R}_+$ *is said to be a comparison function if it satisfies the conditions:*

(i) Υ *is monotone increasing, and*

(ii) $\Upsilon^n(\wp)$ *converges to* 0 *as* $n \to \infty$, *for all* $\wp \in \mathbb{R}_+$.

3. Main Results

In this section, firstly, we demonstrate the *ABPPT* results for various contraction mappings such as the $G-B$ contraction mapping, the G-Bianchini contraction mapping, and their related consequences on G-metric spaces.

Theorem 3.1. *Let* (χ, d_G) *be a G-metric space and* Q_1, Q_2 *and* Q_3 *are three non-empty closed subsets of* χ. *Suppose a cyclic mapping* $Q: Q_1 \cup Q_2 \cup Q_3 \to Q_1 \cup Q_2 \cup Q_3$ *is a contraction* *operator then for all* $\varepsilon > 0$, $P_{B\varepsilon}(Q_1, Q_2, Q_3) \neq \emptyset$. *That is, Q has an approximate best proximity point of the triplets* (Q_1, Q_2, Q_3).

Proof. Let $\wp \in Q_1 \cup Q_2 \cup Q_3 \subseteq \chi$. Then, a sequence $\{\wp_n\}$ is defined by

$$\wp_{n+1} = Q\wp_n, \text{ for all } n \geq 0.$$

That is, $\{\wp_n\}$ is a Cauchy sequence. Thus, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ implies that

$$G(\wp_n, \wp_m, \wp_m) + G(\wp_m, \wp_n, \wp_n) < \varepsilon.$$

In particular, if $n \geq n_0$,

$$G(\wp_n, \wp_{n+1}, \wp_{n+1}) + G(\wp_{n+1}, \wp_n, \wp_n) < \varepsilon.$$

That is,

$$G(\wp_n, Q\wp_n, Qp_n) + G(Q\wp_n, \wp_n, \wp_n) < \varepsilon.$$

Which implies that

$$G(\wp, Q\wp, Q\wp) + G(Q\wp) + G(Q\wp, \wp, \wp) \leq d_G(Q_1, Q_2, Q_3) + \varepsilon$$

Therefore,

$$G(\wp, Q\wp, Q\wp) + G(Q\wp) + G(Q\wp, \wp, \wp) < d_G(Q_1, Q_2, Q_3)$$

Hence,

$$P_{B\varepsilon}(Q_1, Q_2, Q_3) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

□

Theorem 3.2. *Let* Q_1, Q_2 *and* Q_3 *be three non-empty closed subsets of a G-metric space* (χ, d_G). *Suppose that a cyclic mapping* $Q : Q_1 \cup Q_2 \cup Q_3 \to Q_1 \cup Q_2 \cup Q_3$ *be a* $G-B$ *contraction, then for all* $\varepsilon > 0$, $P_{B\varepsilon}(Q_1, Q_2, Q_3) \neq \emptyset$. *That is, Q has an approximate best proximity point of the triplets* (Q_1, Q_2, Q_3) *and*

$$Dtr(P_{B\varepsilon}(Q_1, Q_2, Q_3)) \leq \frac{2(\ell_1 + \ell_3)d_G(Q_1, Q_2, Q_3) + 2\varepsilon(\ell_1 + \ell_3 + 1)}{1 - \ell_2 - 2\ell_3}, \text{ for all } \varepsilon > 0.$$

Proof. Given that a cyclic mapping Q is a $G-B$ contraction operator on a G-metric space (χ, d_G). Let $\varepsilon > 0$ and $\wp \in Q_1 \cup Q_2 \cup Q_3$.

Consider,

$$
\begin{aligned}
& G(Q^n\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(Q^{n+1}\wp, Q^n\wp, Q^n\wp) \\
&\qquad = G(Q(Q^{n-1}\wp), Q(Q^n\wp), Q(Q^n\wp)) + G(Q(Q^n\wp), Q(Q^{n-1}\wp), Q(Q^{n-1}\wp)) \\
&\qquad \le \ell_1[G(Q^{n-1}\wp, Q^n\wp, Q^n\wp) + G(Q^n\wp, Q^{n-1}\wp, Q^{n-1}\wp) \\
&\qquad\quad + G(Q^n\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(Q^{n+1}\wp, Q^n\wp, Q^n\wp)] \\
&\qquad\quad + \ell_2[G(Q^{n-1}\wp, Q^n\wp, Q^n\wp) + G(Q^n\wp, Q^{n-1}\wp, Q^{n-1}\wp)] \\
&\qquad\quad + \ell_3[G(Q^{n-1}\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(G^{n+1}\wp, Q^{n-1}\wp, Q^{n-1}\wp) \\
&\qquad\quad + G(Q^n\wp, Q^n\wp, Q^n\wp) + G(Q^n\wp, Q^n\wp, Q^n\wp)] \\
&\qquad = \ell_1[G(Q^{n-1}\wp, Q^n\wp, Q^n\wp) + G(Q^n\wp, Q^{n-1}\wp, Q^{n-1}\wp)] \\
&\qquad\quad + \ell_1[G(Q^n\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(Q^{n+1}\wp, Q^n\wp, Q^n\wp)] \\
&\qquad\quad + \ell_2[G(Q^{n-1}\wp, Q^n\wp, Q^n\wp) + G(Q^n\wp, Q^{n-1}\wp, Q^{n-1}\wp)] \\
&\qquad\quad + \ell_3[G(Q^{n-1}\wp, Q^n\wp, Q^n\wp) + G(Q^n\wp, Q^{n-1}\wp, Q^{n-1}\wp)] \\
&\qquad\quad + \ell_3[G(Q^n\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(Q^{n+1}\wp, Q^n\wp, Q^n\wp)]
\end{aligned}
$$

On simplifying, we get

$$
\begin{aligned}
& G(Q^n\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(Q^{n+1}\wp, Q^n\wp, Q^n\wp) \\
&\qquad = \left(\frac{\ell_1+\ell_2+\ell_3}{1-\ell_1-\ell_3}\right)[G(Q^{n-1}\wp, Q^n\wp, Q^n\wp) + G(Q^n\wp, Q^{n-1}\wp, Q^{n-1}\wp)] \\
&\qquad = \lambda[G(Q^{n-1}\wp, Q^n\wp, Q^n\wp) + G(Q^n\wp, Q^{n-1}\wp, Q^{n-1}\wp)], \text{ where } \lambda = \frac{\ell_1+\ell_2+\ell_3}{1-\ell_1-\ell_3} \\
&\qquad = \lambda[G(Q(Q^{n-2}\wp), Q(Q^{n-1}\wp), Q(Q^{n-1}\wp)) + G(Q(Q^{n-1}\wp), Q(Q^{n-2}\wp), Q(Q^{n-2}\wp))] \\
&\qquad \le \lambda^2[G(Q^{n-2}\wp, Q^{n-1}\wp, Q^{n-1}\wp) + G(Q^{n-1}\wp, Q^{n-2}\wp, Q^{n-2}\wp)] \\
&\qquad\quad \cdots \\
&\qquad \le \lambda^n[G(\wp, Q\wp, Q\wp) + G(Q\wp, \wp, \wp)]
\end{aligned}
$$

Since ℓ_1, ℓ_2 and $\ell_3 \in (0,1)$ implies that $\lambda \in (0,1)$. Therefore,

$$\lim_{n\to\infty}[G(Q^n\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(Q^{n+1}\wp, Q^n\wp, Q^n\wp)] = 0, \text{ for all } \wp \in Q_1 \cup Q_2 \cup Q_3.$$

It follows that,

$$P_{B\varepsilon}(Q_1, Q_2, Q_3) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

Therefore, Q has an approximate best proximity point of the triplets (Q_1, Q_2, Q_3). For diameter, use condition (ii) of Lemma 2.6. For that, take $\psi > 0$ and $\wp, q \in P_{B\varepsilon}(Q_1, Q_2, Q_3)$. Also,

$$[G(\wp, q, q) + G(q, \wp, \wp)] - [G(Q\wp, Qq, Qq) + G(Qq, Q\wp, Q\wp)] \leq \psi$$

Which implies that

$$[G(\wp, q, q) + G(q, \wp, \wp)] \leq [G(Q\wp, Qq, Qq) + G(Qq, Q\wp, Q\wp)] + \psi.$$

Since $\wp, q \in P_{B\varepsilon}(Q_1, Q_2, Q_3)$, we have

$$[G(\wp, Q\wp, Q\wp) + G(Q\wp, \wp, \wp)] \leq d_G(Q_1, Q_2, Q_3) + \varepsilon_1$$

And

$$[G(q, Qq, Qq) + G(Qq, q, q)] \leq d_G(Q_1, Q_2, Q_3) + \varepsilon_2.$$

Now, choose $\varepsilon = max\{\varepsilon_1, \varepsilon_2\}$, we get

$$\begin{aligned}
G(\wp, q, q) + G(q, \wp, \wp) &\leq d_G[G(Q\wp, Qq, Qq) + G(Qq, Q\wp, Q\wp)] + \psi \\
&\leq \ell_1[2d_G(Q_1, Q_2, Q_3) + 2\varepsilon] + \ell_2[G(\wp, q, q) + G(q, \wp, \wp)] \\
&\quad + \ell_3[2[G(\wp, q, q) + G(q, \wp, \wp)] + 2d_G(Q_1, Q_2, Q_3) + 2\varepsilon] + \psi \\
&\leq \left(\frac{2(\ell_1 + \ell_3)d_G(Q_1, Q_2, Q_3) + 2\varepsilon(\ell_1 + \ell_3) + \psi}{1 - \ell_2 - 2\ell_3}\right) \\
&= \Phi(\psi)
\end{aligned}$$

So, for every $\psi > 0$, there exists $\Phi(\psi) > 0$, such that

$$[G(\wp, q, q) + G(q, \wp, \wp)] - [G(Q\wp, Qq, Qq) + G(Qq, Q\wp, Q\wp)] \leq \psi$$

Implies that

$$G(\wp, q, q) + G(q, \wp, \wp) \leq \Phi(2\varepsilon), \text{ for all } \varepsilon > 0.$$

Hence,

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \frac{2(\ell_1+\ell_3)d_G(Q_1,Q_2,Q_3)+2\varepsilon(\ell_1+\ell_3+1)}{1-\ell_2-2\ell_3}, \text{ for all } \varepsilon > 0.$$

□

Theorem 3.3. *Let Q_1,Q_2 and Q_3 be three non-empty closed subsets of a G-metric space (χ,d_G). Suppose that a cyclic mapping $Q : Q_1\cup Q_2\cup Q_3 \to Q_1\cup Q_2\cup Q_3$ be a G-Bianchini contraction operator, then for all $\varepsilon > 0$, $P_{B\varepsilon}(Q_1,Q_2,Q_3) \neq \emptyset$. That is, Q has an approximate best proximity point of the triplets (Q_1,Q_2,Q_3) and*

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \ell d_G(Q_1,Q_2,Q_3)+\varepsilon(\ell+2), \textit{ for all } \varepsilon > 0.$$

Proof. Given that a cyclic mapping Q is a G-Bianchini contraction operator on a G-metric space (χ,d_G). Let $\varepsilon > 0$ and $\wp \in Q_1\cup Q_2\cup Q_3$. Consider,

Case 1. *Suppose that,*

$$B[G(\wp,q,q)+G(q,\wp,\wp)] = G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)$$

Then, the Definition 2.9 becomes:

$$G(Q\wp,Qq,Qq)+G(Qq,Q\wp,Q\wp) \leq \ell[G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)]$$

Substituting $q = Q\wp$, we get

$$G(Q\wp,Q^2\wp,Q^2\wp)+G(Q^2\wp,Q\wp,Q\wp) \leq \ell[G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)]$$

Again substituting $\wp = Q\wp$, we get

$$\begin{aligned} G(Q^2\wp,Q^3\wp,Q^3\wp)+G(Q^3\wp,Q^2\wp,Q^2\wp) &\leq \ell[G(Q\wp,Q^2\wp,Q^2\wp)+G(Q^2\wp,Q\wp,Q\wp)] \\ &= \ell^2[G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)] \end{aligned}$$

By continuing this process, we have

$$G(Q^n\wp,Q^{n+1}\wp,Q^{n+1}\wp)+G(Q^{n+1}\wp,Q^n\wp,Q^n\wp) \leq \ell^n[G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)]$$

Case 2. *Suppose that,*

$$B[G(\wp,q,q)+G(q,\wp,\wp)] = G(q,Qq,Qq)+G(Qq,q,q)$$

Then, the Definition 2.9 *becomes:*

$$G(Q\wp,Qq,Qq)+G(Qq,Q\wp,Q\wp) \leq \ell[G(q,Qq,Qq)+G(Qq,q,q)]$$

Substituting $q = Q\wp$, we get

$$G(Q\wp,Q^2\wp,Q^2\wp)+G(Q^2\wp,Q\wp,Q\wp) \leq \ell[G(Q\wp,Q^2\wp,Q^2\wp)+G(Q^2\wp,Q\wp,Q\wp)]$$

Which is impossible because $\ell \in (0,1)$. Therefore, Case 2 does not exists. Then, by Case 1 and Theorem 2.4, we have

$$\lim_{n\to\infty}[G(Q^n\wp,Q^{n+1}\wp,Q^{n+1}\wp)+G(Q^{n+1}\wp,Q^n\wp,Q^n\wp)] = 0, \text{ for all } \wp \in Q_1 \cup Q_2 \cup Q_3.$$

It follows that,

$$P_{B\varepsilon}(Q_1,Q_2,Q_3) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

Therefore, Q has an *ABPP* of the triplets (Q_1,Q_2,Q_3). For diameter, use condition (ii) of Lemma 2.6. For that, take $\psi > 0$ and $\wp, q \in P_{B\varepsilon}(Q_1,Q_2,Q_3)$. Also,

$$[G(\wp,q,q)+G(q,\wp,\wp)] - [G(Q\wp,Qq,Qq)+G(Qq,Q\wp,Q\wp)] \leq \psi$$

Which implies that

$$[G(\wp,q,q)+G(q,\wp,\wp)] \leq [G(Q\wp,Qq,Qq)+G(Qq,Q\wp,Q\wp)]+\psi.$$

Since $\wp, q \in P_{B\varepsilon}(Q_1,Q_2,Q_3)$, we have

$$[G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)] \leq d_G(Q_1,Q_2,Q_3)+\varepsilon_1$$

And

$$[G(q,Qq,Qq)+G(Qq,q,q)] \leq d_G(Q_1,Q_2,Q_3)+\varepsilon_2$$

Now, choose $\varepsilon = max\{\varepsilon_1, \varepsilon_2\}$, we get

$$\begin{aligned} G(\wp,q,q)+G(q,\wp,\wp) &\leq [G(Q\wp,Qq,Qq)+G(Qq,Q\wp,Q\wp)]+\psi \\ &\leq \ell[G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)]+\psi \\ &= \ell[d_G(Q_1,Q_2,Q_3)+\varepsilon]+\psi \\ &= \Phi(\psi) \end{aligned}$$

So, for every $\psi > 0$, there exists $\Phi(\psi) > 0$, such that

$$[G(\wp,q,q)+G(q,\wp,\wp)]-[G(Q\wp,Qq,Qq)+G(Qq,Q\wp,Q\wp)] \leq \psi$$

Implies that

$$G(\wp,q,q)+G(q,\wp,\wp) \leq \Phi(\psi)$$

Then by Lemma 2.6 and Theorem 3.2, we have

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \Phi(2\varepsilon), \text{ for all } \varepsilon > 0.$$

Hence,

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \ell d_G(Q_1,Q_2,Q_3)+\varepsilon(\ell+2), \text{ for all } \varepsilon > 0.$$

□

Corollary 3.4. *Let (χ, d_G) be a G-metric space and Q_1, Q_2 and Q_3 are any three nonempty closed subsets of χ. Suppose a cyclic mapping $Q : Q_1 \cup Q_2 \cup Q_3 \subseteq \chi \to Q_1 \cup Q_2 \cup Q_3 \subseteq \chi$ is defined on a G-metric space (χ, d_G) and $\ell \in (0,1)$ such that*

$$\begin{aligned} &G(Q\wp,Qq,Qq)+G(Qq,Q\wp,Q\wp) \\ &\qquad \leq g[G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)], \text{ for all } \wp,q \in Q_1 \cup Q_2 \cup Q_3 \subseteq \chi. \end{aligned}$$

Then, for all $\varepsilon > 0$, $P_{B\varepsilon}(Q_1,Q_2,Q_3) \neq \emptyset$. That is, Q has an approximate best proximity point of the triplets (Q_1,Q_2,Q_3) and

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \ell d_G(Q_1,Q_2,Q_3)+\varepsilon(\ell+2), \textit{ for all } \varepsilon > 0.$$

Proof. Substituting $B[G(\wp,q,q)+G(q,\wp,\wp)] = G(\wp,Q\wp,Q\wp)+G(Q\wp,\wp,\wp)$ in Theorem 3.3 completes this corollary. □

Theorem 3.5. *Let Q_1, Q_2 and Q_3 be three non-empty closed subsets of a G-metric space (χ, d_G). Suppose that a cyclic mapping $Q : Q_1 \cup Q_2 \cup Q_3 \to Q_1 \cup Q_2 \cup Q_3$ be a Hardy-Rogers contraction operator, then for all $\varepsilon > 0$, $P_{B\varepsilon}(Q_1, Q_2, Q_3) \neq \emptyset$. That is, Q has an approximate best proximity point of the triplets (Q_1, Q_2, Q_3) and*

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \frac{(1-\ell_1)d_G(Q_1,Q_2,Q_3)+\varepsilon(3-\ell_1)}{\ell_2-\ell_3}, \text{ for all } \varepsilon > 0.$$

Proof. Given that a cyclic mapping Q is a G-Hardy Roger's contraction operator on a G-metric space (χ, d_G). Let $\varepsilon > 0$ and $\wp \in Q_1 \cup Q_2 \cup Q_3$. Consider,

$$\begin{aligned} &G(Q^n\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(Q^{n+1}\wp, Q^n\wp, Q^n\wp) \\ &\qquad = G(Q(Q^{n-1}\wp), Q(Q^n\wp), Q(Q^n\wp)) + G(Q(Q^n\wp), Q(Q^{n-1}\wp), Q(Q^{n-1}\wp)) \end{aligned}$$

By using Definition and simplifying, we get

$$\begin{aligned} &G(Q^n\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(Q^{n+1}\wp, Q^n\wp, Q^n\wp) \\ &\quad \leq \lambda[G(Q^{n-1}\wp, Q^n\wp, Q^n\wp) + G(Q^n\wp, Q^{n-1}\wp, Q^{n-1}\wp)], \text{ where } \lambda = \frac{\ell_1+\ell_2+\ell_3}{1-\ell_3-\ell_4} \\ &\quad = \lambda[G(Q(Q^{n-2}\wp), Q(Q^{n-1}\wp), Q(Q^{n-1}\wp)) + G(Q(Q^{n-1}\wp), Q(Q^{n-2}\wp), Q(Q^{n-2}\wp))] \\ &\quad \leq \lambda^2[G(Q^{n-2}\wp, Q^{n-1}\wp, Q^{n-1}\wp) + G(Q^{n-1}\wp, Q^{n-2}\wp, Q^{n-2}\wp)] \\ &\qquad \cdots \\ &\quad \leq \lambda^n[G(\wp, Q\wp, Q\wp) + G(Q\wp, \wp, \wp)] \end{aligned}$$

Since ℓ_1, ℓ_2 and $\ell_3 \in (0,1)$ implies that $\lambda \in (0,1)$. Therefore,

$$\lim_{n\to\infty}[G(Q^n\wp, Q^{n+1}\wp, Q^{n+1}\wp) + G(Q^{n+1}\wp, Q^n\wp, Q^n\wp)] = 0, \text{ for all } \wp \in Q_1 \cup Q_2 \cup Q_3.$$

It follows that,

$$P_{B\varepsilon}(Q_1,Q_2,Q_3) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

Therefore, Q has an approximate best proximity point of the triplets (Q_1, Q_2, Q_3). For diameter, use condition (ii) of Lemma 2.6. For that, take $\psi > 0$ and $\wp, q \in P_{B\varepsilon}(Q_1, Q_2, Q_3)$. Also, by using Definition 2.8 and the similar procedure mentioned in Theorem 3.2, we get

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \frac{(\ell_2+\ell_3+\ell_4+\ell_5)d_G(Q_1,Q_2,Q_3)+\varepsilon(\ell_2+\ell_3+\ell_4+\ell_5+2)}{1-(\ell_1+\ell_4+\ell_5)}, \text{ for all } \varepsilon > 0.$$

Which means exactly that

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \frac{(1-\ell_1)d_G(Q_1,Q_2,Q_3)+\varepsilon(3-\ell_1)}{\ell_2-\ell_3}, \text{ for all } \varepsilon > 0.$$

□

Corollary 3.6. *In Theorem 3.2, substitute $\ell_1 = \ell_3 = 0$ and $\ell_2 = \ell = \alpha$ then it becomes $P-\alpha$ contraction operator and for every $\varepsilon > 0$, $P_{B\varepsilon}(Q_1,Q_2,Q_3) \neq \emptyset$ and*

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \frac{2(d_G(Q_1,Q_2,Q_3)+\varepsilon)}{\ell}, \textit{ for all } \varepsilon > 0.$$

Corollary 3.7. *In Theorem 3.2, substitute $\ell_2 = \ell_3 = 0$ then it becomes P-Kannan contraction operator and for every $\varepsilon > 0$, $P_{B\varepsilon}(Q_1,Q_2,Q_3) \neq \emptyset$ and*

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq 2\ell_1 d_G(Q_1,Q_2,Q_3)+2\varepsilon(1+\ell_1), \textit{ for all } \varepsilon > 0.$$

Corollary 3.8. *In Theorem 3.2, substitute $\ell_1 = \ell_2 = 0$ then it becomes P-Chatterjea contraction operator and for every $\varepsilon > 0$, $P_{B\varepsilon}(Q_1,Q_2,Q_3) \neq \emptyset$ and*

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \frac{\ell_3 d_G(Q_1,Q_2,Q_3)+\varepsilon(1+\ell_3)}{1-2\ell_3}, \textit{ for all } \varepsilon > 0.$$

Corollary 3.9. *In Theorem 3.5, substitute $\ell_4 = \ell_5 = 0$ then it becomes P-Reich contraction operator and for every $\varepsilon > 0$, $P_{B\varepsilon}(Q_1,Q_2,Q_3) \neq \emptyset$ and*

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \frac{(1-\ell_1)d_G(Q_1,Q_2,Q_3)+\varepsilon(3-\ell_1)}{1-\ell_1}, \textit{ for all } \varepsilon > 0.$$

Corollary 3.10. *In Theorem 3.5, substitute $\ell_5 = \ell_4$, then it becomes P-Ciric contraction operator and for every $\varepsilon > 0$, $P_{B\varepsilon}(Q_1,Q_2,Q_3) \neq \emptyset$ and*

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \frac{(1-\ell_1)d_G(Q_1,Q_2,Q_3)+\varepsilon(3-\ell_1)}{\ell_2+\ell_3}, \textit{ for all } \varepsilon > 0.$$

Example 3.11. *Let $\chi = [0,1]$ and consider the closed subsets $Q_1 = [0,3/6], Q_2 = [2/6,3/6]$ and $Q_3 = [5/6,1]$ of a G-metric space (χ, d_G) and $Q : Q_1 \cup Q_2 \cup Q_3 \to Q_1 \cup Q_2 \cup Q_3$ is defined* *by:*

$$Q\wp = \begin{cases} \frac{2}{6} + \wp & when\ \wp \in \left[0, \frac{3}{6}\right] \\ \\ \frac{3}{6} + \wp & when\ \wp \in \left[\frac{2}{6}, \frac{3}{6}\right] \\ \\ 1 - \frac{3}{6} & when\ \wp \in \left[\frac{5}{6}, 1\right] \end{cases}$$

This clearly shows that $Q(Q_1) \subseteq Q_2, Q(Q_2) \subseteq Q_3$ and $Q(Q_3) \subseteq Q_1$. Also for every $\wp, q \in Q_1 \cup Q_2 \cup Q_3 \subseteq \chi$ satisfies the Definition 2.5 and Definition 2.7. Thus, Q satisfies all the conditions of the Theorems 3.2, 3.3 and Corollary 3.4 .

Example 3.12. *Let $\chi = \{0, 1, 2, ..., 18\}$ and $G : \chi \times \chi \times \chi \to \mathbb{R}^+$ be defined by:*

$$G(\wp, q, s) = \begin{cases} \wp + q + s & when\ \wp \neq q \neq s \neq 0 \\ \wp + q & when\ \wp = q \neq s; \wp, q, s \neq 0 \\ \wp + s + 1 & when\ \wp = 0, q \neq s; q, s \neq 0 \\ q + 2 & when\ \wp = 0, q = s \neq 0 \\ s + 1 & when\ \wp = 0, q = 0, s \neq 0 \\ 0 & when\ \wp = q = s \end{cases}$$

Consider a closed subsets $Q_1 = \{4, 18\}, Q_2 = \{3, 7, 17\}$ and $Q_3 = \{0\}$ of a metric space (χ, d_G) and $Q : Q_1 \cup Q_2 \cup Q_3 \to Q_1 \cup Q_2 \cup Q_3$ is defined by:

$$Q\wp = \begin{cases} \wp - 1 & when\ \wp \in \{4, 8\} \\ 0 & when\ \wp \in \{3, 7, 17\} \\ 4 & when\ \wp = 0 \end{cases}$$

This clearly shows that $Q(Q_1) \subseteq Q_2, Q(Q_2) \subseteq Q_3$ and $Q(Q_3) \subseteq Q_1$. Also for every $Q_1, Q_2 \in Q_1 \cup Q_2 \cup Q_3 \subseteq \chi$ satisfies the equation $(2.1), (2.2)$ and (2.3). Thus, Q satisfies Theorems 3.2 *and 3.3. Therefore,*

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \ell d_G(Q_1,Q_2,Q_3)+\varepsilon(\ell+2), \textit{ for all } \varepsilon > 0.$$

and

$$Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3)) \leq \frac{2(\ell_1+\ell_3)d_G(Q_1,Q_2,Q_3)+2\varepsilon(\ell_1+\ell_3+1)}{1-\ell_2-2\ell_3}, \textit{ for all } \varepsilon > 0.$$

satisfies respectively.

Remark 3.13. *We have proved many BPPT results by using various operators on G-metric spaces. The diameters of several contraction operators are shown in the table below.*

S. No	*Operator(s)*	*Diameter, for every $\varepsilon > 0$, $Dtr(P_{B\varepsilon}(Q_1,Q_2,Q_3))$*
1	*Contraction*	$\leq \frac{2(d_G(Q_1,Q_2,Q_3)+\varepsilon)}{\ell}$
2	*Kannan*	$\leq 2\ell_1 d_G(Q_1,Q_2,Q_3)+2\varepsilon(1+\ell_1)$
3	*Chatterjea*	$\leq \frac{\ell_3 d_G(Q_1,Q_2,Q_3)+\varepsilon(1+\ell_3)}{1-2\ell_3}$
4	*B-contraction*	$\leq \frac{2(\ell_1+\ell_3)d_G(Q_1,Q_2,Q_3)+2\varepsilon(\ell_1+\ell_3+1)}{1-\ell_2-2\ell_3}$
5	*Bianchini*	$\leq \ell d_G(Q_1,Q_2,Q_3)+\varepsilon(\ell+2)$
6	*Hardy-Rogers*	$\leq \frac{(1-\ell_1)d_G(Q_1,Q_2,Q_3)+\varepsilon(3-\ell_1)}{\ell_2-\ell_3}$
7	*Ćirić*	$\leq \frac{(1-\ell_1)d_G(Q_1,Q_2,Q_3)+\varepsilon(3-\ell_1)}{\ell_2+\ell_3}$
8	*Ćirić-Reich-Rus*	$\leq \frac{2[\ell_2 d_G(Q_1,Q_2,Q_3)+\varepsilon(1+\ell_2)]}{1-\ell_1}$
9	*Reich*	$\leq \frac{(1-\ell_1)d_G(Q_1,Q_2,Q_3)+\varepsilon(3-\ell_1)}{1-\ell_1}$
10	*Zamfirescu*	$\leq \frac{(2[\ell d_G(Q_1,Q_2,Q_3)+\varepsilon(1+\ell)]}{1-\ell}$
11	*Mohseni-saheli*	$\leq \frac{2\ell d_G(Q_1,Q_2,Q_3)+2\varepsilon(1+\ell)}{1-2\ell}$
12	*Mohseni-semi*	$\leq \frac{\ell d_G(Q_1,Q_2,Q_3)+\varepsilon(2+\ell)}{1-\ell}$
13	*Weak contraction*	$\leq \frac{\ell W d_G(Q_1,Q_2,Q_3)+\varepsilon(2+\ell W)}{1-\ell-\ell W}$

4. APPLICATIONS

The *ABPPT* covers a wide range of applications in the domain of mathematics, particularly in differential equations, Fourier series, numerical analysis, and so on. By reading [14] and references therein, one can find a variety of applications involving *ABPPT* results in differential equations. The examples below demonstrate how to apply *ABPPT* results in differential equations.

Example 4.1. *Let $\chi = C([0,1],\mathbb{R})$ and χ is G-metric space defined by $d(\wp,q)=\sup_{\ell\in[0,1]}|\wp - q|^2$. Also, consider $y''(\ell)=3y^2(\ell)/2$, $0\leq \ell\leq 1$ and the initial conditions $y(0)=4$, $y(1)=1$. Here, the exact solution is $y(\ell)=4/(1+\ell)^2$. We have, $y_0(\ell)=c_1\ell+c_2$. By using the initial conditions, we get $y_0(\ell)=4-3\ell$. Now, define the integral operator,*

$$A(y) = y + \int_0^1 G(\ell,\kappa)[y'' - f(\kappa,y,y')]d\kappa \tag{4.1}$$

where

$$G(\ell,\kappa) = \begin{cases} \kappa(1-\ell) & 0\leq\kappa\leq\ell \\ \ell(1-\ell) & \ell\leq\kappa\leq 1 \end{cases}$$

Then, the equation (5.1) *becomes*

$$\begin{aligned} A(y) &= y(\ell) + \int_0^1 G(\ell,\kappa)y''(s)ds - \int_0^1 G(\ell,\kappa)f(\kappa,y,y')ds \\ &= (4-3\ell) - \int_0^1 G(\ell,\kappa)\big[-\frac{3y^2(\kappa)}{2}\big]d\kappa \\ &= 4-3\ell+\frac{3}{2}\left\{\int_0^1 G(\ell,\kappa)y^2(\kappa)d\kappa\right\} \end{aligned}$$

Let us take $G(A\wp,Aq,Aq)+G(Aq,A\wp,A\wp)=d(A\wp,Aq)$ So, we have

$$\begin{aligned} d(A\wp,Aq) &= \sup_{t\in[0,1]}|A\wp - Aq|^2 \\ &= \sup_{\ell\in[0,1]}\left|\frac{3}{2}\int_0^1 G(\ell,\kappa)\wp^2(\kappa)d\kappa - \frac{3}{2}\int_0^1 G(\ell,\kappa)q^2(\kappa)d\kappa\right|^2 \\ &\leq \frac{9}{4}\left(\int_0^1 |G(\ell,\kappa)|^2 ds\right)\left(\int_0^1 |\wp^2(\kappa)-q^2(\kappa)|^2 d\kappa\right) \\ &\leq \frac{3}{4}\frac{\ell^2(1-\ell)^2}{3}\int_0^1 \left|\wp^2(\kappa)-q^2(\kappa)\right|^2 d\kappa \\ &\leq \frac{3}{4}(\frac{1}{4})(\frac{1}{4})\int_0^1 \left|\wp^2(\kappa)-q^2(\kappa)\right|^2 d\kappa \\ &\leq \frac{3}{64}\sup_{\ell\in[0,1]}|\wp(\kappa)-q(\kappa)|^2 \end{aligned}$$

$$\leq \frac{3}{64} d(\wp, q)$$

That is,

$$G(A\wp, Aq, Aq) + G(Aq, A\wp, A\wp) \leq \frac{3}{64}[G(\wp, q, q) + G(q, \wp, \wp)]$$

From this, we get $\ell_2 = 3/64$ *and* $\ell_1 = \ell_3 = 0$. *Hence, it satisfies all the conditions of Theorem* 3.2*. Also, by Theorem* 3.1*, A has ABPP in* $\chi = C([0,1], \mathbb{R})$. *Therefore, the given bounded value problem has ABPP in* χ.

5. CONCLUSION

In this paper, some *ABPPT* results are established on *G*-metric spaces by utilizing various types of contraction mappings. It is worth observing that in the limiting case $\varepsilon \longrightarrow 0$, all the results established in the present paper produces more restricted $ABPPT's$. Also, $ABPP's$ are consequently not less important than $BPP's$. As various future results can be demonstrated in a smaller setting to ensure the existence of the $ABPP's$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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