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EXISTENCE OF SOLUTIONS FOR A NEW CLASS OF HATTAF MIXED FRACTAL-FRACTIONAL EQUATIONS MODELING THE HIV/AIDS DISEASE

HAMZA EL MAMOUNI[1,*], ZAKARIA HAJHOUJI[1], KHALID HATTAF[1,2]

[1]Laboratory of Analysis, Modeling and Simulation (LAMS), Faculty of Sciences Ben M'Scik, Hassan II University of Casablanca, P.O Box 7955 Sidi Othman, Casablanca, Morocco

[2]Equipe de Recherche en Modélisation et Enseignement des Mathématiques (ERMEM), Centre Régional des Métiers de l'Education et de la Formation (CRMEF), 20340 Derb Ghalef, Casablanca, Morocco

Abstract. This paper presents a new definition of Hattaf mixed fractal-fractional derivative which covers many types of fractal and fractional derivatives with singular and non-singular kernels in order to model the complex dynamics in various fields. Furthermore, as an application for our operator, we study the existence of solutions for a model of HIV/AIDS disease by means of fixed point theory.

Keywords: Hattaf mixed fractal-fractional derivative; HIV/AIDS disease; fixed point theory; Krasnoselskii's fixed point theory; Banach contraction.

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1. INTRODUCTION

The fractal derivative represents a novel extension of the classical derivative. This concept has been introduced in multiple forms across the literature. For instance, Chen [1] proposed hypotheses about how a fractal space-time fabric affects physical behaviors, particularly anomalous diffusion, using the Hausdorff derivative as a key tool for modeling these scale-dependent

*Corresponding author
E-mail address: elmamouni89.hamza@gmail.com
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behaviors. In [2], the author treated a discontinuous media by the fractal dimensions. In addition, the Hattaf fractal derivative introduced in [3] was presented as a generalization of the Hausdorff derivative and of the general derivative proposed by Yang in [4].

Fractional calculus extends the classical notions of differentiation and integration to non-integer orders. Far from being a mere mathematical curiosity, this extension has found practical applications in diverse fields, including physics, biology, engineering and economics. In particular, fractional differential operators serve as powerful tools for modeling the behavior of systems with memory or hereditary characteristics. Not long ago, in an effort to model complex nonlocal phenomena, Atangana [5] combined fractal theory with fractional calculus, giving rise in 2017 to a new class of fractal-fractional differential equations (FFDEs). In 2020, the same author [6] applied this new class of FFDEs to study the spread of COVID-19. More recently, Hattaf [3] introduced a new class of differential and integral operators to extend and generalize the eight definitions of fractal-fractional derivatives with non-singular kernels and the five definitions of fractal-fractional integrals previously presented in [5, 6].

In the literature, there exist many types of fractional derivatives, such as the Riemann-Liouville fractional derivative [7, 8], the Caputo fractional derivative [9], the Caputo-Fabrizio (CF) fractional derivative [10], the Atangana-Baleanu (AB) fractional derivative [11], the weighted AB fractional derivative [12], the generalized Hattaf fractional (GHF) derivative [13], as well as the Hattaf mixed fractional (HMF) derivative [14], which combines fractional derivatives with singular and non-singular kernels.

On the other hand, recent epidemiological data indicate that HIV/AIDS remains a major global public health challenge. According to the World Health Organization (WHO) [15], approximately 40.8 million individuals were living with HIV at the end of 2024, with 1.3 million new infections and 630 000 AIDS-related deaths reported that year. It is important to note that HIV primarily infects $CD^{+}4$ T cells, key components of the immune system responsible for initiating and regulating protective responses to viral infections. The primary modes of HIV transmission involve direct contact with body fluids containing high concentrations of the virus. These include unprotected sexual intercourse, sharing contaminated needles or syringes, transfusion of infected blood, and mother-to-child transmission during pregnancy, childbirth, or breastfeeding.

The main objective of this study is to introduce a new class of Hattaf mixed fractal-fractional differential and integral operators and apply it to study the dynamics of HIV/AIDS. To this end, the rest of the paper is organized as follows. Section 2 presents some key findings essential for this work and the formulation of the Hattaf mixed fractal-fractional differential and integral operators. Section 3 focuses on the formulation of an HIV/AIDS fractal-fractional model and the study of its solutions using fixed point theory. Finally, Section 4 concludes the paper.

2. The New Hattaf Mixed Fractal-Fractional Differential and Integral Operators

In this section, we present the fundamental definitions and key mathematical results for the new Hattaf mixed fractal-fractional differential and integral operators. First, we recall the definition of Hattaf fractal derivative.

Definition 1. [3] *Let $\mathscr{I}$ be an open interval of $\mathbb{R}$. The Hattaf fractal derivative of a function $f(t)$ with respect to a fractal measure $g(\varepsilon,t)$ is given by*

$$\frac{d_g}{dt^{\varepsilon}}f(t)=\lim_{\tau\to t}\frac{f(t)-f(\tau)}{g(\varepsilon,t)-g(\varepsilon,\tau)},\ \varepsilon>0. \tag{1}$$

If $\frac{d_g}{dt^{\varepsilon}}f(t)$ exists for all $t\in\mathscr{I}$, then f is fractal differentiable on the interval $\mathscr{I}$ with order ε.

Notice that when $g(\varepsilon,t)=t^{\varepsilon}$, Definition 1 reduced to the Hausdorff fractal derivative [1]. *In addition, if $g(\varepsilon,t)=h(t)$ with $h'(t)>0$ and $f(t)$ is differentiable, then we obtain the general derivative proposed by Yang* [4] *and Definition 1 becomes*

$$\frac{d_g}{dt^{\varepsilon}}f(t)=\frac{1}{h'(t)}\frac{df(t)}{dt}. \tag{2}$$

Next, we introduce the definition of the new Hattaf mixed fractal-fractional derivative in the sense of Caputo and Riemann-Liouville.

Definition 2. *Let $(\alpha,\beta)\in[0,1]^2,\gamma,\eta>0$ and $f(t)$ be differentiable in the interval (a,b) and fractal differentiable on (a,b) with order $0<\varepsilon\leq 1$. The Hattaf mixed fractal-fractional derivative of the function $f(t)$ of order α in Caputo sense with respect to the weight function $w(t)$ is* *defined as follows:*

$$ {}^{FFC}D^{\alpha,\beta,\gamma,\eta,\varepsilon}_{a,t,w,\delta}f(t) = \frac{N(\alpha+\beta-1)}{2-\alpha-\beta}\frac{1}{w(t)}\int_a^t (t-\tau)^{\beta-1}E_{\gamma,\beta}[-\delta\mu_{\alpha,\beta}(t-\tau)^{\eta}]\frac{d_g}{d\tau^{\varepsilon}}(wf)(\tau)d\tau, \tag{3} $$

where $\delta \in \mathbb{R}^*$, $w \in C^1(a,b)$, *with* $w > 0$ *on* $[a,b]$; $N(.)$ *is a normalization function such that* $N(0) = N(1) = 1$, $\mu_{\alpha,\beta} = \frac{\alpha+\beta-1}{2-\alpha-\beta}$; *and* $E_{\gamma,\beta}(t) = \sum_{k=0}^{+\infty}\frac{t^k}{\Gamma(\gamma k+\beta)}$ *is the Mittag-Leffler function* [17] *with two parameters* γ *and* β.

Remark 3. *Definition 2 includes several existing fractal-fractional derivatives involving both singular and non-singular kernels. For example, when* $g(\varepsilon,t) = t^{\varepsilon}$, $w(t) = 1$ *and* $\beta = 1-\alpha$, *we get fractal-fractional derivative with singular kernel* [5] *given by*

$$ {}^{FFC}D^{\alpha,1-\alpha,\gamma,\eta,\varepsilon}_{a,t,1,\delta}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-\tau)^{-\alpha}\frac{d_g}{d\tau^{\varepsilon}}f(\tau)d\tau, $$

where $\frac{d_g}{dt^{\varepsilon}}f(t) = \lim_{\tau\to t}\frac{f(t)-f(\tau)}{t^{\varepsilon}-\tau^{\varepsilon}}$. *Furthermore, Definition 2 is the generalization of the fractal-fractional derivative with exponential decay kernel* [5]*, the fractal-fractional derivative with the generalized Mittag-Leffler kernel* [5]*, as well as the Hattaf fractal-fractional derivative* [3].

Definition 4. *Let* $(\alpha,\beta) \in [0,1]^2, \gamma, \eta > 0$ *and* $f(t)$ *be continuous in the interval* (a,b) *and fractal differentiable on* (a,b) *with order* $0 < \varepsilon \leq 1$. *The Hattaf mixed fractal-fractional derivative of the function* $f(t)$ *of order* α *in Riemann-Liouville sense with respect to the weight function* $w(t)$ *is defined as follows:*

$$ {}^{FFR}D^{\alpha,\beta,\gamma,\eta,\varepsilon}_{a,t,w,\delta}f(t) = \frac{N(\alpha+\beta-1)}{2-\alpha-\beta}\frac{1}{w(t)}\frac{d_g}{d\tau^{\varepsilon}}\int_a^t (t-\tau)^{\beta-1}E_{\gamma,\beta}[-\delta\mu_{\alpha,\beta}(t-\tau)^{\eta}]wf(\tau)d\tau. \tag{4} $$

Clearly, Definition 4 generalizes the fractal-fractional differential operators introduced by Atangana in [6], it suffices to take $\beta = \delta = 1$ and $g(\varepsilon,t) = \frac{t^{2-\varepsilon}}{2-\varepsilon}$.

Based on Theorem 1 of [14], we get easily the following result.

Theorem 5. *Let* $\frac{\partial g(\varepsilon,t)}{\partial t}$ *be exist and not zero. Then*

$$ \begin{aligned} {}^{FFR}D^{\alpha,\beta,\gamma,\eta,\varepsilon}_{a,t,w,\delta}f(t) = \left(\frac{\partial g(\varepsilon,t)}{\partial t}\right)^{-1}\Bigg[& {}^{C}D^{\alpha,\beta,\gamma,\eta}_{a,t,w,\delta}f(t) \\ & + \frac{N(\alpha+\beta-1)}{(2-\alpha-\beta)w(t)}(t-a)^{\beta-1}E_{\gamma,\beta}[-\delta\mu_{\alpha,\beta}(t-a)^{\eta}](wf)(a)\Bigg], \end{aligned} \tag{5} $$

where ${}^{C}D^{\alpha,\beta,\gamma,\eta}_{a,t,w,\delta} f(t)$ is the Hattaf mixed fractional derivative of the function $f(t)$ in Caputo sense defined in [14].

Next, we define the fractal-fractional integral associated with the new Hattaf mixed fractal-fractional derivative. So, we consider the following fractal-fractional differential equation:

$$ {}^{FFR}D^{\alpha,\beta,\gamma,\gamma,\varepsilon}_{0,t,w,\delta} h(t) = f(t), \tag{6}$$

which implies that

$$\left(\frac{\partial g(\varepsilon,t)}{\partial t}\right)^{-1} {}^{R}D^{\alpha,\beta,\gamma,\gamma}_{0,t,w,\delta} h(t) = f(t).$$

Hence,

$$ {}^{R}D^{\alpha,\beta,\gamma,\gamma}_{0,t,w,\delta} h(t) = \frac{\partial g(\varepsilon,t)}{\partial t} f(t). \tag{7}$$

According to Lemma 2 of [14], we deduce that (6) has a unique solution given by

$$ h(t) = \begin{cases} \dfrac{2-\alpha-\beta}{N(\alpha+\beta-1)}\left[{}^{RL}\mathscr{I}^{1-\beta,\varepsilon}_{0,w} f(t) + \delta\mu_{\alpha,\beta}\, {}^{RL}\mathscr{I}^{1+\gamma-\beta,\varepsilon}_{0,w} f(t)\right], & if\ \beta \neq 1; \\ \dfrac{1-\alpha}{N(\alpha)}\dfrac{\partial g(\varepsilon,t)}{\partial t} f(t) + \dfrac{\delta\alpha}{N(\alpha)} {}^{RL}\mathscr{I}^{\gamma,\varepsilon}_{0,w} f(t), & if\ \beta = 1, \end{cases} \tag{8}$$

where ${}^{RL}\mathscr{I}^{\alpha,\varepsilon}_{0,w}$ is the weighted Riemann-Liouville fractal-fractional integral of order α given by

$$ {}^{RL}\mathscr{I}^{\alpha,\varepsilon}_{0,w} f(t) = \frac{1}{\Gamma(\alpha)}\frac{1}{w(t)}\int_0^t (t-\tau)^{\alpha-1}\left(\frac{\partial g(\varepsilon,\tau)}{\partial \tau}\right) w(\tau) f(\tau)d\tau. \tag{9}$$

Definition 6. *If $\eta = \gamma$, then the fractal-fractional integral associated with the new Hattaf mixed fractal-fractional derivative is defined as follows:*

$$ {}^{FF}I^{\alpha,\beta,\gamma,\varepsilon}_{0,t,w,\delta} f(t) = \begin{cases} \dfrac{2-\alpha-\beta}{N(\alpha+\beta-1)}\left[{}^{RL}\mathscr{I}^{1-\beta,\varepsilon}_{0,w} f(t) + \delta\mu_{\alpha\beta}\, {}^{RL}\mathscr{I}^{1+\gamma-\beta,\varepsilon}_{0,w} f(t)\right], & if\ \beta \neq 1; \\ \dfrac{1-\alpha}{N(\alpha)}\left(\dfrac{\partial g(\varepsilon,t)}{\partial t}\right) f(t) + \dfrac{\delta\alpha}{N(\alpha)} {}^{RL}\mathscr{I}^{\gamma,\varepsilon}_{0,w} f(t), & if\ \beta = 1. \end{cases} \tag{10}$$

Remark 7. *Definition 6 is the generalization of numerous integral operators. For example, if we consider*

(i): *$\beta = \delta = 1$ and $g(\varepsilon,t) = t^{\varepsilon}$ in Definition 6, we get*

$$ {}^{FF}I^{\alpha,1,\gamma,\varepsilon}_{0,t,w} f(t) = \frac{\varepsilon(1-\alpha)}{N(\alpha)} t^{\varepsilon-1} f(t) + \frac{\alpha\varepsilon}{N(\alpha)\Gamma(\gamma)w(t)}\int_0^t (t-\tau)^{\gamma-1}\tau^{\varepsilon-1} w(\tau) f(\tau)d\tau. \tag{11}$$

This integral recovers the fractal-fractional cases for exponential decay kernel and Mittag-Leffler kernel given in [5].

(ii): $\beta=\delta=1$ *and* $g(\varepsilon,t)=\frac{t^{2-\varepsilon}}{2-\varepsilon}$ *in Definition 6, we obtain*

$$^{FF}I_{0,t,w}^{\alpha,1,\gamma,\varepsilon}f(t)=\frac{1-\alpha}{N(\alpha)}t^{1-\varepsilon}f(t)+\frac{\alpha}{N(\alpha)\Gamma(\gamma)w(t)}\int_0^t(t-\tau)^{\gamma-1}\tau^{1-\varepsilon}w(\tau)f(\tau)d\tau. \tag{12}$$

This integral includes the fractal-fractional integral for exponential decay kernel and Mittag-Leffler kernel introduced in [6].

(iii): $\beta=\delta=1$ *and* $g(\varepsilon,t)=t$ *in Definition 6, we get the GHF integral* [13].

(iv): $\delta=1$, $\alpha=\gamma=\eta$, $w(t)=1$ *and* $g(\varepsilon,t)=t$ *in Definition 6, we obtain the new fractional integral cited in* [18].

(v): $\alpha=\beta=\delta=1$ *and* $g(\varepsilon,t)=t$ *in Definition 6, we get the weighted Riemann-Liouville fractional integral of order* γ.

3. HIV/AIDS FRACTAL-FRACTIONAL FORMULATION

For convenience, we rewrite $^{FFR}D_{a,t,w,\delta}^{\alpha,\beta,\gamma,\gamma,\varepsilon}$ as $\mathscr{D}_{a,w,\delta}^{\alpha,\beta,\gamma,\varepsilon}$ and $^{FF}I_{a,t,w,\delta}^{\alpha,\beta,\gamma,\varepsilon}=\mathscr{I}_{a,w,\delta}^{\alpha,\beta,\gamma,\varepsilon}$. To describe the dynamics of HIV/AIDS disease, we propose the following fractal-fractional model with the new Hattaf mixed fractal-fractional derivative:

$$\begin{cases}\mathscr{D}_{0,w,\delta}^{\alpha,\beta,\gamma,\varepsilon}x_1=\mathscr{A}-\nu(x_3+\sigma x_4+\zeta x_5)x_1-d_1x_1,\\ \mathscr{D}_{0,w,\delta}^{\alpha,\beta,\gamma,\varepsilon}x_2=\nu\rho(x_3+\sigma x_4+\zeta x_5)x_1-(r+d_1)x_2,\\ \mathscr{D}_{0,w,\delta}^{\alpha,\beta,\gamma,\varepsilon}x_3=(1-\rho)\nu(x_3+\sigma x_4+\zeta x_5)x_1+rx_2-(\tau_1+\tau_2+d_1)x_3+\lambda x_4+\theta x_5,\\ \mathscr{D}_{0,w,\delta}^{\alpha,\beta,\gamma,\varepsilon}x_4=\tau_1x_3-(\lambda+d_1)x_4,\\ \mathscr{D}_{0,w,\delta}^{\alpha,\beta,\gamma,\varepsilon}x_5=\tau_2x_3-(d_1+\theta+d_2)x_4,\end{cases} \tag{13}$$

with initial conditions $x_1(0)=x_{1,0}$, $x_2(0)=x_{2,0}$, $x_3(0)=x_{3,0}$, $x_4(0)=x_{4,0}$ and $x_5(0)=x_{5,0}$. Here, the state variables $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and $x_5(t)$ are the number of susceptible individuals, the number of exposed but not yet infectious individuals, the number of infectious individuals, the number of individuals under care or treatment and the number of individuals in the AIDS stage. The parameters used in the model are given in Table 1.

It very important to note that our model presented by system (13) extends and generalizes the HIV/AIDS model with AB derivative fractional derivative [16], it suffices to choose $\beta=\delta=1$, $\gamma=\alpha$ and $g(\varepsilon,t)=t$.

TABLE 1. Parameter values model (13).

Parameter	Biological meaning
$\mathscr{A}$	Recruitment rate
ν	Contact rate
d_1	Natural mortality rate
d_2	Death rate due to AIDS
ρ	A portion of susceptible enter to x_2
σ, ζ	Relative infectiousness rates
r	Rate of progression from exposed to infected class
τ_1	Treatment rate
τ_2	Rate at which infected individuals progress to the AIDS stage
θ	Rate of recovery from AIDS stage due to treatment
λ	Failure rate of treatment

Let $I=[0,T]$ and rewrite system (13) as

$$\mathscr{D}_{0,w,\delta}^{\alpha,\beta,\gamma,\varepsilon} y(t) = h(t,y(t)), \quad \forall t \in I, \tag{14}$$

with initial condition $y(0)=y_0$, where

$$\begin{aligned} y(t) &= (x_1,x_2,x_3,x_4,x_5)^T, \\ y_0 &= (x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0})^T, \\ h(t,y(t)) &= \mathscr{G}_n(t,x_1,x_2,x_3,x_4,x_5)^T, n=1,2,3,4,5, \end{aligned} \tag{15}$$

and

$$\begin{cases} \mathscr{G}_1(t,x_1,x_2,x_3,x_4,x_5) = \mathscr{A} - \nu(x_3+\sigma x_4+\zeta x_5)x_1 - d_1x_1, \\ \mathscr{G}_2(t,x_1,x_2,x_3,x_4,x_5) = \nu\rho(x_3+\sigma x_4+\zeta x_5)x_1 - (r+d_1)x_2, \\ \mathscr{G}_3(t,x_1,x_2,x_3,x_4,x_5) = (1-\rho)\nu(x_3+\sigma x_4+\zeta x_5)x_1 + rx_2 - (\tau_1+\tau_2+d_1)x_3 + \lambda x_4 + \theta x_5, \\ \mathscr{G}_4(t,x_1,x_2,x_3,x_4,x_5) = \tau_1 x_3 - (\lambda+d_1)x_4, \\ \mathscr{G}_5(t,x_1,x_2,x_3,x_4,x_5) = \tau_2 x_3 - (d_1+\theta+d_2)x_4, \end{cases} \tag{16}$$

For simplicity, we study the case of $\beta=1$. By a simple computation, we have

$$
\begin{aligned}
y(t) = {} & \frac{w(0)y(0)}{w(t)} + \frac{1-\alpha}{N(\alpha)}\frac{\partial g(\varepsilon,t)}{\partial t}h(t,y(t)) \\
& + \frac{\delta\alpha}{N(\alpha)\Gamma(\gamma)}\frac{1}{w(t)}\int_0^t (t-\tau)^{\gamma-1}\frac{\partial g(\varepsilon,\tau)}{\partial \tau}w(\tau)h(\tau,y(\tau))d\tau.
\end{aligned} \tag{17}
$$

For the existence and uniqueness of solution of our fractal-fractional differential model for HIV/AIDS, we need the following lemma.

Lemma 8. *(Krasnoselskii's fixed point theorem* [19, 20]*) Let E be a nonempty closed convex subset of a Banach space $(\mathscr{B}, \|.\|)$. Suppose that $\mathscr{F}_1$ and $\mathscr{F}_2$ map E into $\mathscr{B}$ such that*

- **(i):** *$\mathscr{F}_1\varphi_1 + \mathscr{F}_2\varphi_2 \in E$, for all $\varphi_1, \varphi_2 \in E$;*
- **(ii):** *$\mathscr{F}_1$ is a contraction with constant $k < 1$;*
- **(iii):** *$\mathscr{F}_2$ is continuous and $\mathscr{F}_2(E)$ is contained in a compact subset of $\mathscr{B}$.*

Then $\mathscr{F}_1 + \mathscr{F}_2$ has a fixed point $\varphi \in E$.

To satisfy the condition of Lemma 8, let $\mathscr{B} = C([0,T], \mathbb{R}^5)$ be the Banach space of continuous functions from $[0,T]$ to $\mathbb{R}^5$ defined with the norm

$$\|\varphi(t)\| = \sup_{t\in[0,T]} |\varphi(t)|.$$

In addition, we consider the following hypotheses:

($\mathscr{H}_1$): There exist positive constants ϕ, ψ and $m \in [0,1)$ such that

$$h(t,y(t)) \leq \phi\|y\|^m + \psi.$$

($\mathscr{H}_2$): There exists a positive constant $M_1 > 0$ for all y, $\tilde{y}$, such that

$$|h(t,y(t)) - h(t,\tilde{y}(t)| \leq M_1\|y - \tilde{y}\|.$$

($\mathscr{H}_3$): There exists a positive constant $M_2 > 0$ such that for all $\varepsilon > 0$ and $t \in [0,T]$ we have

$$\left|\frac{\partial g(\eta,t)}{\partial t}\right| \leq M_2.$$

Furthermore, we define the following operator $\mathscr{F} : C([0,T], \mathbb{R}^5) \longrightarrow C([0,T], \mathbb{R}^5)$ such that

$$\mathscr{F}y(t) = \mathscr{F}_1 y(t) + \mathscr{F}_2 y(t),$$

where

$$\begin{cases} \mathscr{F}_1 y(t) = \dfrac{w(0)y(0)}{w(t)} + \dfrac{1-\alpha}{N(\alpha)}\dfrac{\partial g(\varepsilon,t)}{\partial t}h(t,y(t)), \\ \mathscr{F}_2 y(t) = \dfrac{\delta\alpha}{N(\alpha)\Gamma(\gamma)}\dfrac{1}{w(t)}\displaystyle\int_0^t (t-\tau)^{\gamma-1}\dfrac{\partial g(\varepsilon,\tau)}{\partial \tau}w(\tau)h(\tau,y(\tau))d\tau. \end{cases} \tag{18}$$

Consequently, (17) can be written as

$$\begin{aligned} \mathscr{F} y(t) = & \frac{w(0)y(0)}{w(t)} + \frac{1-\alpha}{N(\alpha)}\frac{\partial g(\varepsilon,t)}{\partial t}h(t,y(t)) \\ & + \frac{\delta\alpha}{N(\alpha)\Gamma(\gamma)}\frac{1}{w(t)}\int_0^t (t-\tau)^{\gamma-1}\frac{\partial g(\varepsilon,\tau)}{\partial \tau}w(\tau)h(\tau,y(\tau))d\tau. \end{aligned} \tag{19}$$

Theorem 9. *Suppose that* $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ *and* $(\mathscr{H}_3)$ *hold such that* $\frac{M_1M_2(1-\alpha)}{N(\alpha)} < 1$. *Then model (13) has at least one solution.*

Proof. Firstly, we prove that $\mathscr{F}_1$ is a contraction. Consider

$$E = \{y \in \mathscr{B} : \ \|y\| \leq \mathscr{L}, \mathscr{L} > 0\},$$

which is closed and convex set. For all $y, \tilde{y} \in E$, we have

$$|\mathscr{F}_1 y(t) - \mathscr{F}_1 \tilde{y}(t)| \leq \frac{M_1M_2(1-\alpha)}{N(\alpha)}\|y-\tilde{y}\|.$$

Then

$$\|\mathscr{F}_1 y - \mathscr{F}_1 \tilde{y}\| \leq \frac{M_1M_2(1-\alpha)}{N(\alpha)}\|y-\tilde{y}\|.$$

Since $\frac{M_1M_2(1-\alpha)}{N(\alpha)} < 1$, we deduce that $\mathscr{F}_1$ is a contraction.

Secondly, we prove that $\mathscr{F}_2$ is compact. We have

$$\begin{aligned} \|\mathscr{F}_2 y\| & \leq \max_{t\in[0,T]}\left|\frac{\delta\alpha}{N(\alpha)\Gamma(\gamma)}\frac{1}{w(t)}\int_0^t (t-\tau)^{\gamma-1}\frac{\partial g(\varepsilon,\tau)}{\partial \tau}w(\tau)h(\tau,y(\tau))d\tau\right| \\ & \leq \frac{\delta\alpha M_2 T^\gamma}{N(\alpha)\Gamma(\gamma+1)}\left(\phi\|y\|^m + \psi\right). \end{aligned}$$

Hence, for all $y \in E$, $\mathscr{F}_2$ is bounded. For equicontinuity, let $t_1, t_2 \in [0,T]$ such that $t_2 < t_1$, then

$$\begin{aligned} |\mathscr{F}_2 y(t_1) - \mathscr{F}_2 y(t_2)| = & \frac{\delta\alpha}{N(\alpha)\Gamma(\gamma)}\left|\int_0^{t_1}\frac{(t_1-\tau)^{\gamma-1}}{w(t_1)}\frac{\partial g(\varepsilon,\tau)}{\partial \tau}w(\tau)h(\tau,y(\tau))d\tau\right. \\ & \left. - \int_0^{t_2}\frac{(t_2-\tau)^{\gamma-1}}{w(t_2)}\frac{\partial g(\varepsilon,\tau)}{\partial \tau}w(\tau)h(\tau,y(\tau))d\tau\right|. \end{aligned}$$

Therefore, $\lim_{t_2 \longrightarrow t_1} |\mathscr{F}_2 y(t_1) - \mathscr{F}_2 y(t_2)| = 0$. Thus, $\mathscr{F}_2$ is equicontinuous. It follows from Arzela-Ascoli theorem that $\mathscr{F}_2$ is compact.

Therefore, $\mathscr{F}_2$ is continuous since y is continuous.

Finally to verify the item (i) of Lemma 8, we notice that $\mathscr{F}_1$ is a contraction, then

$$\|y\| \leq \|(I - \mathscr{F}_1)y\|,$$

and since $\mathscr{F}_2(E)$ is contained in a compact subset of $\mathscr{B}$ we deduce by appropriate construction of the subset E that for fixed y in E the contraction $y \rightarrow \mathscr{F}_1 y + \mathscr{F}_2 y$ has a fixed point in E, then $y \in E$. According to Lemma 8, we deduce that model (13) has at least one solution. □

Theorem 10. *If* $\left(\frac{1-\alpha}{N(\alpha)} + \frac{\delta\alpha T^{\gamma}}{N(\alpha)\Gamma(\gamma+1)}\right) M_1 M_2 < 1$, *then model (13) has a unique solution.*

Proof. Let $y, \tilde{y} \in C([0,T], \mathbb{R}^5)$. We have

$$\begin{aligned}
\|\mathscr{F} y - \mathscr{F} \tilde{y}\| &\leq \|\mathscr{F}_1 y - \mathscr{F}_1 \tilde{y}\| + \|\mathscr{F}_2 y - \mathscr{F}_2 \tilde{y}\| \\
&\leq \frac{(1-\alpha)M_2}{N(\alpha)} \max_{t\in[0,T]} |h(t, y(t)) - h(t, \tilde{y}(t))| \\
&\quad + \frac{\delta\alpha M_2}{N(\alpha)\Gamma(\gamma)} \max_{t\in[0,T]} \left| \int_0^t (t-\tau)^{\gamma-1} \frac{w(\tau)}{w(t)} h(\tau, y(\tau)) d\tau \right. \\
&\quad \left. - \int_0^t (t-\tau)^{\gamma-1} \frac{w(\tau)}{w(t)} h(\tau, \tilde{y}(\tau)) d\tau \right| \\
&\leq \left(\frac{1-\alpha}{N(\alpha)} + \frac{\delta\alpha T^{\gamma}}{N(\alpha)\Gamma(\gamma+1)} \right) M_1 M_2 \|y - \tilde{y}\|.
\end{aligned}$$

Since $\left(\frac{1-\alpha}{N(\alpha)} + \frac{\delta\alpha T^{\gamma}}{N(\alpha)\Gamma(\gamma+1)}\right) M_1 M_2 < 1$, we deduce that $\mathscr{F}$ is a contraction. Consequently, the Hattaf mixed fractal-fractional differential HIV/AIDS model (13) has a unique solution. □

4. CONCLUSION

In this work, we have introduced a new Hattaf mixed fractal-fractional in the sense of Caputo and Riemann-Liouville, which includes many definitions of fractal-fractional derivatives, both with singular and non-singular kernels. In addition, the fractal-fractional integral operator associated with the new Hattaf mixed fractal-fractional derivative covers various types existing in the literature. Furthermore, we have applied our operator to an HIV/AIDS model. By means of Krasnoselskii's fixed point theorem and Banach contraction, we studied the existence and uniqueness of solutions.

Conflict of Interests

The authors declare that there is no conflict of interests.

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