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A COMMON SOLUTION OF VARIATIONAL INCLUSION PROBLEM INVOLVING MAXIMAL $\eta-$RELAXED MONOTONE MAPPING AND FIXED POINT PROBLEM OF NONLINEAR MAPPING ON BANACH SPACES

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Abstract. Let E be a real uniformly convex and uniformly smooth Banach space with the dual space E^*. Let $J : E \to E^*$ be the normalized duality mapping and $A : E \to 2^{E^*}$ a maximal $\eta-$relaxed monotone mapping. The purpose of this article is two fold. First we present the characterizations of $\eta-$relaxed monotone mappings on uniformly convex and uniformly smooth Banach spaces, and also some properties of the resolvent mapping associated with maximal $\eta-$relaxed monotone mapping are proved. Secondly, iterative algorithms for approximating a common element of the set of fixed points of nonlinear mapping and the set of solution of variational inclusion problems involving maximal $\eta-$relaxed mappings is proposed, and then strong convergence theorems are proved. The results improve and extend some recent results in the literature.

Keywords: maximal $\eta-$relaxed monotone mapping; fixed points; variational inclusion problem.

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1. Introduction

Let E be a real Banach space and E^* be its dual space. The natural pairing between $x \in E$ and $x^* \in E^*$ is denoted by $\langle x, x^* \rangle$. In what follows the symbol 2^{E^*} denotes the collection of

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all subsets of E^*; $\|.\|$ denotes the norms of both E and E^*, and an underlying space will be understood from the context in which the norm symbol is used. For mappings $T : E \to E^*$ and $A : E \to 2^{E^*}$, a variational inclusion problem (VIP) (see Xia and Huang [21]) is a problem of finding $x \in E$ such that

$$0 \in Tx + Ax. \tag{1.1}$$

The set of solutions of VIP in (1.1) is denoted by $VIP(T,A)$. Variational inclusion problem is an important extension of the classical variational inequality problem introduced by Stampacchia [19]. Variational inequality theory has had a great impact and influence on the development of several branches of pure and applied sciences. Because of its wide applications in the nonlinear analysis and optimization, during the past six decades, the classical variational inequality has been generalized in various directions. An important extension of the classical variational inequality is a variational inclusion. Variational inclusions have become a rich source of inspiration and motivation for the study of a large number of problems, arising, for example, in convex optimization, economics and finance, image and signal Processing, and engineering sciences, etc. An existence of a solution for a given variational inclusion problem depends mainly on the geometric structures of the involved Banach space, and properties of the mappings T and A. Many authors imposed suitable conditions on the mappings T and A for having at least one solution. It is well-known that the monotonicity of the operator plays a prominent role in constructing an algorithm for solving variational inequalities or variational inclusions. Due to rich geometric properties of Hilbert spaces, many authors introduced different iterative algorithms to approximate a solution of the problem in (1.1). To mention some, see Alakoya and Mewomo [1], Ansari et al.[2], Balooee [3], Byrne [6], Ezeafulukwe et al. [9], Fang et al. [10], Han and Lo [12], Jung [13], Moudafi and Théra[14], Plubtieg and Punpaeng [15], Peng and Yao [16], Razani and Yazdi [17], Sangago et al. [18], Tian and Liu [20], Xu [22, 23, 24], Zegeye et al. [26] and the references therein.

For a multi-valued mapping $A : E \to 2^{E^*}$, throughout this paper we use the following definitions and notations:

(1) The effective domain of A, denoted by $D(A)$, is the set

$$D(A) = \{x \in E : A(x) \neq \emptyset\};$$

(2) The graph of A, denoted by $Gr(A)$, is the set

$$Gr(A) = \{(x,u) \in E \times E^* : u \in A(x)\};$$

(3) The range of A, denoted by $R(A)$, is the set

$$R(A) = \{u \in E^* : (x,u) \in Gr(A) \text{ for some } x \in D(A)\};$$

(4) The inverse of A, denoted by A^{-1}, is the set

$$A^{-1} = \{(u,x) \in E^* \times E : (x,u) \in Gr(A)\};$$

(5) For an arbitrary real constant ρ we define ρA by

$$\rho A = \{(x,\rho u) : (x,u) \in Gr(A)\}.$$

(6) For a multi-valued mapping $B : E \to 2^{E^*}$, we define $A+B$ by

$$A+B = \{(x,u+v) : (x,u) \in Gr(A) \text{ and } (x,v) \in Gr(B)\}.$$

Definition 1.1. *Let E be a real Banach space. A multi-valued mapping $A : E \to 2^{E^*}$ is said to be*

(1) ***monotone*** *if $\langle u-v, x-y\rangle \geq 0$ for all $(x,u),(y,v) \in Gr(A)$.*

*(2) $\gamma-$**strongly monotone** if there is $\gamma > 0$ such that*

$$\langle u-v, x-y\rangle \geq \gamma \|x-y\|^2 \text{ for all } (x,u),(y,v) \in Gr(A).$$

The generalization of monotone mappings, called $m-$relaxed monotone mappings, was introduced in the Hilbert space setting by Ansari et al. [2]. The following definition extends the definition to general real Banach space.

Definition 1.2. *Let E be a real Banach space. A multi-valued mapping $A : E \to 2^{E^*}$ is said to be $\eta-$**relaxed monotone** if there is $\eta > 0$ such that*

$$\langle u-v, x-y\rangle \geq -\eta \|x-y\|^2 \text{ for all } (x,u),(y,v) \in Gr(A).$$

Ansari et al. [2] studied some characteristics of $\eta-$relaxed monotone mapping in the setting of Hilbert spaces. They also proposed algorithms to approximate solutions of the variational inclusion problem and fixed point problem of nonlinear mappings.

Motivated by the results obtained in the Hilbert setting for variational inclusion problems involving $\eta-$relaxed monotone mappings, we further analyze characteristics of $\eta-$relaxed monotone mappings and discuss solutions of variational inclusion problem in (1.1) when $\eta-$relaxed monotone mapping A, in more general Banach space setting; particularly focusing uniformly convex and uniformly smooth Banach spaces. Our results generalize the results in Ansari et al. [2] and Balooee [3].

2. Preliminaries

Let E be a real Banach space with its dual space E^* and $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}.$$

It can be shown easily that if E is a real Hilbert space, then $J = I$, where I is the identity mapping on E. S_E denotes the unit sphere of E; that is, $S_E = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be strictly convex if for all $x, y \in S_E$ with $x \neq y$ imply $\|\lambda x + (1-\lambda)y\| < 1$ for all $\lambda \in (0,1)$. It is said to be uniformly convex if for any two sequences $\{x_n\}$ and $\{y_n\}$ in the unit sphere S_E

$$\lim_{n\to\infty} \left\| \frac{x_n + y_n}{2} \right\| = 1 \text{ implies } \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

The Banach space E is said to be smooth if $\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in S_E$. It is also said to be uniformly smooth if the above limit is attained uniformly for $x, y \in S_E$. For more on these notions refer to Chidume [7], Cioranescu [8], Goebel and Kirk [11], and the references therein.

Theorem 2.1 (Goebel and Kirk [11])**.** *Let E be a real Banach space with its dual space E^* and $J : E \to 2^{E^*}$ be the normalized duality mapping.*

(i) E is uniformly smooth if and only if E^* is uniformly convex.

(ii) If E is either uniformly smooth or uniformly convex, then J is single valued.

(iii) *If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.*

Theorem 2.2 (Berinde [5], Zegeye et al. [25])**.** *Let E be a real uniformly convex Banach space. Then there exists $\mu > 0$ such that*

$$\langle Jx - Jy, x - y\rangle \geq \mu \|x - y\|^2. \tag{2.1}$$

Lemma 2.3 (Ansari et al.[2])**.** *Let $\{a_n\} \subseteq [0,\infty)$, $\{\gamma_n\} \subseteq [0,1)$, and $\{d_n\} \subseteq [0,\infty)$ satisfy*

$$(i)\ a_{n+1} \leq \gamma_n a_n + d_n,$$

$$(ii)\ \limsup_{n\to\infty} \gamma_n < 1\ ,\ \text{and}$$

$$(iii)\ \lim_{n\to\infty} d_n = 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

3.1. Properties of $\eta-$relaxed monotone mappings on Banach Spaces. In the sequel we derive characterizations of $\eta-$**relaxed monotone** mappings in a real uniformly convex Banach space.

Theorem 3.1. *Let E be a real uniformly convex Banach space and $\mu > 0$ satisfies the inequality in Theorem 2.2. A multi-valued mapping $A : E \to 2^{E^*}$ is $\eta-$relaxed monotone if and only if for every $\rho \in (0, \frac{\mu}{2\eta})$,*

$$-2\eta \|x - y\|^2 \leq \rho \|u - v\|^2 + 2\langle u - v, x - y\rangle\ \forall (x,u),(y,v) \in Gr(A). \tag{3.1}$$

Proof. Assume that A is $\eta-$relaxed monotone. Let $\rho \in (0, \frac{\mu}{2\eta})$, and $(x,u),(y,v) \in Gr(A)$. Then

$$-\eta \|x - y\|^2 \leq \langle u - v, x - y\rangle$$

$$\Rightarrow\ -2\eta \|x - y\|^2 \leq 2\langle u - v, x - y\rangle$$

$$\Rightarrow\ -2\eta \|x - y\|^2 \leq \rho \|u - v\|^2 + 2\langle u - v, x - y\rangle$$

Conversely assume that (3.1) holds for every $\rho \in (0, \frac{\mu}{2\eta})$. Let $(x,u),(y,v) \in Gr(A)$. Then it follows from (3.1) that

$$-2\eta \|x-y\|^2 \leq \rho \|u-v\|^2 + 2\langle u-v, x-y\rangle \tag{3.2}$$

Letting $\rho \to 0$ in (3.2) we get

$$-2\eta \|x-y\|^2 \leq 2\langle u-v, x-y\rangle;$$

Hence we have

$$-\eta \|x-y\|^2 \leq \langle u-v, x-y\rangle,$$

and so that A is $\eta-$relaxed monotone. □

We obtain the following characterization of $\eta-$relaxed monotone operators on uniformly convex Banach spaces.

Theorem 3.2. *Let E be a real uniformly convex Banach space and $\mu > 0$ satisfies the inequality in Theorem 2.2. Let $A : E \to 2^{E^*}$ be a multi-valued $\eta-$relaxed monotone mapping. Then for every $\rho \in (0, \frac{\mu}{2\eta})$, the mapping $(J+\rho A)^{-1} : R(J+\rho A) \to E$ is single valued.*

Proof. Let $\rho \in (0, \frac{\mu}{2\eta})$. Let $u \in R(J+\rho A)$ and $x, y \in (J+\rho A)^{-1}(u)$. Then there exist $w \in A(x)$ and $v \in A(y)$ such that $u = Jx + \rho w$ and $u = Jy + \rho v$. Therefore,

$$w = \frac{1}{\rho}(u - Jx) \text{ and } v = \frac{1}{\rho}(u - Jy). \tag{3.3}$$

By $\eta-$relaxed monotonicity of A, we have

$$-\eta \|x-y\|^2 \leq \langle w-v, x-y\rangle. \tag{3.4}$$

Substituting the expressions of w and v in (3.3) in (3.4) and applying Theorem 2.2, we get

$$-\eta \|x-y\|^2 \leq -\frac{1}{\rho}\langle Jx-Jy, x-y\rangle \leq -\frac{\mu}{\rho} \|x-y\|^2 \tag{3.5}$$

Because $\frac{\mu}{\rho} - \eta > \eta > 0$, we have $x = y$. Therefore, $(J+\rho A)^{-1}(u)$ is single-valued. This completes the proof. □

Definition 3.3. *Let E be a real Banach space with its dual space E^*. A multi-valued mapping $A : E \to 2^{E^*}$ is said to be maximal $\eta-$relaxed monotone if A is $\eta-$relaxed monotone and $R(J+\rho A) = E^*$ for every $\rho > 0$.*

We note that every monotone mapping is $\eta-$relaxed monotone for any real constant $\eta > 0$, it follows that every maximal monotone mapping is maximal $\eta-$relaxed monotone for any real constant $\eta > 0$.

Definition 3.4. *Let E be a real uniformly convex Banach space with its dual space E^* and $\mu > 0$ satisfies the inequality in Theorem 2.2. Let $A : E \to 2^{E^*}$ be a maximal $\eta-$relaxed monotone mapping and let $\rho \in \left(0, \dfrac{\mu}{2\eta}\right)$. The resolvent mapping $\mathscr{R}_A^\rho : E^* \to E$ associated with A and ρ is defined by*

$$\mathscr{R}_A^\rho(u) = (J+\rho A)^{-1}(u),\ \forall u \in E^*. \tag{3.6}$$

The following theorem states one of the important characterization of maximal $\eta-$relaxed monotone mappings.

Theorem 3.5. *Let E be a real uniformly convex Banach space with its dual space E^* and $\mu > 0$ satisfies the inequality in Theorem 2.2. A multi-valued mapping $A : E \to 2^{E^*}$ is maximal $\eta-$relaxed monotone if and only if for given points $x \in E$ and $u \in E^*$, the property*

$$\langle u - v, x - y\rangle + \eta \|x-y\|^2 \geq 0,\ \forall (y,v) \in Gr(A), \tag{3.7}$$

implies that $(x,u) \in Gr(A)$.

Proof. Assume that A is a maximal $\eta-$relaxed monotone mapping. Let $x \in E$ and $u \in E^*$ satisfies (3.7). Let $\rho \in \left(0, \dfrac{\mu}{2\eta}\right)$. Because A is a maximal $\eta-$relaxed monotone mapping, $R(J+\rho A) = E^*$, $Jx + \rho u \in R(J+\rho A)$; and so that there is $(y,v) \in Gr(A)$ such that

$$Jx + \rho u = Jy + \rho v. \tag{3.8}$$

It follows from (3.7) that

$$\langle u - v, x - y\rangle + \eta \|x-y\|^2 \geq 0. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$-\frac{1}{\rho}\langle Jx-Jy,x-y\rangle+\eta\,\|x-y\|^2\geq 0. \tag{3.10}$$

We get from (3.10) and Theorem 2.2 that

$$\left(\eta-\frac{\mu}{\rho}\right)\|x-y\|^2\geq 0. \tag{3.11}$$

Because $\left(\eta-\frac{\mu}{\rho}\right)<0$, we have $x=y$. Hence $Jx=Jy$ and it follows from (3.8) that $u=v$. Therefore, $(x,u)\in Gr(A)$.

Conversely, assume that for any $x\in E$ and $u\in E^*$, (3.7) implies that $(x,u)\in Gr(A)$. Let $B:E\to 2^{E^*}$ be an $\eta-$relaxed monotone mapping such that $Gr(A)\subseteq Gr(B)$. If $(x,u)\in Gr(B)$, then

$$\langle u-v,x-y\rangle+\eta\,\|x-y\|^2\geq 0,\ \forall(y,v)\in Gr(A). \tag{3.12}$$

It follows from the assumption that $(x,u)\in Gr(A)$. Therefore, $Gr(A)=Gr(B)$, and thus A is a maximal $\eta-$ relaxed monotone mapping. This completes the proof. □

As an immediate consequence of Theorem 3.2 and Theorem 3.5, the following corollary characterizes the resolvent mapping $\mathscr{R}_A^\rho:E^*\to E$.

Corollary 3.6. *Let E be a real uniformly convex Banach space with its dual space E^* and $\mu>0$ satisfies the inequality in Theorem 2.2. Let $A:E\to 2^{E^*}$ be a maximal $\eta-$relaxed monotone mapping. Then, for any $\rho\in\left(0,\frac{\mu}{2\eta}\right)$, the mapping $\mathscr{R}_A^\rho:E^*\to 2^E$ is single-valued.*

Theorem 3.7. *Let E be a real uniformly convex Banach space with its dual space E^* and $\mu>0$ satisfies the inequality in Theorem 2.2. Let $A:E\to 2^{E^*}$ be a maximal $\eta-$relaxed monotone mapping and let $\rho\in\left(0,\frac{\mu}{2\eta}\right)$. The resolvent mapping $\mathscr{R}_A^\rho$ is Lipschitz continuous.*

Proof. Let $u,v\in E^*$. Because A is maximal $\eta-$relaxed monotone, there exist $x,y\in E$ such that $x=\mathscr{R}_A^\rho(u)$ and $y=\mathscr{R}_A^\rho(v)$. Thus $u\in Jx+\rho A(x)$ and $v\in Jx+\rho A(y)$. Then there exist $u'\in A(x)$ and $v'\in A(y)$ such that $u'=\frac{1}{\rho}(u-Jx)$ and $v'=\frac{1}{\rho}(v-Jy)$. Thus we get

$$-\eta\,\|x-y\|^2\leq\langle u'-v',x-y\rangle$$

$$= \frac{1}{\rho}\langle u-v, x-y\rangle - \frac{1}{\rho}\langle Jx-Jy, x-y\rangle$$
$$\leq \frac{1}{\rho}\langle u-v, x-y\rangle - \frac{\mu}{\rho}\|x-y\|^2$$

and hence

$$\|x-y\|^2 \leq \frac{1}{\mu-\rho\eta}\langle u-v, x-y\rangle. \tag{3.13}$$

Since $\|x-y\| = \left\|\mathscr{R}_A^\rho(u) - \mathscr{R}_A^\rho(v)\right\|$, we have

$$\left\|\mathscr{R}_A^\rho(u) - \mathscr{R}_A^\rho(v)\right\| \leq \frac{1}{\mu-\rho\eta}\|u-v\|. \tag{3.14}$$

This completes the proof. □

3.2. Variational inclusion problem involving $\eta-$relaxed monotone mapping.

Theorem 3.8. *Let E be a real uniformly convex Banach space with its dual space E^* and $\mu > 0$ satisfies the inequality in Theorem 2.2. Given a mapping $T : E \to E^*$ and a maximal $\eta-$relaxed monotone mapping $A : E \to 2^{E^*}$ such that $\rho \in \left(0, \frac{\mu}{2\eta}\right)$. Then, $x \in E$ is a solution of the VIP (1.1) if and only if*

$$x = \mathscr{R}_A^\rho(Jx - \rho Tx).$$

Proof. Let $x \in E$ be a solution of the VIP (1.1). Then there exists $v \in Ax$ such that $v = -Tx$. Hence

$$Jx + \rho v = Jx - \rho Tx$$
$$Jx - \rho Tx \in (J+\rho A)x$$
$$x = \mathscr{R}_A^\rho(Jx - \rho Tx).$$

Conversely, assume that $x = \mathscr{R}_A^\rho(Jx-\rho Tx)$. Then $(Jx-\rho Tx) \in (J+\rho A)x$, and so that for some $v \in Ax$ we have

$$Jx - \rho Tx = Jx + \rho v \Rightarrow v = -Tx \Rightarrow Tx + v = 0 \Rightarrow 0 \in (Tx + Ax).$$

□

The Hilbert space version of Theorem 3.8 was proved by Ansari et al. [2].

Theorem 3.9. *Let E be a real uniformly convex Banach space with its dual space E^* and $\mu > 0$ satisfies the inequality in Theorem 2.2. Let $T : E \to E^*$ be a mapping, and let $A : E \to 2^{E^*}$ be a maximal $\eta-$relaxed monotone mapping. Suppose further that there exists a real constant $\rho \in \left(0, \frac{\mu}{2\eta}\right)$ such that $J - \rho T$ is $r-$Lipschitz continuous mapping for some $r > 0$ and*

$$\frac{r}{\mu - \rho\eta} < 1. \tag{3.15}$$

Then, the VIP (1.1) has a unique solution.

Proof. Define $S : E \to E$ by

$$S(x) = \mathscr{R}_A^\rho(Jx - \rho Tx),\ x \in E.$$

Then for $x, y \in E$ it follows from Theorem 3.6 that

$$\begin{aligned}
\|Sx - Sy\| &= \left\|\mathscr{R}_A^\rho(Jx - \rho Tx) - \mathscr{R}_A^\rho(Jy - \rho Ty)\right\| \\
&\leq \frac{1}{\mu - \rho\eta} \|(J - \rho T)x - (J - \rho T)y\| \\
&\leq \frac{r}{\mu - \rho\eta} \|x - y\|.
\end{aligned}$$

Since $\frac{r}{\mu - \rho\eta} = k < 1$, S is a contraction mapping and hence has a unique fixed point by the Banach Contraction Mapping Principle [4]; say $Sz = z$. It follows from Theorem 3.8 that z is the only solution of the (VIP). □

3.3. Iterative Algorithm to Approximate a Common Solution of VIP and Fixed Point Problem.

Through out this section we have the following assumptions:

- **(A1)** Let E be a real uniformly convex Banach space with its dual space E^* and $\mu > 0$ satisfies the inequality in Theorem 2.2.
- **(A2)** Let $A : E \to 2^{E^*}$ be a maximal $\eta-$relaxed monotone mapping. Assume that
$$\rho \in \left(0, \frac{\mu}{2\eta}\right).$$
- **(A3)** Let $T : E \to E^*$ be a mapping for which $(J - \rho T) : E \to E^*$ is $r-$ Lipschitz continuous for some $r > 0$ that satisfies $\frac{r}{\mu - \rho\eta} < 1$.
- **(A4)** Let $U : E \to E$ be a nonexpansive mapping with nonempty fixed point set $Fix(U)$.

(A5) Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequence in $[0,1]$ such that

$$(i)\ \lim_{n\to\infty}\alpha_n = 0 \text{ and } \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty.$$

$$(ii)\ \lim_{n\to\infty}\beta_n = 0 \text{ and } \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty.$$

(A6) Assume that $\Gamma = VIP(T,A)\cap Fix(U) \neq \emptyset$.

Define a Halpern-type iterative algorithm that generates the sequence $\{x_n\}$ in E as

$$\begin{cases} x_0, w \in E \\ y_n = \alpha_n x_n + (1-\alpha_n)\mathscr{R}_A^{\rho}(Jx_n - \rho T x_n) \\ x_{n+1} = \beta_n w + (1-\beta_n) U y_n \end{cases} \tag{3.16}$$

Theorem 3.10. *If all the assumptions (A1)-(A6) hold, then the sequence $\{x_n\}$ generated by (3.16) converges strongly to a common solution of the variational inclusion problem (VIP(T,A)) and the fixed point problem of the mapping U.*

Proof. It follows from Theorem 3.8 and Theorem 3.9 that the mapping $\mathscr{R}_A^{\rho}(J-\rho T)$ has a unique fixed point, say $z \in E$; that is,

$$z = \mathscr{R}_A^{\rho}(Jz - \rho T z). \tag{3.17}$$

From the assumption in (A6), $\Gamma = \{z\}$. Let us first show that the sequence $\{x_n\}$ generated by (3.16) is bounded. For each $n = 0,1,2,...$, it follows from (3.17) and conditions in (A2), (A3) and (A4) that

$$\begin{aligned} \|x_{n+1}-z\| &\leq \beta_n\|w-z\| + (1-\beta_n)\|Uy_n - z\| \\ &\leq \beta_n\|w-z\| + (1-\beta_n)\|y_n - z\| \\ &\leq \beta_n\|w-z\| + (1-\beta_n)\alpha_n\|x_n - z\| \\ &\quad +(1-\beta_n)(1-\alpha_n)\left\|\mathscr{R}_A^{\rho}(Jx_n-\rho T x_n) - \mathscr{R}_A^{\rho}(Jz-\rho T z)\right\| \\ &\leq \beta_n\|w-z\| + (1-\beta_n)\alpha_n\|x_n - z\| \\ &\quad +(1-\beta_n)(1-\alpha_n)\frac{r}{\mu-\rho\eta}\|x_n - z\| \\ &\leq \beta_n\|w-z\| + (1-\beta_n)\|x_n - z\| \end{aligned}$$

$$\leq \max\{\|w-z\|,\|x_n-z\|\}. \tag{3.18}$$

By induction on n, it follows from (3.18) that

$$\|x_n-z\| \leq \max\{\|w-z\|,\|x_0-z\|\} \text{ for all } n\in\mathbb{N}. \tag{3.19}$$

Therefore, the sequence $\{x_n\}$ generated by (3.16) is bounded; and consequently, the sequences $\{Ux_n\}$, $\{y_n\}$, $\{Uy_n\}$, $\{\mathscr{R}_A^{\rho}(J-\rho T)x_n\}$ are all bounded.

Put $\gamma = \dfrac{r}{\mu-\rho\eta} < 1$, and $M_n = \max\{\|w\|,\|Ux_n\|,\|y_n\|,\|Uy_n\|,\left\|\mathscr{R}_A^{\rho}(J-\rho T)x_n\right\|\}$. Then the sequence $\{M_n\}$ is bounded, and so that $M=\sup_{n\geq 1} M_n$ is real number. We note that for each $n=0,1,2,\ldots$

$$\begin{aligned}\|y_n-z\| &\leq \alpha_n\|x_n-z\|+(1-\alpha_n)\left\|\mathscr{R}_A^{\rho}(J-\rho T)x_n-\mathscr{R}_A^{\rho}(J-\rho T)z\right\| \\ &\leq [\gamma+(1-\gamma)\alpha_n]\|x_n-z\|. \qquad (3.20)\end{aligned}$$

On the other hand for each $n\in\mathbb{N}$ we have

$$\begin{aligned}\|x_{n+1}-z\| &\leq \beta_n\|w-z\|+(1-\beta_n)\|Uy_n-z\| \\ &\leq (1-\beta_n)\|y_n-z\|+M\beta_n. \qquad (3.21)\end{aligned}$$

We get from (3.20) and (3.21) that

$$\|x_{n+1}-z\| \leq (1-\beta_n)[\gamma+(1-\gamma)\alpha_n]\|x_n-z\|+M\beta_n. \tag{3.22}$$

If we put $a_n=\|x_n-z\|$, $\lambda_n=(1-\beta_n)[\gamma+(1-\gamma)\alpha_n]$, and $d_n=M\beta_n$, then it follows from (3.22) and the condition (A5)(i,ii) on the parameters that

$$\begin{aligned}&(i)\ \lim_{n\to\infty} d_n = 0,\\ &(ii)\ \limsup_{n\to\infty}\lambda_n \leq \gamma < 1 \qquad (3.23)\\ &(iii)\ a_{n+1}\leq \lambda_n a_n + d_n.\end{aligned}$$

It follows from (3.22), (3.23) and Lemma 2.3 that

$$\lim_{n\to\infty}\|x_n-z\| = 0. \tag{3.24}$$

This completes the proof. □

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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