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A NEW ITERATIVE SCHEME FOR SOLVING DELAY DIFFERENTIAL EQUATION AND OXYGEN DIFFUSION PROBLEMS

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Abstract. The purpose of this paper is to introduce a new five steps iterative algorithm for approximating the solution of a delay differential equation and an oxygen diffusion problem. In addition, using our proposed iteration process, we state and prove some convergence results for approximating the fixed points of generalized (α,β)-nonexpansive type I mapping. In addition, we show that our proposed iterative scheme converges faster than some existing iterative schemes in the literature, data dependency, and stability results for our proposed iterative scheme are established with an analytical and numerical example given to justify our claim. Lastly, we established that the D-iterative scheme introduced by [8] and the AG-iterative scheme introduced by [21] have the same rate of convergence.

Keywords: iterative scheme; contraction; eneralized (α,β)-nonexpansive type I mapping; delay differential equation; oxygen diffusion problem.

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1. INTRODUCTION

The study of fixed point and optimization theory is becoming a more significant field in nonlinear analysis. Its successful applications in practically every field and mathematical research are the reason for this. It is well known that fixed point or optimization problems are extremely difficult or practically impossible to solve analytically, hence approximation methods of resolution must be taken into consideration. Because of this, scholars in this area have put forth a number of methods for solving optimization and fixed point problems. Proximal-like techniques, fixed point techniques, auxiliary principles, decomposition techniques, extra-gradient techniques, sub-gradient techniques, projection contraction techniques, and normal map equations are a few examples; refer to [15, 16, 17, 18] and the references therein. Finding a solution to an equation of the type

$$x = Tx, \tag{1}$$

where T is a single-valued nonlinear operator defined on a nonempty set X, is the focus of fixed point problems. To approximate the answer (1) for various operators and spaces, scholars have developed a number of iterative approaches throughout the years; for instance, see ([8, 10, 11, 12, 13, 21]). Research is still ongoing to create iterative algorithms that solve (1) more quickly and effectively. At the very least, a good and dependable fixed point iteration must have the following qualities: (1) it must converge to an operator's fixed point; (2) it must be T-stable; (3) it must be quick in comparison to other iterations already in the literature; (4) it must display data dependent results; and (5) it must be used to approximate some real-life problems. One of the prominent iterative scheme over the years is the Picard [19] iterative process is defined by the sequence $\{a_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases} a_0 \in C, \\ a_{n+1} = Ta_n, \quad \forall n \in \mathbb{N}. \end{cases} \tag{1.1}$$

It is well-known that the Picard iterative process only converges to the fixed point if and only if the initial guess is close to the fixed point. In addition, if it does converges, it does that at a linear rate. In light of this, we have Ullah and Arshad [20] introduced the iterative process

$$(1.3)\qquad \begin{cases} c_0 \in C \\ a_n = (1-\alpha_n)c_n + \alpha_n T c_n \\ b_n = T b_n \\ c_{n+1} = T b_n, \quad \forall n \in \mathbb{N} \end{cases}$$

where $\{\alpha_n\}$ are real sequences in $(0,1)$. Also, Karakaya [10] introduced the iterative sequence $\{x_n\}$ defined by

$$(1.4)\qquad \begin{cases} w_0 \in C \\ u_n = T w_n \\ v_n = (1-\alpha_n)u_n + \alpha_n T u_n \\ w_{n+1} = T v_n, \quad \forall n \in \mathbb{N} \end{cases}$$

where $\{\alpha_n\}$ are real sequences in $(0,1)$. More so, Abass eta al., [1] introduced the iterative sequence $\{x_n\}$ defined by

$$(1.5)\qquad \begin{cases} p_0 \in C \\ p_n = T r_n \\ q_n = T p_n, \\ r_{n+1} = (1-\alpha_n)q_n + \alpha_n T q_n \\ \quad \forall n \in \mathbb{N} \end{cases}$$

where $\{\alpha_n\}$ are real sequences in $(0,1)$. It was established by the authors that the iterative processes (1.3), (1.4), and (1.5) have the same convergence rate.

Recently, Okeke et.al [21] introduced an iterative scheme named AG-iterative scheme defined by the sequence $\{u_n\}_{n=0}^{\infty}$ as follows:

$$(1.10)\qquad \begin{cases} u_0 = x \in C \\ x_n = (1-\gamma_n)u_n + \gamma_n T u_n \\ w_n = (1-\beta_n)Tu_n + \beta_n T x_n \\ v_n = T[(1-\alpha_n)w_n + \alpha_n T w_n] \\ u_{n+1} = T v_n, \quad \forall n \in \mathbb{N}. \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. They established that the above iterative process converges faster than existing iterative schemes in the literature. Furthermore, In the light of this, Hussain et al., [8] introduced a new iterative method called the *D*-iteration. The *D*-iterative method is defined as follows

$$(1.11)\qquad \begin{cases} x_0 = x \in C \\ y_n = T[(1-\alpha_n)x_n + \alpha_n T x_n] \\ v_n = T[(1-\beta_n)Tx_n + \beta_n T y_n] \\ x_{n+1} = T v_n, \quad \forall n \in \mathbb{N}. \end{cases}$$

They established that the D-iterative process is faster than M* iterative process in [20] and some other existing iterative process in the literature. Furthermore, Hussain et al. [8] presented the stability, data dependency and errors estimation results for D-iteration method. More so, they establish that the error in D-iterative process is controllable. However, the convergence, stability, and data dependence results were obtained under some strong assumptions imposed on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$. The authors in [2] provide and affirmative answer to the setback of the *D*-iterative scheme. They also applied the iterative scheme to a delay differential equations and two point second order-boundary value problem.

Inspired and motivated by the above works and the ongoing research interest in this direction, our purpose of this work is to introduce a new five steps iteration process 3.1 for approximating the solution of a delay differential equation and an oxygen diffusion problem. In addition, using our proposed iteration process, we state and prove some convergence results for approximating the fixed points of generalized (α,β)-nonexpansive type I mapping. In addition, we show that our proposed iterative scheme converges faster than some existing iterative schemes in the literature, data dependency ,and stability results for our proposed iterative scheme are established with an analytical and numerical example given to justify our claim.

2. Preliminaries

Let X be a Banach space and $S_X = \{x \in X : \|x\| \leq 1\}$ be a unit ball in X. For all $\alpha \in (0,1)$ and $x, y \in S_X$ such that $x \neq y$, if $\|(1-\alpha)x + \alpha y\| < 1$, then we say X is strictly convex. If X is a strictly convex Banach space and $\|x\| = \|y\| = \|(1-\lambda)y + \lambda x\|$ $\forall x, y \in X$ and $\lambda \in (0,1)$, then $x = y$.

Definition 2.1 (2.1)**.** A Banach space X is said to be smooth if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2}$$

exists for all $x, y \in S_X$.

In the above definition, the norm of X is called Gateaux differentiable. For all $y \in S_X$, if the limit (2) is attained uniformly for $x \in S_X$, then the norm is said to be uniformly Gateaux differentiable or Fréchet differentiable.

Definition 2.2. A Banach space X satisfies Opial's condition [14], if for any sequence $\{x_n\} \in X$, $x_n \rightharpoonup x$ implies that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|,$$

for all $y \in X$ such that $x \neq y$.

Definition 2.3 (2.3)**.** Let C be a subset of a normed space X. A mapping $T : C \to C$ is said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ such that $f(0) = 0$ and $f(t) > 0$ $\forall t \in (0,\infty)$ and that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T))$ denotes the distance from x to $F(T)$.

Berinde [5] proposed a method to compare the fastness of two sequences.

Lemma 2.4. [5] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b respectively. If $\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0$, then $\{a_n\}$ converges faster than $\{b_n\}$.*

Lemma 2.5. [5] *Suppose that for two fixed point iteration processes* $\{u_n\}$ *and* $\{v_n\}$ *both converging to the same fixed point* x^**, the error estimates*

$$\|u_n - x^*\| \leq a_n \quad n \geq 1,$$

$$\|v_n - x^*\| \leq b_n \quad n \geq 1,$$

are available where $\{a_n\}$ *and* $\{b_n\}$ *are two sequences of positive numbers converging to zero. If* $\{a_n\}$ *converges faster than* $\{b_n\}$*, then* $\{u_n\}$ *converges faster than* $\{v_n\}$ *to* x^**.*

Definition 2.6. [7] Let $\{t_n\}$ be any arbitrary sequence in C. Then, an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point p, is said to be T-stable or stable with respect to T, if for $\varepsilon_n = \|t_{n+1} - f(T, t_n)\|$, $\forall n \in \mathbb{N}$, we have $\lim_{n\to\infty} \varepsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = p$.

Lemma 2.7. [25] *Let* $\{\delta_n\}$ *and* $\{\theta_n\}$ *be nonnegative real sequence satisfying the following inequality:*

$$\delta_{n+1} \leq (1 - \phi_n)\delta_n + \theta_n,$$

where $\phi_n \in (0,1)$ *for all* $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \phi_n = \infty$ *and* $\lim_{n\to\infty} \frac{\theta_n}{\phi_n} = 0$*, then* $\lim_{n\to\infty} \delta_n = 0$.

Lemma 2.8. [23] *Let* $\{\delta_n\}$ *and* $\{\theta_n\}$ *be nonnegative real sequence satisfying the following inequality:*

$$\delta_{n+1} \leq (1 - \phi_n)\delta_n + \phi_n \theta_n,$$

where $\phi_n \in (0,1)$ *for all* $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \phi_n = \infty$ *and* $\theta_n \geq 0$ *for all* $n \in \mathbb{N}$ *then* $0 \leq \limsup_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \theta_n$.

Lemma 2.9. [6] *Let* X *be a uniformly convex Banach space and* $0 < p \leq t_n \leq q < 1$ $\forall n \in \mathbb{N}$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of* X *such that* $\limsup_{n\to\infty} \|x_n\| \leq c$, $\limsup_{n\to\infty} \|y_n\| \leq c$ *and* $\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = c$ *hold for some* $c \geq 0$*. Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Definition 2.10 (2.5)**.** Let C be a nonempty subset of a Banach space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to C if for all $x \in C$ and $n \geq 1$,

$$\|x_{n+1} - x\| \leq \|x_n - x\|.$$

Proposition 2.11 (2.1)**.** Let $\{x_n\}$ be a sequence in X and C be a nonempty subset of X. Suppose that $T : C \to C$ is any nonlinear mapping and the sequence $\{x_n\}$ is Fejér monotone with respect to C, then we have the following:

(i) $\{x_n\}$ is bounded;

(ii) The sequence $\{\|x_n - x^*\|\}$ is decreasing and converges for all $x^* \in F(T)$.

Lemma 2.12. [3] *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping with $F(T) \neq \emptyset$. Then T is quasi-nonexpansive.*

Theorem 2.13 (2.1, [3])**.** *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Theorem 2.14. [3] *Let C be a nonempty closed subset of a Banach space X with Opial property and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping with $\lambda = \gamma/2$, $\gamma \in [0, 1)$. If $\{x_n\}$ converges weakly to x and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tx = x$. That is $I - T$ is demiclosed at zero, where I is the identity mapping on X.*

Definition 2.15. [7] Let $T, \tilde{T} : C \to C$ be two operators. We say that $\tilde{T}$ is an approximate operator for T if for some $\varepsilon > 0$, we have

$$\|Tx - \tilde{T}x\| \leq \varepsilon,$$

for all $x \in C$.

Lemma 2.16. [24] *Let $\{x_n\}$ be a sequence of positive real numbers including zero satisfying*

$$\theta_{n+1} \leq (1 - \eta_n)\theta_n$$

such that $\{\eta_n\} \subset (0, 1)$ and $\sum_{n=0}^{\infty} \eta_n = \infty$, then

$$\lim_{n\to\infty} \theta_n = 0.$$

3. Proposed Algorithm

In this section, we present our proposed method for approximating the solution of a Delay differential equation and oxygen diffusion problem. We present the following iterative algorithm defined by the sequence:

Algorithm 3.1.

$$\begin{aligned}
x_0 &\in C \\
u_n &= (1-\alpha_n)x_n + \alpha_n T x_n \\
v_n &= T u_n \\
y_n &= T[(1-\beta_n)v_n + \beta_n T v_n] \\
w_n &= T[(1-\gamma_n)y_n + \gamma_n T y_n] \\
x_{n+1} &= T w_n
\end{aligned}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$.

4. Rate of Convergence, Stability, and Data Dependence

In this section, we establish the rate of convergence, stability and data dependency results for the iterative process 3.1. In addition, we show that the proposed iterative scheme perform faster than some existing iterative schemes in literature.

Theorem 4.1. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let T be a contraction mapping with constant k such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the iterative process defined in* (3.1) *with real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying*

$$\sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = \infty.$$

Then, $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Suppose that $F(T) \neq \emptyset$, let $x^* \in F(T)$, it follows that $x^* = Tx^*$. Using (3.1), we have that

$$\begin{aligned}
\|u_n - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n T x_n - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|Tx_n - Tx^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n k \|x_n - x^*\| \\
&= (1-\alpha_n(1-k))\|x_n - x^*\|. \qquad (3)
\end{aligned}$$

Also, using (3.1) and (3), we have

$$\begin{aligned}
\|v_n - x^*\| &= \|Tu_n - Tx^*\| \\
&\leq k\|u_n - x^*\| \\
&\leq k(1-\alpha_n(1-k))\|x_n - x^*\|. \qquad (4)
\end{aligned}$$

Furthermore, using (3.1) and (4), we have

$$\begin{aligned}
\|y_n - x^*\| &= \|T[(1-\beta_n)v_n + \beta_n T v_n] - Tx^*\| \\
&\leq k\|(1-\beta_n)v_n + \beta_n T v_n - x^*\| \\
&\leq k[(1-\beta_n)\|v_n - x^*\| + \beta_n k \|v_n - x^*\|] \\
&= k(1-\beta_n(1-k))\|v_n - x^*\| \\
&\leq k^2(1-\beta_n(1-k))(1-\alpha_n(1-k))\|x_n - x^*\| \\
&\leq k^2(1-\alpha_n\beta_n(1-k))\|x_n - x^*\|. \qquad (5)
\end{aligned}$$

In addition, using (3.1) and (5), we have

$$\begin{aligned}
\|w_n - x^*\| &= \|T[(1-\gamma_n)y_n + \gamma_n T y_n] - Tx^*\| \\
&\leq k[(1-\gamma_n)\|y_n - x^*\| + \gamma_n k \|y_n - x^*\|] \\
&= k(1-\gamma_n(1-k))\|y_n - x^*\| \\
&\leq k^3(1-\alpha_n\beta_n\gamma_n(1-k))\|x_n - x^*\|. \qquad (6)
\end{aligned}$$

Lastly, using (3.1) and (6), we have

$$\|x_{n+1} - x^*\| = \|Tw_n - Tx^*\|$$

$$\leq k\|w_n - x^*\|$$

$$\leq k^4(1-\alpha_n\beta_n\gamma_n(1-k))\|x_n - x^*\|. \tag{7}$$

From (7), we have the inequality

$$\|x_{n+1} - x^*\| \leq k^4(1-\alpha_n\beta_n\gamma_n(1-k))\|x_n - x^*\|$$

$$\|x_n - x^*\| \leq k^4(1-\alpha_{n-1}\beta_{n-1}\gamma_{n-1}(1-k))\|x_{n-1} - x^*\|$$

$$\vdots$$

$$\|x_1 - x^*\| \leq k^4(1-\alpha_0\beta_0\gamma_0(1-k))\|x_0 - x^*\|. \tag{8}$$

From (8), we have that

$$\|x_{n+1} - x^*\| \leq k^{4(n+1)}\|x_0 - x^*\|\Pi_{m=0}^{n}(1-\alpha_m\beta_m\gamma_m(1-k)). \tag{9}$$

Since $\{\alpha_m\}, \{\gamma_m\}$ and $\{\beta_m\}$ are sequences in $(0,1)$ for all $m \in \mathbb{N}$. We have that $(1-\alpha_m\beta_m\gamma_m(1-k)) \in (0,1)$. Using the inequality that $1-x \leq e^{-x}$ for all $x \in [0,1]$, hence from (9), we have

$$\|x_{n+1} - x^*\| \leq \frac{k^{4(n+1)}\|x_0 - x^*\|}{e^{(1-k)\sum_{m=0}^{n}\alpha_m\gamma_m\beta_m}}. \tag{10}$$

Taking the limit of both sides of the above inequalities, we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Hence, $\{x_n\}$ converges strongly to the fixed point of T. □

Theorem 4.2. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let T be a contraction mapping with constant k such that $F(T) \neq \emptyset$. Let $\{x_n\}, \{v_n\}, \{r_n\}, \{w_n\}$ and $\{n_n\}$ be the iterative process defined in* (1.11)*,* (1.10)*,* (1.5)*,* (1.4)*, and* (1.3) *with sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying*

$$\sum_{n=0}^{\infty}\alpha_n = \infty,$$

$$\sum_{n=0}^{\infty}\alpha_n\beta_n = \infty \text{ and}$$

$$\sum_{n=0}^{\infty}\alpha_n\beta_n\gamma_n = \infty.$$

Then, $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Using a similar approach as in Theorem 4.1, we obtain the desired result. □

Theorem 4.3. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let T be a contraction mapping with constant k such that $F(T) \neq \emptyset$. Suppose that each of the iterative schemes* (3.1), (1.11), (1.10), (1.5), (1.4), *and* (1.3) *converge to same fixed point x^* of T, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in (0,1) satisfying $\alpha \leq \alpha_n < 1$, $\beta\gamma \leq \beta_n\gamma_n < 1$ and $\alpha\beta\gamma \leq \alpha_n\beta_n\gamma_n < 1$. Then, the* (3.1) *scheme converges faster than all other iterative schemes above.*

Proof. From Theorem 3, we have that

$$\|x_{n+1} - x^*\| \leq k^{4(n+1)}\|x_0 - x^*\|\Pi_{m=0}^{n}(1 - \alpha_m\beta_m\gamma_m(1-k)). \tag{11}$$

Using $\alpha\beta\gamma \leq \alpha_n\beta_n\gamma_n < 1$, we have

$$\|x_{n+1} - x^*\| \leq k^{4(n+1)}\|x_0 - x^*\|(1 - \alpha\beta\gamma(1-k))^{n+1}. \tag{12}$$

Let

$$\theta = k^{4(n+1)}\|x_0 - x^*\|(1 - \alpha\beta\gamma(1-k))^{n+1}.$$

Using a similar approach as in Theorem 3, we obtain that the iterative process (1.5) takes the form

$$\|r_{n+1} - x^*\| \leq k^{2(n+1)}\|r_0 - x^*\|(1 - \alpha(1-k))^{n+1}. \tag{13}$$

Let

$$\overline{\theta} = k^{2(n+1)}\|r_0 - x^*\|(1 - \alpha(1-k))^{n+1}.$$

Similarly, we obtain that iterative process (1.4) takes the form

$$\|w_{n+1} - x^*\| \leq k^{2(n+1)}\|w_0 - x^*\|(1 - \alpha(1-k))^{n+1}. \tag{14}$$

Let

$$\omega = k^{2(n+1)}\|w_0 - x^*\|(1 - \alpha(1-k))^{n+1}.$$

Similarly, we obtain that iterative process (1.3) takes the form

$$\|c_{n+1} - x^*\| \leq k^{2(n+1)}\|c_0 - x^*\|(1 - \alpha(1-k))^{n+1}. \tag{15}$$

Let

$$\overline{\omega} = k^{2(n+1)}\|c_0 - x^*\|(1 - \alpha(1-k))^{n+1}.$$

Similarly, we obtain that iterative process (1.10) takes the form

$$\|u_{n+1}-x^*\| \leq k^{3(n+1)}\|p_0-x^*\|(1-\beta\alpha\gamma(1-k))^{n+1}. \tag{16}$$

Let

$$\pi = k^{3(n+1)}\|p_0-x^*\|(1-\beta\alpha\gamma(1-k))^{n+1}.$$

Similarly, we obtain that iterative process (1.11) takes the form

$$\|x_{n+1}-x^*\| \leq k^{3(n+1)}\|x_0-x^*\|(1-(\beta+k\alpha\gamma)(1-k))^{n+1}. \tag{17}$$

Let

$$\Pi = k^{3(n+1)}\|x_0-x^*\|(1-(\beta+k\alpha\gamma)(1-k))^{n+1}.$$

Using Lemma 4, we establish the rate of convergence

$$\begin{aligned}\frac{\theta}{\overline{\overline{\theta}}} &= \frac{k^{4(n+1)}\|x_0-x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}{k^{2(n+1)}\|r_0-x^*\|(1-\alpha(1-k))^{n+1}}\\ &= \frac{k^{2(n+1)}\|x_0-x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}{\|r_0-x^*\|(1-\alpha(1-k))^{n+1}} \to 0 \text{ as } n\to\infty. \end{aligned} \tag{18}$$

Also,

$$\begin{aligned}\frac{\theta}{\omega} &= \frac{k^{4(n+1)}\|x_0-x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}{k^{2(n+1)}\|w_0-x^*\|(1-\alpha(1-k))^{n+1}}\\ &= \frac{k^{2(n+1)}\|x_0-x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}{\|w_0-x^*\|(1-\alpha(1-k))^{n+1}} \to 0 \text{ as } n\to\infty. \end{aligned} \tag{19}$$

Also,

$$\begin{aligned}\frac{\theta}{\overline{\omega}} &= \frac{k^{4(n+1)}\|x_0-x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}{k^{2(n+1)}\|c_0-x^*\|(1-\alpha(1-k))^{n+1}}\\ &= \frac{k^{2(n+1)}\|x_0-x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}{\|c_0-x^*\|(1-\alpha(1-k))^{n+1}} \to 0 \text{ as } n\to\infty. \end{aligned} \tag{20}$$

Also,

$$\begin{aligned}\frac{\theta}{\pi} &= \frac{k^{4(n+1)}\|x_0-x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}{k^{3(n+1)}\|p_0-x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}\\ &= \frac{k^{(n+1)}\|x_0-x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}{\|p_0-x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}} \to 0 \text{ as } n\to\infty. \end{aligned} \tag{21}$$

Lastly, we have Also,

$$\begin{aligned}\frac{\theta}{\Pi} &= \frac{k^{4(n+1)}\|x_0 - x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}{k^{3(n+1)}\|p_0 - x^*\|(1-(\beta+k\alpha\gamma)(1-k))^{n+1}} \\ &= \frac{k^{(n+1)}\|x_0 - x^*\|(1-\alpha\beta\gamma(1-k))^{n+1}}{\|p_0 - x^*\|(1-(\beta+k\alpha\gamma)(1-k))^{n+1}} \to 0 \text{ as } n \to \infty. \qquad (22)\end{aligned}$$

□

Remark 4.4. In the light of the above results, we claim that the iterative schemes (1.3), (1.4) and (1.5) have the same rate of convergence. In addition, the iterative process (1.10) and (1.11) have the same rate of convergence.

Remark 4.5. Clearly, this shows that the iterative scheme (3.1) converges faster than (1.3)-(1.11). We are going to complement our analytical method with numerical examples. See Section 6.

Theorem 4.6. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let T be a contraction mapping with constant k such that $F(T) \neq \emptyset$. Suppose that the iterative schemes 3.1 satisfies*

$$\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n = \infty \quad \textit{and} \quad \alpha_n\beta_n\gamma_n \leq \alpha\beta\gamma \in (0,1) \quad \textit{for each } n \in \mathbb{N},$$

for $\{x_n\}_{n=1}^{\infty}$ converges to x^*. Hence, the iterative scheme (3.1) *is T-stable.*

Proof. Suppose that $\{t_n\}$ is an arbitrary sequence. Let $\varepsilon_n = \|t_n - Tx_{n+1}\|$, where $x_{n+1} = Tw_n$. Now using Definition 2.6 and (3.1), observe that

$$\begin{aligned}\|t_n - x^*\| &\leq \|t_n - Tx_{n+1}\| + \|Tx_{n+1} - x^*\| \\ &= \varepsilon_n + \|Tx_{n+1} - Tx^*\| \\ &\leq \varepsilon_n + k\|x_{n+1} - x^*\| \\ &= \varepsilon_n + k\|Tw_n - Tx^*\| \\ &\leq \varepsilon_n + k^2\|w_n - x^*\| \\ &\leq \varepsilon_n + k^2[k^3(1-\alpha_n\beta_n\gamma_n(1-k))\|x_n - x^*\|] \\ &= \varepsilon_n + k^5(1-\alpha_n\beta_n\gamma_n(1-k))\|x_n - x^*\|. \qquad (23)\end{aligned}$$

Note that $\alpha_n\beta_n\gamma_n \leq \alpha\beta\gamma \in (0,1)$ for all $n \in \mathbb{N}$. If follows form Lemma 2.7 that

$$\lim_{n\to\infty} \|t_n - x^*\| = 0.$$

Conversely, suppose that $\lim_{n\to\infty} \|t_n - x^*\| = 0$. We have that

$$\begin{aligned}
\varepsilon_n &= \|t_n - Tx_{n+1}\| \\
&\leq \|t_n - x^*\| + \|x^* - Tx_{n+1}\| \\
&= \|t_n - x^*\| + \|Tx_{n+1} - Tx^*\| \\
&\leq \|t_n - x^*\| + k\|x_{n+1} - x^*\| \\
&\leq \|t_n - x^*\| + k^5(1 - \alpha_n\beta_n\gamma_n(1-k))\|x_n - x^*\|. \qquad (24)
\end{aligned}$$

Taking limit of both sides, we have that

$$\lim_{n\to\infty} \varepsilon_n = 0.$$

□

Theorem 4.7. *Let $\tilde{T}$ be an approximate operator of a mapping T which is a contraction. Let $\{x_n\}$ be an iterative sequence generated by* (3.1) *for T and define an iterative scheme $\overline{\{x_n\}}$ as follows:*

$$\begin{aligned}
\bar{x}_0 &\in C \\
\bar{u}_n &= (1-\alpha_n)\bar{x}_n + \alpha_n\overline{T}\bar{x}_n \\
\bar{v}_n &= \overline{T}\bar{u}_n \\
\bar{y}_n &= \overline{T}[(1-\beta_n)\bar{v}_n + \beta_n\overline{T}\bar{v}_n] \qquad (25)\\
\overline{w}_n &= \overline{T}[(1-\gamma_n)\bar{y}_n + \gamma_n\overline{T}\bar{y}_n] \\
\bar{x}_{n+1} &= \overline{T}\overline{w}_n
\end{aligned}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ and $\frac{1}{2} \leq \gamma_n\alpha_n\beta_n$ for all $n \in \mathbb{N}$ such that $\sum_{n=0}^{\infty} \alpha_n\gamma_n\beta_n = \infty$. If $Tx^* = x^*$ and $\tilde{T}x^* = x^*$ such that $\lim_{n\to\infty} x_n = x^*$, then we have

$$\|x^* - \overline{x^*}\| \leq \frac{14\varepsilon\,'}{1-k}$$

where $\varepsilon > 0$ is a fixed number.

Proof. Using Algorithm 3.1, Algorithm 25 and the definition of contraction, we have

$$\begin{aligned}
\|u_n - \overline{u}_n\| &= \|(1-\alpha_n)x_n + \alpha_n T x_n - [(1-\alpha_n)\overline{x}_n + \alpha_n \overline{T}\overline{x}_n]\| \\
&\leq (1-\alpha_n)\|x_n - \overline{x}_n\| + \alpha_n \|T x_n - \overline{T}\overline{x}_n\| \\
&\leq (1-\alpha_n)\|x_n - \overline{x}_n\| + \alpha_n \|T x_n - T\overline{x}_n\| + \alpha_n \|T\overline{x}_n - \overline{T}\overline{x}_n\| \\
&\leq (1-\alpha_n(1-k))\|x_n - \overline{x}_n\| + \alpha_n \varepsilon \\
&\leq (1-\alpha_n(1-k))\|x_n - \overline{x}_n\| + \varepsilon. \qquad (26)
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\|v_n - \overline{v}_n\| &= \|T u_n - \overline{T}\overline{u}_n\| \\
&\leq \|T u_n - T\overline{u}_n\| + \|T\overline{u}_n - \overline{T}\overline{u}_n\| \\
&\leq k\|u_n - \overline{u}_n\| + \varepsilon_n \\
&\leq (1-\alpha_n(1-k))\|x_n - \overline{x}_n\| + \varepsilon + \varepsilon_n \\
&= (1-\alpha_n(1-k))\|x_n - \overline{x}_n\| + 2\varepsilon \qquad (27)
\end{aligned}$$

More so, we have

$$\begin{aligned}
\|y_n - \overline{y}_n\| &= \|T[(1-\beta_n)v_n + \beta_n T v_n] - \overline{T}[(1-\beta_n)\overline{v}_n + \beta_n \overline{T}\overline{v}_n] \\
&\leq \|T[(1-\beta_n)v_n + \beta_n T v_n] - T[(1-\beta_n)\overline{v}_n + \beta_n \overline{T}\overline{v}_n]\| \\
&+ \|T[(1-\beta_n)\overline{v}_n + \beta_n \overline{T}\overline{v}_n] - \overline{T}[(1-\beta_n)\overline{v}_n + \beta_n \overline{T}\overline{v}_n] \\
&\leq (1-\beta_n(1-k))\|v_n - \overline{v}_n\| + \beta_n \varepsilon_n + \varepsilon_n \\
&\leq (1-\alpha_n\beta_n(1-k))\|x_n - \overline{x}_n\| + 4\varepsilon \qquad (28)
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\|w_n - \overline{w}_n\| &= \|T[(1-\gamma_n)y_n + \gamma_n T y_n] - \overline{T}[(1-\gamma_n)\overline{y}_n + \gamma_n \overline{T}\overline{y}_n]\| \\
&\leq \|T[(1-\gamma_n)y_n + \gamma_n T y_n] - T[(1-\gamma_n)\overline{y}_n + \gamma_n \overline{T}\overline{y}_n]\| \\
&+ \|T[(1-\gamma_n)\overline{y}_n + \gamma_n \overline{T}\overline{y}_n] - \overline{T}[(1-\gamma_n)\overline{y}_n + \gamma_n \overline{T}\overline{y}_n]\|
\end{aligned}$$

$$\leq (1-\gamma_n(1-k))\|y_n - \overline{y}_n\| + \gamma_n\varepsilon + \varepsilon$$

$$\leq (1-\alpha_n\beta_n\gamma_n(1-k))\|x_n - \overline{x}_n\| + 6\varepsilon. \tag{29}$$

Lastly, we have that

$$\begin{aligned}
\|x_{n+1} - \overline{x}_{n+1}\| &= \|Tw_n - \overline{T}\overline{w}_n\| \\
&\leq \|Tw_n - T\overline{w}_n\| + \|T\overline{w}_n - \overline{T}\overline{w}_n\| \\
&\leq k\|w_n - \overline{w}_n\| + \varepsilon \\
&\leq \|w_n - \overline{w}_n\| + \varepsilon \\
&\leq (1-\alpha_n\beta_n\gamma_n(1-k))\|x_n - \overline{x}_n\| + 7\varepsilon. \qquad (30)
\end{aligned}$$

Using our assumption that $\frac{1}{2} \leq \alpha_n\beta_n\gamma_n$, we have that

$$\begin{aligned}
&1 - \alpha_n\beta_n\gamma_n \leq \alpha_n\beta_n\gamma_n \\
\Rightarrow &1 = 1 - \alpha_n\beta_n\gamma_n + \alpha_n\beta_n\gamma_n \leq \alpha_n\beta_n\gamma_n + \alpha_n\beta_n\gamma_n = 2\alpha_n\beta_n\gamma_n.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|x_{n+1} - \overline{x}_{n+1}\| &\leq (1-\alpha_n\beta_n\gamma_n(1-k))\|x_n - \overline{x}_n\| + 14\alpha_n\gamma_n\beta_n\varepsilon \\
&= (1-\alpha_n\beta_n\gamma_n(1-k))\|x_n - \overline{x}_n\| + \alpha_n\gamma_n\beta_n(1-k)\frac{14\varepsilon}{(1-k)} \qquad (31)
\end{aligned}$$

Let

$$\Gamma_n = \|x_n - x^*\|, \quad \Phi_n = (1-k)\alpha_n\beta_n\gamma_n,$$

and

$$\Psi_n = \frac{14\varepsilon}{1-k}.$$

From Theorem 4.1, we have that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Also, using Lemma 2.8, we have that

$$0 \leq \limsup_{n\to\infty}\|x_n - x^*\| \leq \limsup_{n\to\infty}\frac{14\varepsilon}{1-k}. \tag{26}$$

Using our hypothesis that $\lim_{n\to\infty} x_n = x^*$, (26) and from Theorem 4.1, we conclude that

$$\|x^* - \overline{x^*}\| \leq \frac{14\varepsilon}{1-k}.$$

□

5. Convergence

In this section, we establish some convergence results for Suzuki generalized nonexpansive mapping. We recall from [3] that, a mapping $T : C \to C$ is said to be generalized (α,β)-nonexpansive type I mapping if for all $x, y \in C$; we have

$$\lambda\|x-Tx\| \le \|x-y\| \implies \|Tx-Ty\| \le \alpha\|y-Tx\| + \beta\|x-Ty\| + (1-(\alpha+\beta))\|x-y\|.$$

Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ be ageneralized (α,β)-nonexpansive type I mapping. Let $\{x_n\}$ be the sequence defined in Algorithm 3.1. We first state and prove the following lemmas which will be needed in the proof of our main theorems. Now, observe that for all $x^* \in F(T)$; we have

$$\lambda\|x^*-Tx^*\| = \frac{1}{2}\|x^*-x^*\| \le \|x^*-z_n\|;$$
$$\lambda\|x^*-Tx^*\| = \frac{1}{2}\|x^*-x^*\| \le \|x^*-x_n\| \quad \text{and}$$
$$\lambda\|x^*-Tx^*\| = \frac{1}{2}\|x^*-x^*\| \le \|x^*-y_n\|;$$

which implies that

$$\|Tx^*-Tz_n\| \le \alpha\|z_n-Tx^*\| + \beta\|x^*-Tz_n\| + (1-(\alpha+\beta))\|x^*-z_n\|$$
$$\|Tx^*-Tx_n\| \le \alpha\|x_n-Tx^*\| + \beta\|x^*-Tx_n\| + (1-(\alpha+\beta))\|x^*-x_n\| \quad \text{and}$$
$$\|Tx^*-Ty_n\| \le \alpha\|y_n-Tx^*\| + \beta\|x^*-Ty_n\| + (1-(\alpha+\beta))\|x^*-y_n\|.$$

Lemma 5.1. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T : C \to C$ be a (α,β)-nonexpansive type I mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by* (3.1)*, where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$. Then the following hold:*

(1) $\{x_n\}$ is bounded.
(2) $\lim_{n\to\infty}\|x_n-x^*\|$ exists for all $x^* \in F(T)$.

Proof. Using (3.1) and and Lemma 2.12, we have

$$\|u_n-x^*\| \le (1-\gamma_n)\|x_n-x^*\| + \gamma_n\|Tx_n-x^*\|$$
$$\le (1-\gamma_n)\|x_n-x^*\| + \gamma_n\|x_n-x^*\| \tag{32}$$

$$= \|x_n - x^*\|.$$

Using (32) and Lemma 2.12, we obtain

$$\begin{aligned}
\|v_n - x^*\| &= \|Tu_n - x^*\| \\
&\leq \|u_n - x^*\| \\
&\leq \|x_n - x^*\|. \qquad (33)
\end{aligned}$$

In addition, using (3.1) and (33), we have

$$\begin{aligned}
\|y_n - x^*\| &= \|T[(1-\beta_n)v_n + \beta_n T v_n] - x^*\| \\
&\leq= \|(1-\beta_n)v_n + \beta_n T v_n - x^*\| \\
&\leq (1-\beta_n)\|v_n - x^*\| + \beta_n \|T v_n - x^*\| \\
&\leq (1-\beta_n)\|v_n - x^*\| + \beta_n \|v_n - x^*\| \\
&= \|v_n - x^*\| \\
&\leq \|u_n - x^*\| \\
&\leq \|x_n - x^*\|. \qquad (34)
\end{aligned}$$

Also, using (34), we obtain

$$\begin{aligned}
\|w_n - x^*\| &= \|T[(1-\beta_n)y_n + \beta_n T y_n] - x^*\| \\
&\leq= \|(1-\beta_n)y_n + \beta_n T y_n - x^*\| \\
&\leq (1-\beta_n)\|y_n - x^*\| + \beta_n \|T y_n - x^*\| \\
&\leq (1-\beta_n)\|y_n - x^*\| + \beta_n \|y_n - x^*\| \\
&= \|y_n - x^*\| \\
&\leq \|v_n - x^*\| \\
&\leq \|u_n - x^*\| \\
&\leq \|x_n - x^*\|. \qquad (35)
\end{aligned}$$

Lastly, using (35), we have

$$\begin{aligned} \|x_{n+1}-x^*\| &= \|Tw_n - x^*\| \\ &\leq \|w_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{36}$$

This shows that $\{\|x_n - x^*\|\}$ is bounded and non-increasing for all $x^* \in F(T)$. Thus, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. □

Theorem 5.2. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T : C \to C$ be a (α, β)-nonexpansive type I mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by* (3.1)*, where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$, then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. Since $F(T) \neq \emptyset$, then we can find $x^* \in F(T)$. We have established in Lemma 5.1 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Suppose that $\lim_{n\to\infty} \|x_n - x^*\| = c$. If we take $c = 0$, then we are done. Thus, we consider the case where $c > 0$. From (32), we have $\|u_n - x^*\| \leq \|x_n - x^*\|$; it then follows that

$$\limsup_{n\to\infty} \|u_n - x^*\| \leq c.$$

Also, using Lemma 2.12, we have $\|Tx_n - x^*\| \leq \|x_n - x^*\|$; it then follows that

$$\limsup_{n\to\infty} \|Tx_n - x^*\| \leq c.$$

Using (36) and (35), we have

$$\begin{aligned} \|x_{n+1}-x^*\| &\leq \|w_n - x^*\| \\ &\leq \|u_n - x^*\|. \end{aligned}$$

Taking the $\liminf_{n\to\infty}$ of both sides and rearranging the inequalities, we have

$$c \leq \liminf_{n\to\infty} \|u_n - x^*\|.$$

Clearly, we obtain that $\lim_{n\to\infty} \|u_n - x^*\| = c$. That is,

$$\lim_{n\to\infty} \|(1-\alpha_n)x_n + \alpha_n T x_n - x^*\| = c.$$

Thus, by Lemma 2.9, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

□

Theorem 5.3. *Let X be a uniformly convex Banach space which satisfies the Opial's condition and C a nonempty closed convex subset of X. Let $T : C \to C$ be a (α,β)-nonexpansive type I mapping with $F(T) \neq \emptyset$ and $\{x_n\}$ be a sequence defined by iteration* (3.1)*. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. In Lemma 5.1, we established that $\lim_{n\to\infty} \|x_n - x^*\|$ exists and that $\{x_n\}$ is bounded. Now, since X is uniformly convex, we can find a subsequence say $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly in C. We now establish that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Let u and v be weak limits of the subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Theorem 5.2, we have that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero by Theorem 2.14, we therefore have that $Tu = u$. Using similar approach, we can show that $v = Tv$. In what follows, we establish uniqueness. From Lemma 5.1, we have that $\lim_{n\to\infty} \|x_n - v\|$ exists. Now, suppose that $u \neq v$; then by Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - u\| &= \lim_{k\to\infty} \|x_{n_k} - u\| < \lim_{k\to\infty} \|x_{n_k} - v\| \\
&= \lim_{n\to\infty} \|x_n - v\| = \lim_{j\to\infty} \|x_{n_j} - v\| \\
&< \lim_{j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|.
\end{aligned}$$

This is a contradiction, so $u = v$. Hence, $\{x_n\}$ converges weakly to a fixed point of $F(T)$ and this completes the proof. □

Theorem 5.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a (α,β)-nonexpansive type I mapping on C; $\{x_n\}$ defined by* (3.1) *and $F(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ where $d(x, F(T)) = \inf \|x - x^*\| : x^* \in F(T)$.*

Proof. Suppose that $\{x_n\}$ converges to a fixed point, say x^* of T. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$, and since $0 \leq d(x_n, F(T)) \leq d(x_n, x^*)$; it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore,

$\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Lemma 5.1, we have that $\lim_{n\to\infty} \|x_n - x^*\|$ exists and that $\lim_{n\to\infty} d(x_n, F(T))$ exists for all $x^* \in F(T)$. By our hypothesis, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$; so for any given $\varepsilon > 0$; there exists $n_0 \in \mathbb{N}$; such that for all $n \geq n_0$; we have $d(x_n, F(T)) \leq \varepsilon$. We now show that $\{x_n\}$ is a Cauchy sequence in C. Since, $\lim_{n\to\infty} d(x_n, F(T)) = 0$; for any given $\varepsilon > 0$; there exist $n_0 \in \mathbb{N}$ such that for $n, m \geq n_0$; we have $d(x_m, F(T)) \leq \frac{\varepsilon}{2}$; and $d(x_n, F(T)) \leq \frac{\varepsilon}{2}$. Therefore, we have

$$\begin{aligned}
\|x_m - x_n\| &\leq \|x_m - x^*\| + \|x_n - x^*\| \\
&\leq d(x_m, F(T)) + d(x_n, F(T)) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in C. Since C is closed, then there exists a point $x_1 \in C$ such that $\lim_{n\to\infty} x_n = x_1$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$; it follows that $\lim_{n\to\infty} d(x_1, F(T)) = 0$. Since, $F(T)$ is closed, $x_1 \in F(T)$. □

Theorem 5.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a (α, β)-nonexpansive type I mapping, $\{x_n\}$ defined by (13) and $F(T) \neq \emptyset$. Let T satisfy condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Lemma 5.1, we have $\lim_{n\to\infty} \|x_n - F(T)\|$ exists and by Theorem 5.2, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Using the fact that

$$0 \leq \lim_{n\to\infty} f(d(x, F(T))) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0 \quad \forall x \in C;$$

we have that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$; it then follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Hence, by Theorem 5.4 $\{x_n\}$ converges strongly to $x^* \in F(T)$. □

6. Numerical Example and Application

6.1. Numerical Example.

Example 6.1. Let $X = \mathbb{R}$ and $C = [0, 50]$. Let $T : C \to C$ be a mapping defined by

$$T(x) = \frac{x}{2} + 1.$$

It is clear that the fixed point of T is 2. Choose $\alpha_n = \beta_n = \gamma_n = \frac{2}{3}$, with an initial value of $x_1 = 10$. The comparison of the proposed iterative scheme with existing methods is summarized in Table 1.

TABLE 1. Comparison of Iteration Values for Different Schemes

Alg 1.3	**Alg 1.4**	**Alg 1.5**	**Alg 1.10**	**Alg 1.11**	**Alg 3.1**
10.0000	10.000	10.0000	10.0000	10.0000	10.0000
3.3333	3.3333	3.3333	2.5185	2.5556	2.1481
2.2222	2.2222	2.2222	2.0336	2.0386	2.0027
2.0370	2.0370	2.0370	2.0022	2.0027	2.0001
2.0061	2.0061	2.0061	2.0001	2.0002	2.0000
2.0010	2.0010	2.0010	2.0000	2.0000	2.0000
2.0001	2.0001	2.0001	2.0000	2.0000	2.0000
2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
2.0000	2.0000	2.0000	2.0000	2.0000	2.0000

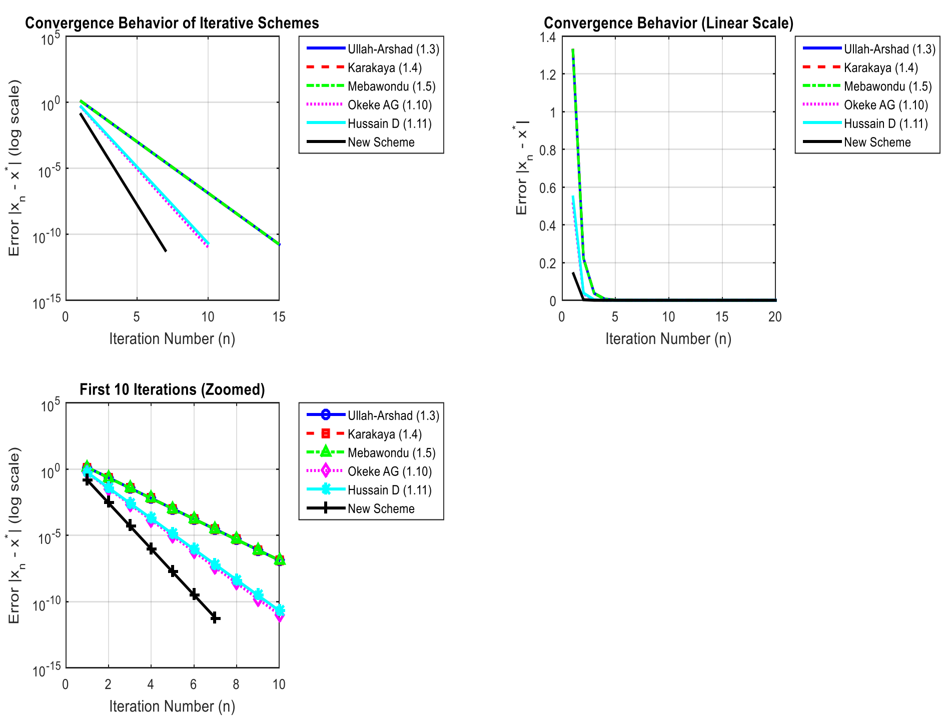

FIGURE 1. The comparison graphs of the rate of convergence

Example 6.2. It is well known that finding the roots of a cubic equation

$$x^3 + x^2 - 1 = 0$$

is equivalent to finding the fixed point of the function

$$T(x) = \sqrt{1 - x^3},$$

since the equation can be rewritten as

$$\sqrt{1 - x^3} = x.$$

The fixed point of T is

$$x^* = 0.7548777.$$

For the numerical experiment, we choose the control parameters

$$\alpha_n = \beta_n = \gamma_n = 0.5,$$

and take the initial guess $x_0 = 0.8$. The comparison of the proposed iterative scheme with existing methods is summarized in Table 2.

Alg (1.3)	Alg (1.4)	Alg (1.5)	Alg (1.10)	Alg (1.11)	Alg (3.1)
0.800000	0.800000	0.800000	0.800000	0.800000	0.800000
0.7476963	0.7545020	0.7468466	0.7560966	0.7138734	0.7548330
0.7554101	0.7549097	0.7554375	0.7549318	0.7828061	0.7548777
0.7548321	0.7548749	0.7548296	0.7548801	0.7539425	0.7548777
0.7548815	0.7548779	0.7548817	0.7548777	0.7548777	0.7548777
0.7548777	0.7548777	0.7548777	0.7548777	0.7548777	0.7548777
0.7548777	0.7548777	0.7548777	0.7548777	0.7548777	0.7548777
0.7548777	0.7548777	0.7548777	0.7548777	0.7548777	0.7548777
0.7548777	0.7548777	0.7548777	0.7548777	0.7548777	0.7548777

TABLE 2. Comparison of Iteration Values for Different Schemes

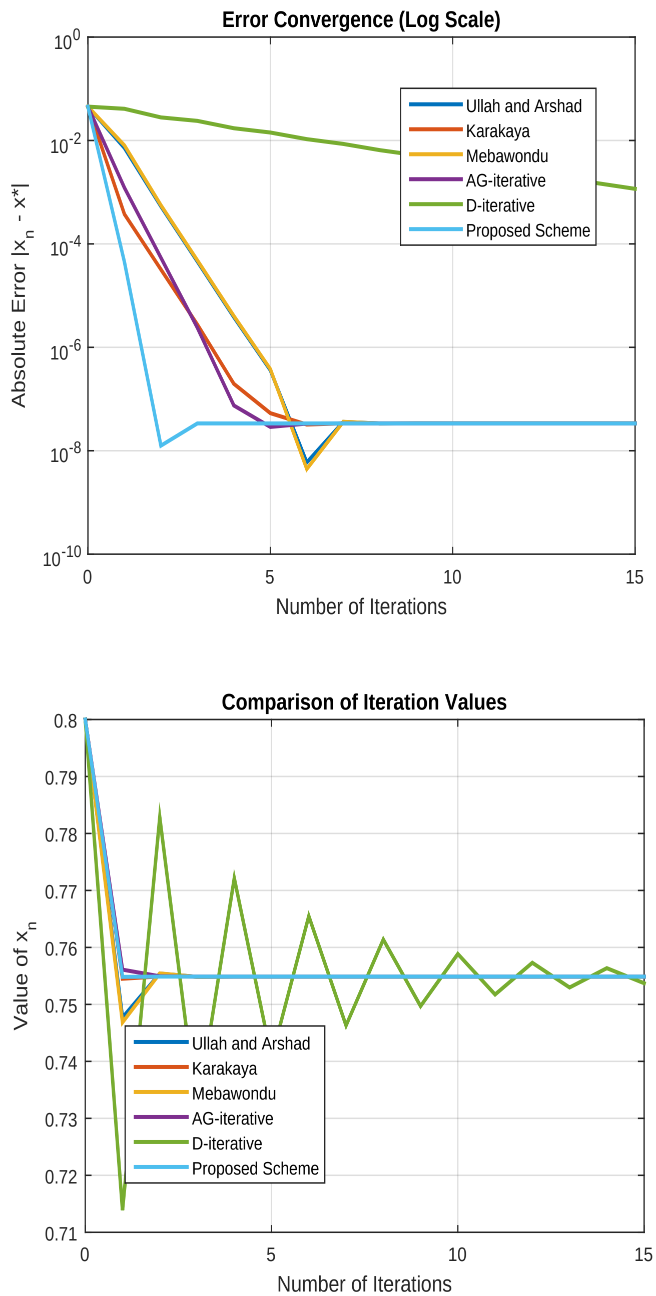

FIGURE 2. The comparison graphs of the rate of convergence

The above numerical result justify our claim that the iterative schemes (1.3), (1.4) and (1.5) have the same rate of convergence. In addition, the iterative process (1.10) and (1.11) have the same rate of convergence.

6.2. Application.

Oxygen Diffusion Problem. Sarivastava and Rai in [22] introduced a model for oxygen diffusion. They proposed that,the general equation for conveying oxygen from the capillary to the surrounding tissue is

$$\frac{\partial^\alpha C}{\partial t^\alpha} - \mu \frac{\partial^\beta C}{\partial t^\beta} = \nabla(d \cdot \nabla C) - k, \quad 0 < \beta < \alpha \leq 1 \tag{9}$$

where $C(r,z,t)$ is the concentration of oxygen, $k(r,z,t)$ is the rate of consumption per volume of tissue and d is the diffusion coefficient of oxygen, and this d may be a function of C. No flux on the boundary yields a boundary condition:

$$\left.\frac{\partial C}{\partial r}\right|_{r=R} = 0. \tag{10}$$

Consider at $t = 0$, $C(r,z,0) = g(r,z)$, and according to the condition $z \to \infty$, $C \to 0$.

Equation (9) above is equivalent to the integral equation

$$C(r,z,t) = g(r,z)\left(1 - \mu \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right) + \mu D_t^{-\alpha}\left[D_t^{-(\alpha-\beta)}(\nabla(d \cdot \nabla C))\right] - D_t^{-\alpha} k.$$

So as to apply our iterative algorithm, we define

$$C(r,z,t) = \eta(C_0) + \frac{1}{\Gamma(\alpha)} \int_0^t H(s, C(s), k) ds, \tag{37}$$

whaere

$$\eta(C_0) = g(r,z)\left(1 - \mu \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right) - D_t^{-\alpha} k \tag{38}$$

and

$$H(s, C(s), k) = \mu D_t^{-\alpha}\left[D_t^{-(\alpha-\beta)}(\nabla(d \cdot \nabla C))\right]. \tag{39}$$

Thus, we have that

$$TC(r,z,t) = \eta(C_0) + \frac{1}{\Gamma(\alpha)} \int_0^t H(s, C(s), k) ds, \tag{40}$$

Theorem 6.3. *Let the space $X = ([0,T], \mathbb{R})$, with a supremum norm be defined as*

$$\|C\| = \sup_{t \in [0,T]} \{|C(t)| : C \in X\}$$

and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0,1)$ of the iterative process (3.1) *such that*

(1)

$$\left[1-\beta_n\alpha_n\gamma_n\left(1-\frac{B_H T}{\Gamma(\alpha)}\right)\right] < 1$$

(2) There exists a constant $B_H > 0$ such that $|H(t,C_1(t),k)-H(t,C_2(t),k)| \leq B_H|C_1-C_2|$, for each $C \in X$ and $t \in [0,T]$.

(3) $\frac{B_H T}{\Gamma(\alpha)} < 1.$

Then, the diffusion model (9) *has a solution x^* and the iterative process* (3.1) *converges to the solution x^**

Proof. Note that the term $|x_n(s)-x^*(s)|$ is always less than or equal to the supremum norm of the difference, which is defined as $\|x_n-x^*\| = \sup_{s\in[0,T]}|x_n(s)-x^*(s)|$.

$$\int_0^t |x_n(s)-x^*(s)|ds \leq \int_0^t \|x_n-x^*\|ds$$

Using Algorithm (3.1), we have

$$\|u_n-x^*\| = \|(1-\alpha_n)x_n+\alpha_n Tx_n-x^*\|$$

$$\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|Tx_n-Tx^*\|$$

$$\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\sup_{t\in[0,T]}|Tx_n(t)-Tx^*(t)|$$

$$\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\sup_{t\in[0,T]}\left|\eta(C_0)+\frac{1}{\Gamma(\alpha)}\int_0^t H(s,x_n(s),k)ds-\left[\eta(C_0)+\frac{1}{\Gamma(\alpha)}\int_0^t H(s,x^*(s),k)ds\right]\right|$$

$$\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\sup_{t\in[0,T]}\left|\frac{1}{\Gamma(\alpha)}\int_0^t H(s,x_n(s),k)ds-\frac{1}{\Gamma(\alpha)}\int_0^t H(s,x^*(s),k)ds\right|$$

$$\leq (1-\alpha_n)\|x_n-x^*\|+\frac{\alpha_n}{\Gamma(\alpha)}\sup_{t\in[0,T]}\left|\int_0^t H(s,x_n(s),k)ds-\int_0^t H(s,x^*(s),k)ds\right|$$

$$\leq (1-\alpha_n)\|x_n-x^*\|+\frac{\alpha_n}{\Gamma(\alpha)}\sup_{t\in[0,T]}\int_0^t |H(s,x_n(s),k)-H(s,x^*(s),k)|ds$$

$$\leq (1-\alpha_n)\|x_n-x^*\|+\frac{\alpha_n}{\Gamma(\alpha)}\sup_{t\in[0,T]}\int_0^t B_H|x_n(s)-x^*(s)|ds$$

$$\leq (1-\alpha_n)\|x_n-x^*\|+\frac{\alpha_n B_H T}{\Gamma(\alpha)}\|x_n-x^*\|$$

$$\leq \left[1-\alpha_n\left(1-\frac{B_H T}{\Gamma(\alpha)}\right)\right]\|x_n-x^*\|. \tag{41}$$

Also, using (3.1) and (41), we obtain

$$
\begin{aligned}
\|v_n - x^*\| &= \|Tu_n - x^*\| \\
&\leq \sup_{t\in[0,T]} |Tu_n(t) - Tx^*(t)| \\
&\leq \sup_{t\in[0,T]} \left| \eta(C_0) + \frac{1}{\Gamma(\alpha)} \int_0^t H(s,u_n(s),k)ds - \left[\eta(C_0) + \frac{1}{\Gamma(\alpha)} \int_0^t H(s,x^*(s),k)ds \right] \right| \\
&\leq \sup_{t\in[0,T]} \left| \frac{1}{\Gamma(\alpha)} \int_0^t H(s,u_n(s),k)ds - \frac{1}{\Gamma(\alpha)} \int_0^t H(s,x^*(s),k)ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \sup_{t\in[0,T]} \left| \int_0^t H(s,u_n(s),K)ds - \int_0^t H(s,x^*(s),k)ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \sup_{t\in[0,T]} \int_0^t |H(s,u_n(s),k) - H(s,x^*(s),k)| ds \\
&\leq \frac{B_H T}{\Gamma(\alpha)} \|u_n - x^*\| \\
&\leq \frac{B_H T}{\Gamma(\alpha)} \left[1 - \alpha_n \left(1 - \frac{B_H T}{\Gamma(\alpha)} \right) \right] \|x_n - x^*\| \\
&\leq \left[1 - \alpha_n \left(1 - \frac{B_H T}{\Gamma(\alpha)} \right) \right] \|x_n - x^*\|. \qquad (42)
\end{aligned}
$$

Let $\zeta_n = (1-\beta_n)v_n + \beta_n T v_n$, (3.1) and (42), we obtain

$$
\begin{aligned}
&\|\zeta_n - x^*\| = \|(1-\beta_n)v_n + \beta_n T v_n - x^*\| \\
&\leq (1-\beta_n)\|v_n - x^*\| + \beta_n \|Tv_n - Tx^*\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n \max_{t\in[0,T]} |Tv_n(t) - Tx^*(t)| \\
&\leq (1-\beta_n)\|v_n - x^*\| + \beta_n \max_{t\in[0,T]} \left| \eta(C_0) + \frac{1}{\Gamma(\alpha)} \int_0^t H(s,v_n(s),k)ds \right. \\
&\left. - \left[\eta(C_0) + \frac{1}{\Gamma(\alpha)} \int_0^t H(s,x^*(s),k)ds \right] \right| \\
&\leq (1-\beta_n)\|v_n - x^*\| + \beta_n \max_{t\in[0,T]} \left| \frac{1}{\Gamma(\alpha)} \int_0^t H(s,v_n(s),k)ds - \frac{1}{\Gamma(\alpha)} \int_0^t H(s,x^*(s),k)ds \right| \\
&\leq (1-\beta_n)\|v_n - x^*\| + \frac{\beta_n}{\Gamma(\alpha)} \max_{t\in[0,T]} \left| \int_0^t H(s,v_n(s),K)ds - \int_0^t H(s,x^*(s),k)ds \right| \\
&\leq (1-\beta_n)\|v_n - x^*\| + \frac{\beta_n}{\Gamma(\alpha)} \max_{t\in[0,T]} \int_0^t |H(s,v_n(s),K) - H(s,x^*(s),K)| ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|v_n - x^*\| + \frac{\beta_n B_H T}{\Gamma(\phi)}\|v_n - x^*\| \\
&\leq \left[1 - \beta_n\left(1 - \frac{B_H T}{\Gamma(\alpha)}\right)\right]\|v_n - x^*\| \\
&\leq \left[1 - \beta_n\left(1 - \frac{B_H T}{\Gamma(\alpha)}\right)\right]\left[1 - \alpha_n\left(1 - \frac{B_H T}{\Gamma(\alpha)}\right)\right]\|x_n - x^*\| \\
&\leq \left[1 - \beta_n \alpha_n\left(1 - \frac{B_H T}{\Gamma(\alpha)}\right)\right]\|x_n - x^*\|. \qquad (43)
\end{aligned}
$$

Using (43), we obtain

$$
\begin{aligned}
\|y_n - x^*\| &= \|T\zeta_n - x^*\| \\
&\leq \sup_{t\in[0,T]} |T\zeta_n(t) - Tx^*(t)| \\
&\leq \sup_{t\in[0,T]} \left|\eta(C_0) + \frac{1}{\Gamma(\alpha)}\int_0^t H(s,\zeta_n(s),k)ds - \left[\eta(C_0) + \frac{1}{\Gamma(\alpha)}\int_0^t H(s,x^*(s),k)ds\right]\right| \\
&\leq \sup_{t\in[0,T]} \left|\frac{1}{\Gamma(\alpha)}\int_0^t H(s,\zeta_n(s),k)ds - \frac{1}{\Gamma(\alpha)}\int_0^t H(s,x^*(s),k)ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\sup_{t\in[0,T]} \left|\int_0^t H(s,\zeta_n(s),K)ds - \int_0^t H(s,x^*(s),k)ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\sup_{t\in[0,T]} \int_0^t |H(s,\zeta_n(s),k) - H(s,x^*(s),k)|ds \\
&\leq \frac{B_H T}{\Gamma(\alpha)}\|\zeta_n - x^*\| \\
&\leq \|\zeta_n - x^*\| \\
&\leq \left[1 - \beta_n \alpha_n\left(1 - \frac{B_H T}{\Gamma(\alpha)}\right)\right]\|x_n - x^*\|. \qquad (44)
\end{aligned}
$$

Similarly, let $\rho_n = (1-\gamma_n)y_n + \gamma_n T y_n$ and using similar approach as in (43), we hav

$$
\|\rho_n - x^*\| \leq \left[1 - \beta_n \alpha_n \gamma_n\left(1 - \frac{B_H T}{\Gamma(\alpha)}\right)\right]\|x_n - x^*\| \qquad (45)
$$

Furthermore, we have

$$
\begin{aligned}
\|w_n - x^*\| &= \|T\rho_n - x^*\| \\
&\leq \sup_{t\in[0,T]} |T\rho_n(t) - Tx^*(t)|
\end{aligned}
$$

$$\leq \sup_{t\in[0,T]}\left|\eta(C_0)+\frac{1}{\Gamma(\alpha)}\int_0^t H(s,\rho_n(s),k)ds-\left[\eta(C_0)+\frac{1}{\Gamma(\alpha)}\int_0^t H(s,x^*(s),k)ds\right]\right|$$

$$\leq \sup_{t\in[0,T]}\left|\frac{1}{\Gamma(\alpha)}\int_0^t H(s,\rho_n(s),k)ds-\frac{1}{\Gamma(\alpha)}\int_0^t H(s,x^*(s),k)ds\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\sup_{t\in[0,T]}\left|\int_0^t H(s,\rho_n(s),K)ds-\int_0^t H(s,x^*(s),k)ds\right|$$

$$\leq \frac{1}{\Gamma(\phi)}\sup_{t\in[0,T]}\int_0^t |H(s,\rho_n(s),K)-H(s,x^*(s),k)|ds$$

$$\leq \frac{B_HT}{\Gamma(\alpha)}\|\rho_n-x^*\|$$

$$\leq \left[1-\beta_n\alpha_n\gamma_n\left(1-\frac{B_HT}{\Gamma(\alpha)}\right)\right]\|x_n-x^*\|. \tag{46}$$

Lastly, using (46), we obtain

$$\|x_{n+1}-x^*\| = \|Tw_n-x^*\|$$

$$\leq \sup_{t\in[0,T]}|Tw_n(t)-Tx^*(t)|$$

$$\leq \sup_{t\in[0,T]}\left|\eta(C_0)+\frac{1}{\Gamma(\alpha)}\int_0^t H(s,w_n(s),k)ds-\left[\eta(C_0)+\frac{1}{\Gamma(\alpha)}\int_0^t H(s,x^*(s),k)ds\right]\right|$$

$$\leq \sup_{t\in[0,T]}\left|\frac{1}{\Gamma(\alpha)}\int_0^t H(s,w_n(s),k)ds-\frac{1}{\Gamma(\alpha)}\int_0^t H(s,x^*(s),k)ds\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\max_{t\in[0,T]}\left|\int_0^t H(s,w_n(s),k)ds-\int_0^t H(s,x^*(s),k)ds\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\max_{t\in[0,T]}\int_0^t |H(s,w_n(s),k)-H(s,x^*(s),k)|ds$$

$$\leq \frac{L_HT}{\Gamma(\alpha)}\|w_n-x^*\|$$

$$\leq \left[1-\beta_n\alpha_n\gamma_n\left(1-\frac{B_HT}{\Gamma(\alpha)}\right)\right]\|x_n-x^*\| \quad (7.14)$$

Hence, taking $\theta_n = \left[1-\beta_n\alpha_n\gamma_n\left(1-\frac{B_HT}{\Gamma(\alpha)}\right)\right] < 1$ and $p_n = \|x_n-x^*\|$, then the conditions of Lemma 2.16 are satisfied. Hence $\lim_{n\to\infty}\|x_n-x^*\| = 0$. Therefore the iterative process (3.1) converges to the solution x^* of the Oxygen diffusion model. □

Application to a Delay Differential Equation and Oxygen Diffusion Problem. The application to delay differential equation presented in our earlier work [2] provides a clear illustration of a similar problem, and we apply our new method to it here for comparison.

Let $C[a,b]$ denote the space of all continuous real valued functions on a closed interval $[a,b]$ with the norm

$$\|x-y\|_\infty = \max_{t\in[a,b]} |x(t)-y(t)|.$$

It is well-known that $(C[a,b], \|\cdot\|_\infty)$ is a Banach space. In what follows, we apply our result to the following delay differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\rho)), \quad t \in [t_0, t_1], \tag{47}$$

with initial condition

$$x(t) = \psi(t) \quad t \in [t_0-\rho, t_0]. \tag{48}$$

We suppose that the following conditions hold:

(1) $t_0, t_1 \in \mathbb{R}$ and $\rho > 0$;

(2) $f \in C([t_0,t_1]\times\mathbb{R}^2, \mathbb{R})$ such that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq k(|x_1-y_1| + |x_2-y_2|), \tag{49}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $t \in [t_0,t_1]$ and $2k(t_1-t_0) < 1$;

(3) $\psi \in C([t_0-\rho, t_1], \mathbb{R})$.

It is well-known that problems (47) and (48) can be formulated as follows;

$$x(t) = \begin{cases} \psi(t), & t \in [t_0-\rho, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\rho))ds, & t \in [t_0, t_1]. \end{cases} \tag{50}$$

Theorem 6.4. *Suppose that the assumptions (1) – (3) hold. Then the iterative process* (3.1) *converges strongly to the solution of problem* (47) – (48) *if* $\sum_{n=1}^{\infty} \gamma_n \alpha_n \beta_n = \infty$.

Proof. Let $\{x_n\}$ be an iterative sequence (3.1) for an operator T defined by

$$Tx(t) = \begin{cases} \psi(t), & t \in [t_0-\rho, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\rho))ds, & t \in [t_0, t_1]. \end{cases}$$

Let $\{x_n\}$ be an iterative sequence (3.1) for an operator T defined by

$$Tx(t) = \begin{cases} \psi(t), & t \in [t_0-\rho, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\rho))ds, & t \in [t_0, t_1]. \end{cases}$$

Let x^* be the fixed point of T. That is $Tx^* = x^*$. Using (3.1), we have

$$\begin{aligned}
&\|u_n - x^*\|_\infty = \|(1-\alpha_n)x_n + \alpha_n Tx_n - x^*\|_\infty \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n\|Tx_n - Tx^*\|_\infty \\
&= (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\rho,t_1]} |Tx_n(t) - Tx^*(t)| \\
&= (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0,t_1]} \left| \left(\psi(t_0) + \int_{t_0}^{t} f(s, x_n(s), x_n(s-\rho))ds\right) \right. \\
&\left. - \left(\psi(t_0) + \int_{t_0}^{t} f(s, x^*(s), x^*(s-\rho))ds\right) \right| \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0,t_1]} \int_{t_0}^{t} |f(s, x_n(s), x_n(s-\rho)) - f(s, x^*(s), x^*(s-\rho))|ds \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \int_{t_0}^{t_1} k(|x_n(s) - x^*(s)| + |x_n(s-\rho) - x^*(s-\rho)|)ds \\
&\leq (1-\alpha_n)\|x_n - x^*\|_\infty + \alpha_n \int_{t_0}^{t_1} k(\|x_n - x^*\|_\infty + \|x_n - x^*\|_\infty)ds \\
&= (1-\alpha_n)\|x_n - x^*\|_\infty + 2\alpha_n k\|x_n - x^*\|_\infty(t_1 - t_0) \\
&= (1-\alpha_n(1-2k(t_1-t_0)))\|x_n - x^*\|_\infty. \qquad (51)
\end{aligned}$$

Also, using (3.1) and (51), we obtain

$$\begin{aligned}
&\|v_n - x^*\|_\infty = \|Tu_n - Tx^*\|_\infty \\
&= \max_{t\in[t_0-\rho,t_1]} \left| \left(\psi(t_0) + \int_{t_0}^{t} f(s, u_n(s), u_n(s-\rho))ds\right) - \left(\psi(t_0) + \int_{t_0}^{t} f(s, x^*(s), x^*(s-\rho))ds\right) \right| \\
&\leq \max_{t\in[t_0-\rho,t_1]} \int_{t_0}^{t} |f(s, u_n(s), u_n(s-\rho)) - f(s, x^*(s), x^*(s-\rho))|ds \\
&\leq k\int_{t_0}^{t_1} (\|u_n - x^*\|_\infty + \|u_n - x^*\|_\infty)ds \\
&= 2k\|u_n - x^*\|_\infty(t_1 - t_0) \\
&\leq (1-\alpha_n(1-2k(t_1-t_0)))\|x_n - x^*\|_\infty. \qquad (52)
\end{aligned}$$

In addition, let $\zeta_n = (1-\beta_n)v_n + \beta_n T v_n$. Then using a similar argument as in $\|u_n - x^*\|_\infty$, we have

$$\begin{aligned}\|\zeta_n - x^*\|_\infty &\leq (1-\beta_n(1-2k(t_1-t_0)))\|v_n - x^*\|_\infty \\ &\leq \|v_n - x^*\|_\infty \\ &\leq (1-\alpha_n\beta_n(1-2k(t_1-t_0)))\|x_n - x^*\|_\infty. \quad (53)\end{aligned}$$

It then follows that

$$\begin{aligned}&\|y_n - x^*\|_\infty = \|T\zeta_n - Tx^*\|_\infty \\ &= \max_{t\in[t_0-\rho,t_1]}\left|\left(\psi(t_0) + \int_{t_0}^{t} f(s,\zeta_n(s),\zeta_n(s-\rho))ds\right) - \left(\psi(t_0) + \int_{t_0}^{t} f(s,x^*(s),x^*(s-\rho))ds\right)\right| \\ &\leq \max_{t\in[t_0-\rho,t_1]}\int_{t_0}^{t}|f(s,u_n(s),u_n(s-\rho)) - f(s,x^*(s),x^*(s-\rho))|ds \\ &\leq k\int_{t_0}^{t_1}(\|\zeta_n - x^*\|_\infty + \|\zeta_n - x^*\|_\infty)ds \\ &= 2k\|\zeta_n - x^*\|_\infty(t_1-t_0) \\ &\leq (1-\alpha_n\beta_n(1-2k(t_1-t_0)))\|x_n - x^*\|_\infty. \quad (54)\end{aligned}$$

Furthermore, let $\phi_n = (1-\gamma_n)y_n + \gamma_n T y_n$. Then using a similar argument as in $\|u_n - x^*\|_\infty$, we have

$$\begin{aligned}\|\phi_n - x^*\|_\infty &\leq (1-\beta_n(1-2k(t_1-t_0)))\|y_n - x^*\|_\infty \\ &\leq \|y_n - x^*\|_\infty \\ &\leq (1-\alpha_n\beta_n\gamma_n(1-2k(t_1-t_0)))\|x_n - x^*\|_\infty. \quad (55)\end{aligned}$$

Using (55), we have

$$\begin{aligned}\|w_n - x^*\|_\infty &= \|T\phi_n - Tx^*\|_\infty \\ &\leq 2k(t_1-t_0)\|\phi_n - x^*\|_\infty \\ &\leq \|\phi_n - x^*\|_\infty \\ &\leq (1-\alpha_n\beta_n\gamma_n(1-2\lambda(t_1-t_0)))\|x_n - x^*\|_\infty. \quad (56)\end{aligned}$$

Finally, using similar approach and (56), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|_\infty &= \|Tw_n - Tx^*\|_\infty \\ &\leq 2\lambda(t_1-t_0)\|w_n - x^*\|_\infty \\ &\leq \|v_n - x^*\|_1 \\ &\leq (1-\alpha_n\beta_n\gamma_n(1-2k(t_1-t_0)))\|x_n - x^*\|_\infty.\end{aligned}$$

Having

$$\|x_{n+1}-x^*\|_\infty \leq (1-\alpha_n\beta_n\gamma_n(1-2k(t_1-t_0)))\|x_n - x^*\|_\infty,$$

we suppose that $\theta_n = \alpha_n\beta_n\gamma_n(1-2k(t_1-t_0)) < 1$, thus, $\theta_n \in [0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n\beta_n\gamma_n = \infty$ and $\zeta_n = \|x_n - x^*\|_1$. Hence, we have

$$\zeta_{n+1} \leq (1-\theta_n)\zeta_n.$$

It is easy to see that the conditions in Lemma 1.3 are satisfied. Hence, applying Lemma 2.16, we have that $\lim_{n\to\infty}\|x_n - x^*\|_\infty = 0$. □

Example 3.2. Consider the following first order delay differential equation

(57) $$\dot{x}(t) = \frac{1}{2}(x(t) - x(t-1)), \quad t \in [0,2],$$

with initial condition

(58) $$x(t) = \psi(t) = e^{2t}, \quad t \in [-1-\rho, 0].$$

It is easy to see that the conditions (1) – (3) above are satisfied. We have

(1) $t_0 = 0$, $t_1 = 2$ and $\rho = 1$;
(2) $f : [0,2] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and

$$f(t, x(t), x(t-\delta)) = \frac{1}{8}(x(t) - x(t-1)), \quad t \in [0,2]$$

and for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$, $t \in [0,2]$, we have

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| = \frac{1}{8}(|x_1 - y_1| + |x_2 - y_2|).$$

It is clear that $k = \frac{1}{8}$, thus, we have $2\lambda(t_1 - t_0) = 2 \times \frac{1}{8} \times 2 = \frac{1}{2} < 1$. The problems (3.14) and (3.15) can be reformulated as the following integral equation

$$x(t) = \begin{cases} e^{2t}, & t \in [t_0 - \rho, t_0] \\ \phi(t_0) + \frac{1}{8}\int_{t_0}^{t}(x(s) - x(s-1))ds, & t \in [0,2]. \end{cases} \tag{59}$$

Thus, the exact solution of the problems (3.14) and (3.15) is

$$x(t) = \begin{cases} e^{2t}, & t \in [t_0 - \rho, t_0] \\ 1 + \frac{1}{16}[e^{2t} - 1 - e^{2t-2} + e^{-2}], & t \in [0,2]. \end{cases} \tag{60}$$

Author Contributions

The authors acknowledge and agree with the content, accuracy and integrity of the manuscript and take absolute accountability for the same. All authors read and approved the final manuscript.

Availability of Data and Materials

Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

Conflict of Interests

The authors declare that there is no conflict of interests.

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