Available online at http://scik.org
Adv. Fixed Point Theory, 2026, 16:4
https://doi.org/10.28919/afpt/9661
ISSN: 1927-6303

UNIQUENESS OF COINCIDENCE AND COMMON FIXED POINTS IN FUZZY B-METRIC SPACES WITH APPLICATIONS TO INTEGRAL EQUATIONS

THANESHOR BHANDARI[1,2,*], K. B. MANANDHAR[2], KANHAIYA JHA[2]

[1]Department of Mathematics, Tribhuvan University, Kathmandu, Nepal

[2]Department of Mathematics, School of Science, Kathmandu University, Dhulikhel, Nepal

Abstract. This paper investigates the existence of coincident points and common fixed points for two self-mappings in a fuzzy b-metric space. We establish that under the conditions of weak compatibility and suitable contractive criteria, these mappings possess a unique common fixed point. Illustrative examples are provided to demonstrate the validity of our results, along with an application to determine the unique solution of an integral equation. The application underscores the practical relevance of our findings, particularly in addressing conditions for dynamic market equilibrium in economics.

Keywords: fuzzy b-metric space; coincident point; common fixed point; uniqueness.

2020 AMS Subject Classification: 47H09, 47H10, 46S40.

1. INTRODUCTION

The theory of coincident and fixed points is one of the most active and developing areas in pure mathematics. Their issues can be used to explain a wide variety of nonlinear problems that occurs in many scientific domains. One useful technique for handling issues of this nature is the Banach contraction principle [4]. Generally speaking, coincident point and fixed point theory has been effective in posing and addressing a wide range of issues and has significantly aided in

*Corresponding author

E-mail address: tnbhandari2072@gmail.com

Received October 24, 2025

the solution of numerous real-world issues. Nonetheless, numerous strong fixed point theorems have been established using a few strong assumptions. Understanding their fundamentals issues and reducing the constraints on them by using lessor condition of these initial, strong assumptions are the main areas of research in recent years.

In 1965, Zadeh [18] introduced a new idea that represents the defense of vagueness, imprecision and deceit. This theory is much more useful and interesting than classical set theory. Several technical and scientific domains, such as image processing, navigation, fractals and many more use these techniques. In order to apply this idea in nonlinear analysis, other researchers have greatly expanded the research on fuzzy sets and its uses. Fuzzy metric space was introduced by Karmosil and Michalek in [11] in 1975. There are several expansions of the metric and metric space, including fuzzy metric spaces. With this change, fuzzy scenarios are included in the probabilistic metric space.

The concept of a fuzzy metric space was first refined by George and Veeramani [8] and it has important ramifications for quantum particle physics, especially with regard to the E-infinity and string theories [14]. A strong basis for applying fixed-point theory to fuzzy metric space is established by this study. In 1983, Grabiec [7] expanded the Banach contraction theorem in these spaces and described the completeness property of the fuzzy metric. Since then, several researchers have provided numerous expansions and generalizations. The ideas of Bakhtin [3] and Bourbaki served as the foundation for the concept of b-metric space. The notion of b-metric space was later introduced and formally defined by Czerwik [6]. Jain and Kaur established Some fixed point results in b-metric space and b-metric-like space with new contraction mappings intuitionistic fuzzy metric spaces in 2021 [10]. Many researchers have provided examples and fixed point results about these spaces [2]. In 1986, Jungck [9] introduced the notion of compatible mappings with common fixed points. In which, by substituting a weaker inequality for the triangle inequality, Sedghi and Shobe [15] created the concept of fuzzy b-metric space, which is actually much broader than fuzzy metric spaces. Nabadan [12] modified the idea of Sedghi and Shobe and established new results. Oner and Sostak [13] defined strong fuzzy b-metric spaces and their attributes in 2020. In 2024, Bhandari et al. [5] introduced the fixed point theorems in strong fuzzy b-metric space.

Our objective is to obtain a coincidence point by using the two self mappings in fuzzy b-metric space and if these mappings satisfy the cotractive condition then they have a unique common fixed point. As an application of our result, we have established the unique solution of an integral equation to show that how the conditions for dynamic market equilibrium are addressed in economics.

2. Preliminaries

Definition 2.1 [16] *A mapping* $\odot : I \times I \to I$*, where* $I = [0,1]$ *is known as a continuous triangular norm if it satisfies the the properties given as below:*

(i) *Symmetry:* $x \odot y = y \odot x$*, for* $x, y \in I$*;*

(ii) *Monotonocity:* $x \odot y \leq z \odot w$ *whenever* $x \leq z$ *and* $y \leq w$*;*

(iii) *Associativity:* $(x \odot y) \odot z = x \odot (y \odot z)$*, where* $x, y, z \in I$*;*

(iv) *Boundary condition:* $1 \odot x = x$*, for all* $x \in I$*.*

Definition 2.2. [11] *Let* ξ *be an arbitrary set,* $\mathbb{H}$ *be a fuzzy set defined on* $\xi \times \xi \times (0, \infty)$ *and* $\odot$ *be a continuous t-norm. Then the order tuple* $(\xi, \mathbb{H}, *)$ *is said to be a fuzzy metric space (FMS) satisfying the properties given as below,* $\forall\ r, s, t \in \xi$ *and* $a, b > 0$

(i) $\mathbb{H}(r, s, 0) = 0$;

(ii) $\mathbb{H}(r, s, a) = 1$ *for all* $a > 0$ *iff* $r = s$

(iii) $\mathbb{H}(r, s, a) = \mathbb{H}(s, r, a)$;

(iv) $\mathbb{H}(r, s, a) \odot \mathbb{H}(s, t, b) \leq \mathbb{H}(r, t, a + b)$ *for all* $a, b > 0$;

(v) $\mathbb{H}(r, s, .) : (0, \infty) \to I$ *is continuous from left.*

Where the expression $\mathbb{H}(r, s, a)$ represents the degree of closeness between r and s depending upon the parameter $a > 0$.

This idea is modified by George and Veeramani in 1994 and defined as follow:

Definition 2.3. [8] *The order tuple* $(\xi, \mathbb{H}, \odot)$ *is known as a FMS if* ξ *is an arbitrary set,* $\odot$ *is a continuous t-norm and* $\mathbb{H}$ *is a fuzzy set defined on* $\xi \times \xi \times (0, \infty)$ *with the properties indicated as below, for all* $r, s, t \in \xi$ *and* $a, b > 0$

(i) $\mathbb{H}(r, s, a) > 0$;

(ii) $\mathbb{H}(r, s, a) = 1$*,* $a > 0$ *iif* $r = s$;

(iii) $\mathbb{H}(r, s, a) = \mathbb{H}(s, r, a)$;

(iv) $\mathbb{H}(r,s,a)\odot\mathbb{H}(s,t,b)\leq\mathbb{H}(r,t,a+b)$ *for all* $a,b>0$;

(v) $\mathbb{H}(r,s,.):(0,\infty)\rightarrow I$ *is continuous.*

Example 2.4 . [8] *Let* $\xi=\mathbb{R}$*, the set of all real numbers and t-norm is defined in terms of the product* $x\odot y=x.y\ \forall\ x,y\in I$ *and* $r,s\in\xi,a>0$ *defined as:*

$$\mathbb{H}(r,s,a)=\begin{cases}\dfrac{a}{a+|r-s|} & \text{if } r,s\in\xi,a>0,\\ 0 & \text{if } r,s\in\xi,a=0.\end{cases}$$

Then $\mathbb{H}$ is a fuzzy metric on $\mathbb{R}$.

Definition 2.5. [12] *Let* ξ *be an arbitrary set,* $\mathbb{H}$ *be a fuzzy set defined on* $\xi\times\xi\times(0,\infty)$ *where* $\odot$ *be a continuous t-norm and given* $p\geq1$*. Then the order tuple* $(\xi,\mathbb{H},\odot)$ *is known a fuzzy b-metric space (FbMS) with the properties given as below,* $\forall\ r,s,t\in\xi$ *and* $a,b>0$

(i) $\mathbb{H}(r,s,a)>0$;

(ii) $\mathbb{H}(r,s,a)=1\ \forall\ a>0$ *iff* $r=s$;

(iii) $\mathbb{H}(r,s,a)=\mathbb{H}(s,r,a)$;

(iv) $\mathbb{H}(r,s,a)\cdot\mathbb{H}(s,r,b)\leq\mathbb{H}(r,t,p(a+b))\ \forall\ a,b>0$;

(v) $\mathbb{H}(r,s,.):(0,\infty)\rightarrow I$ *is continuous;*

(vi) $\lim_{t\to\infty}\mathbb{H}(r,s,a)=1$

Example 2.6 [12] *Let* $\mathbb{H}(r,s,a)=e^{\frac{-d(r,s)}{a^2}}$ *where* d *is a b-metric on* ξ*, and t-norm is defined in terms of the product. Then it is a fuzzy b-metric space.*

Definition 2.7. [7] *Let* $(\xi,\mathbb{H},\odot)$ *be a FbMS. If we take a sequence* $\{r_n\}$ *in* ξ *then it is said to be convergent in* $r\in\xi$ *if* $\lim_{n\to\infty}\mathbb{H}(r_n,r,a)=1$ for each $a>0$.

Equivalently,

A sequence $\{r_n\}$ in ξ is said to converge to $r\in\xi$ if

$$\forall\varepsilon>0,\ \forall a>0,\ \exists N\in\mathbb{N}\text{ such that } n\geq N\implies\mathbb{H}(r_n,r,a)>1-\varepsilon.$$

That is, as $n\rightarrow\infty$, the degree of nearness between $\{r_n\}$ and r approaches 1 uniformly.

Definition 2.8. [7] *A sequence* $\{r_n\}$ *in* ξ *is called to be Cauchy sequence in* ξ *if* $\lim_{n\to\infty}\mathbb{H}(r_n,r_{m+n},a)=1$ *where* $a>0$ *and* $m,n\geq n'$ *where* $n'\in N$*. Equivalently,* *A sequence* $\{r_n\}$ *in* ξ *is called a Cauchy sequence if*

$$\forall \varepsilon > 0,\ \forall a > 0,\ \exists n' \in \mathbb{N} \text{ such that } m,n \geq n' \implies \mathbb{H}(r_m, r_n, a) > 1-\varepsilon.$$

A FbMS is considered complete if every Cauchy sequence in that space converges to the same limit.

Definition 2.9 [1] *Let* ξ *be a nonempty set and* $\phi, \psi : \xi \to \xi$ *be two mappings on* ξ.

(i) A point $r \in \xi$ *is called a coincidence point of* ϕ *and* ψ *if* $\phi r = \psi r$.

(ii) A point $z \in \xi$ *is called a point of coincidence of* ϕ *and* ψ *if there exists* $r \in \xi$ *such that* $z = \phi r = \psi r$.

(iii) A point $w \in \xi$ *is called a common fixed point of* ϕ *and* ψ *if* $w = \phi w = \psi w$.

Definition 2.10. [17] *Two self mappings* ϕ *and* ψ *defined on a FbMS* $(\xi, \mathbb{H}, \odot)$ *where* $p \geq 1$ *are said to be*

(a) compatible if, for all $a > 0$, $\lim_{n\to\infty} \mathbb{H}(\phi\psi r_n, \psi\phi r_n, a) = 1$ *where* $\{r_n\} \in \xi$ *which gives,* $\lim_{n\to\infty} \phi r_n = \lim_{n\to\infty} \psi r_n = z$, *where* $z \in \xi$.

(b) weakly compatible when they commute at the point where they coincide i.e. $\phi r = \psi r$ *which gives that* $\phi\psi r = \psi\phi r$.

(c) semi-compatible if for all $a > 0$, $\lim_{n\to\infty} \mathbb{H}(\phi\psi r_n, \psi r_n, a) = 1$ *where* $\{r_n\} \in \xi$ *with the property,* $\lim_{n\to\infty} \phi r_n = \lim_{n\to\infty} \psi r_n = z$, *where* $z \in \xi$.

3. Main Results

Theorem 3.1. *Suppose* $\xi \neq \phi$ *and* $(\xi, \mathbb{H}, \odot)$ *be a FbMS, and let* $\phi, \psi : \xi \to \xi$ *be mappings with the following properties:*

(i) $\phi(\xi) \subseteq \psi(\xi)$.

(ii) There exists $k \in \left(0, \frac{1}{p}\right)$, *such that for all* $r, s \in \xi$,

$$\mathbb{H}(\psi r, \psi s, ka) \geq \mathbb{H}(\phi r, \phi s, a), \tag{1}$$

If $\phi(\xi)$ *or* $\psi(\xi)$ *is complete, then there exists a point* $z \in \xi$ *such that* $\phi(z) = \psi(z)$. *Also,* ϕ *and* ψ *have a unique point of coincidence.*

Proof: Let $r_0 \in \xi$, then using (1), we can find $r_1 \in \xi$ such that $\phi(r_1) = \psi(r_0)$. taking $k = 0$,

$$\mathbb{H}(\psi(r_0), \psi(r_1), 0a) \geq \mathbb{H}(\phi(r_0), \phi(r_1), a)$$

which gives

$$\mathbb{H}(\psi(r_0), \psi(r_1), 0) = 1$$

Hence,

$$\psi(r_0) = \psi(r_1) \quad \Rightarrow \phi(r_1) = \psi(r_1).$$

Hence we can say r_1 is the coincidence point of ϕ and ψ. If $k \neq 0$, then by induction, we can define a sequence $\{r_n\}$ in ξ such that

$$\phi(r_n) = \psi(r_{n-1}).$$

Then,

$$\begin{aligned}
\mathbb{H}(\phi(r_n), \phi(r_{n+1}), a) &= \mathbb{H}(\psi(r_{n-1}), \psi(r_n), a) \\
&\geq \mathbb{H}\left(\phi(r_{n-1}), \phi(r_n), \frac{a}{k}\right) \\
&\geq \dots \\
&\geq \mathbb{H}\left(\phi(r_0), \phi(r_1), \frac{a}{k^n}\right).
\end{aligned}$$

Clearly,

$$1 \geq \mathbb{H}(\phi(r_n), \phi(r_{n+1}), a) \geq \mathbb{H}\left(\phi(r_0), \phi(r_1), \frac{a}{k^n}\right) \to 1 \quad \text{as } n \to \infty.$$

Thus,

$$\lim_{n\to\infty} \mathbb{H}(\phi(r_n), \phi(r_{n+1}), a) = 1.$$

Let $u_n(a) = \mathbb{H}(\phi(r_n), \phi(r_{n+1}), a)$ for all $n \in \mathbb{N} \cup \{0\}$ and $a > 0$.

Clearly,

$$\lim_{n\to\infty} r_n(a) = 1$$

Now we will show that $\{\phi(r_n)\}$ is a Cauchy sequence, assume that it is not true. Then there exist $0 < \varepsilon < 1$ and two sequences $\{p_n\}$ and $\{q_n\}$ such that, for every $n \in \mathbb{N}$ containing 0 and $a > 0$, with $p_n > q_n \geq n$, we have

$$\mathbb{H}(\phi(r_{p_n}), \phi(r_{q_n}), a) \leq 1 - \varepsilon,$$

Then,

$$\mathbb{H}(\phi(r_{p_{n-1}}),\phi(r_{q_{n-1}}),a) > 1-\varepsilon$$

and

$$\mathbb{H}(\phi(r_{p_{n-1}}),\phi(r_{q_n}),a) > 1-\varepsilon,$$

Now,

$$\begin{aligned} 1-\varepsilon &\geq \mathbb{H}(\phi(r_{p_n}),\phi(r_{q_n}),a) \\ &\geq \mathbb{H}\left(\phi(r_{p_{n-1}}),\phi(r_{p_n}),\frac{a}{2k}\right) \odot \mathbb{H}\left(\phi(r_{p_{n-1}}),\phi(r_{q_n}),\frac{a}{2k}\right) \\ &> u_{p_{n-1}}\left(\frac{a}{2k} \odot 1-\varepsilon\right), \end{aligned}$$

Since $u_{p_{n-1}}\left(\frac{a}{2k}\right) \to 1$ as $n \to \infty$ and also we have,

$$1-\varepsilon \;\geq\; \mathbb{H}(\phi(r_{p_n}),\phi(r_{q_n}),a) \;>\; 1-\varepsilon,$$

Which leads a contradiction. Hence $\phi(r_n)$ is a Cauchy sequence in $\phi(\xi)$.

Now let us assume that $\phi(\xi)$ is complete, then there exists a point $z \in \phi(\xi)$ such that

$$\lim_{n\to\infty} \phi(r_n) = z.$$

This implies that there exists $w \in \xi$ such that $z = \phi(w)$.

Now,

$$\begin{aligned} \mathbb{H}(\phi(w),\psi(w),a) &\geq \mathbb{H}\left(\phi(w),\phi(r_n),\frac{a}{k}\right) \odot \mathbb{H}\left(\phi(r_n),\psi(w),\frac{a}{k}\right) \\ &= \mathbb{H}\left(\phi(w),\phi(r_n),\frac{a}{k}\right) \cdot \mathbb{H}\left(\psi(r_{n-1}),\psi(w),\frac{a}{k}\right) \\ &\geq \mathbb{H}\left(\phi(w),\phi(r_n),\frac{a}{k}\right) \cdot \mathbb{H}\left(\phi(r_{n-1}),phi(w),\frac{a}{k}\right) \\ &\geq 1\cdot 1 = 1, \quad \text{as } n \to \infty \end{aligned}$$

Then by using the definition and the given conditions, we have

$$\phi(w) = \psi(w).$$

Hence w is the point at which ϕ and ψ coincide and is a coincidence point. Again assume that $\psi(\xi)$ is complete then there exists a point $z \in g(\xi)$ such that

$$\lim_{n\to\infty} \phi(r_n) = z.$$

However $\psi(\xi) \subseteq \phi(\xi)$ this implies that $z \in \phi(\xi)$, so there exists $w \in \xi$ such that $z = \phi(w)$.

Now it requires to show that the point of coincidence of ϕ and ψ is unique.

Let z_1 be another point of coincidence of ϕ and ψ.

Then,

$$z_1 = \phi(w_1) = \psi(w_1) \quad \text{for some } w_1 \in \xi.$$

$$\begin{aligned}
1 \geq \mathbb{H}(z, z_1, a) =& \mathbb{H}(\psi(w), \psi(w_1), a) \\
& \geq \mathbb{H}\left(\phi(w), \phi(w_1), \frac{a}{k}\right) \\
& = \mathbb{H}\left(z, z_1, \frac{a}{k}\right) \\
& \geq \dots \\
& \geq \mathbb{H}\left(z, z-1, \frac{a}{k^n}\right)
\end{aligned}$$

Hence by using the given conditions, we have

$$\lim_{n\to\infty} \mathbb{H}\left(z, z_1, \frac{a}{k^n}\right) = 1.$$

which gives

$$1 \geq \mathbb{H}(z, z_1, a) \geq 1$$

Hence $\mathbb{H}(z, z_1, a) = 1$, so $z_1 = z$.

This completes the proof.

Theorem 3.2 Let $(\xi, \mathbb{H}, \odot,)$ be a complete FbMS, for a given $p \geq 1, k \in \left(0, \frac{1}{p}\right)$ and $\phi, \psi : \xi \to \xi$ be the self mappings satisfying the following conditions

(i) $\psi(\xi) \subseteq \phi(\xi)$ such that

$$\mathbb{H}(\psi r, \psi s, ka) \geq \mathbb{H}(\phi(r), \phi(s), a) \tag{2}$$

(ii) ϕ and ψ are weakly compatible mappings.

Then, ϕ and ψ have a unique common fixed point in ξ.

Proof: By using the above theorem, there exist $z, w \in \xi$ such that $w = \phi(z) = \psi(z)$.

Since $w = \phi(z)$ and ϕ and ψ are weakly compatible then we can say that

$$\psi(w) = \psi(\phi(z)) = \phi(\psi(z)) = \phi(w).$$

Let $x = \phi(w) = \psi(w)$, then x is a point of coincidence of ϕ and ψ.

Since the point of coincidence is unique, this implies $x = w$.

Which gives $w = \phi(w) = \psi(w)$.

Hence w is the unique common fixed point of ϕ and ψ.

Now we the following example to justify the above theorem:

Example 3.3. Let $\xi = \mathbb{R}$ and define

$$\mathbb{H}(r,s,a) = \frac{a}{a+|r-s|} \qquad ,a > 0,$$

which is the standard fuzzy metric derived from the usual metric $d(r,s) = |r-s|$. This changes $(\xi, \mathbb{H}, \odot)$ into a complete fuzzy b-metric space.

For this let us choose

$$a = 1, \qquad k = \tfrac{1}{2},$$

so that $a \geq 1$ and $0 < k < \frac{1}{a} = 1$.

If we define the mappings $\phi, \psi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(r) = \frac{r}{2}, \qquad \psi(r) = \frac{r}{4}.$$

As we see that

$$\phi(\mathbb{R}) = \mathbb{R}, \qquad \psi(\mathbb{R}) = \mathbb{R},$$

and hence $\psi(\xi) \subseteq \phi(\xi)$.

To satisfy the given inequality for all $r, s \in \mathbb{R}$ and $a > 0$,

$$\mathbb{H}\big(\psi(r), \psi(s), ka\big) = \frac{\frac{1}{2}a}{\frac{1}{2}a + \frac{|r-s|}{4}} = \frac{2a}{2a+|r-s|},$$

while

$$\mathbb{H}\big(\phi(r), \phi(s), a\big) = \frac{a}{a + \frac{|r-s|}{2}} = \frac{2a}{2a+|r-s|}.$$

Hence

$$\mathbb{H}\big(\psi(r),\psi(s),ka\big)=\mathbb{H}\big(\phi(r),\phi(s),a\big)\geq\mathbb{H}\big(\phi(r),\phi(s),a\big),$$

Therefore the required inequality holds.

In particular, if we take $r=2, s=6, a=4$, then we have,

$$\phi(2)=1,\quad \phi(6)=3,\quad \psi(2)=0.5,\quad \psi(6)=1.5,$$

$$\mathbb{H}(\phi(2),\phi(6),4)=\frac{4}{4+|1-3|}=\frac{4}{6}=\frac{2}{3},$$

$$\mathbb{H}(\psi(2),\psi(6),2)=\frac{2}{2+|0.5-1.5|}=\frac{2}{3}.$$

Now for weak compatibility, we have $\phi(r)=\psi(r)\Rightarrow\frac{r}{2}=\frac{r}{4}\Rightarrow r=0$ at that point.

Also

$$\phi\big(\psi(0)\big)=\phi(0)=0,\quad \psi\big(\phi(0)\big)=\psi(0)=0.,$$

So ϕ and ψ commute at the coincidence point. The only common fixed point is

$$\phi(r)=r\implies\tfrac{r}{2}=r\implies r=0,$$

$$\psi(r)=r\implies\tfrac{r}{4}=r\implies r=0.$$

Hence $r=0$ is the unique common fixed point.

4. APPLICATION

An application to dynamic market equilibrium is provided in this section to bolster our findings. Supply and demand are influenced by current price and price trends in various market places. The first and second order differential coefficients are involved to the present price $P(t)$. Assume that

$$Q_S=u_1+v_1P(t)+r_1\frac{dP(t)}{dt}+s_1\frac{d^2P(t)}{dt^2}=u_1+v_1P+r_1P'+s_1P'',$$

$$Q_D=u_2+v_2P(t)+r_2\frac{dP(t)}{dt}+s_2\frac{d^2P(t)}{dt^2}=u_2+v_2P+r_2P'+s_2P''.$$

Here, $u_1,u_2;v_1,v_2;r_1$, r_2 and s_1,s_2 are taken as constants. For the dynamic stability of the market equilibrium in any time t, there must be $Q_S=Q_D$. Then we have

$$u_1+v_1P+r_1P'+s_1P''=u_2+v_2P+r_2P'+s_2P'',$$

After simplifying we get,

$$(v_1 - v_2)P + (r_1 - r_2)P' + (s_1 - s_2)P'' = -(u_1 - u_2).$$

Assume $u = u_1 - u_2$, $v = v_1 - v_2$, $r = r_1 - r_2$ and $s = s_1 - s_2$ then dividing by y, $P(t)$ is shown as with the given initial conditions:

$$\begin{cases} P'' + \frac{r}{s}P' + \frac{v}{s}P = -\frac{a}{s}, \\ P(0) = 0, \\ P'(0) = 0, \end{cases} \tag{3}$$

As $\frac{r^2}{s} = \frac{4v}{s}$ as well as $\frac{v}{r} = \delta$ posses the property of continuity. Now we will show the above equation (3) is converted to the following integral equation

$$\lambda(u) = \int_0^A \Gamma(u,v)\mathbb{H}(u,v,P(u))\,du,$$

where $\Gamma(u,v)$ is Green's function given by

$$\Gamma(u,v) = \begin{cases} ue^{\frac{\mu}{2}(u-v)} & \text{if } 0 \le u \le v \le A, \\ ve^{\frac{\mu}{2}(u-v)} & \text{if } 0 \le u \le u \le A. \end{cases}$$

At first it requires to verify the existence part of the solution to the given equation,

$$P(v) = \int_0^A G(v,u,P(u))\,du. \tag{4}$$

Let $\beta = C([0,A])$ represent the collection of continuous functions with real values that are defined over the specified interval $[0,A]$ and let us define a function

$$\mathbb{H}(m,n,a) = \sup_{a\in[0,A]} \frac{\min\{m,n\}+a}{\max\{m,n\}+a},$$

for each $a > 0$, and $m,n \in \beta$, with continuous t-norm $\odot$ such that $r \odot s = rs$. It can be easily verified that $(\xi,\mathbb{H},\odot)$ is a FbMS. Let us define a mapping $f : \beta \to \beta$ defined by

$$P(v) = \int_0^A G(u,v,P(u))\,du.$$

Theorem 4.1. *Let us assume that $P(v) = \int_0^A G(v,u,P(u))\,du$ and assume the following conditions are satisfied* *(i)* $K:[0,A]\times[0,A]\to(0,\infty)$ *is continuous,*

(ii) There exists a mapping $\Gamma:[0,A]\times[0,A]\to\mathbb{R}^+$ *which is continuous and*

$$\sup_{a\in[0,A]}\int_0^A \Gamma(a,u)\,du\geq 1,$$

(iii) $\int_0^A \min\{K(u,v,m(u)),K(u,v,n(u))\}\,du+kv\geq \int_0^A \Gamma(u,v)\min\{m(u),n(u)\}\,du+v,$

$$\int_0^A \max\{K(u,v,m(u)),K(u,v,n(u))\}\,du+kv\geq \int_0^A \Gamma(u,v)\max\{m(u),n(u)\}\,du+v.$$

Consequently, the integral equation (4) *has a distinct solution.*

proof. Let $m,n\in\beta$, then by applying the given conditions of the theorem, we obtain

$$\begin{aligned}
\mathbb{H}(fm,fn,ka) &= \sup_{a\in[0,A]}\frac{\min\left\{\int_0^A K(u,v,m(u))\,du,\int_0^A K(u,v,n(u))\,du\right\}+ka}{\max\left\{\int_0^A K(u,v,m(u))\,du,\int_0^A K(u,v,n(u))\,du\right\}+ka}\\
&= \sup_{a\in[0,A]}\frac{\int_0^A \min\{K(u,v,m(u)),K(u,v,n(u))\}\,du+ka}{\int_0^A \max\{K(u,v,m(u)),K(u,v,n(u))\}\,du+ka}\\
&\geq \sup_{a\in[0,A]}\frac{\int_0^A \Gamma(u,v)\min\{m(u),n(u)\}\,du+a}{\int_0^A \Gamma(u,v)\max\{m(u),n(u)\}\,du+a}\\
&\geq \sup_{a\in[0,A]}\frac{\min\{m(u),n(u)\}\int_0^A \Gamma(u,v)\,du+a}{\max\{m(u),n(u)\}\int_0^A \Gamma(u,v)\,du+a}\\
&\geq \frac{\min\{m(u),n(u)\}+a}{\min\{m(u),n(u)\}+a}=\mathbb{H}(m,n,a).
\end{aligned}$$

Thus, $\mathbb{H}(fm,fn,ka)\geq\mathbb{H}(m,n,a)$ for all $m,n\in\beta$. From this result we conclude that all the properties of theorem (3.4) are also verified.

This theorem can be verified with the following example.

Example 3.5. Consider the integral equation

$$P(v)=\int_0^A G(v,u,P(u))\,du,$$

with $A=1$ and kernel

$$G(v,u,x)=K(u,v,x)=\alpha x+\beta v,\qquad \alpha=2,\ \beta=0.5.$$

(i) $\Gamma(u,v)\equiv\alpha=2$ is continuous, and

$$\sup_{a\in[0,1]}\int_0^1 \Gamma(a,u)\,du=\int_0^1 2\,du=2\geq 1.$$

(ii) $K(u,v,x) = 2x + 0.5v$ is continuous on $[0,1] \times [0,1] \times \mathbb{R}$.

(iii) Choose $k = 1 - \beta = 0.5$ so that the inequalities of the theorem are satisfied.

The integral equation becomes

$$P(v) = \int_0^1 \big(2P(u) + 0.5v\big)\, du = 2\int_0^1 P(u)\, du + 0.5v.$$

Let

$$C = \int_0^1 P(u)\, du.$$

Then

$$P(v) = 2C + 0.5v.$$

Integrating both sides:

$$C = \int_0^1 P(u)\, du = \int_0^1 (2C + 0.5u)\, du = 2C + \tfrac{1}{4}.$$

Thus,

$$C(1-2) = \tfrac{1}{4} \quad \Rightarrow \quad C = -\tfrac{1}{4}.$$

Hence the explicit solution is

$$P(v) = -\tfrac{1}{2} + \tfrac{1}{2}v, \qquad v \in [0,1].$$

Take $m(u) = u$ and $n(u) = 1 - u$ for $u \in [0,1]$. Then for any $v \in [0,1]$:

$$\int_0^1 \min\{K(u,v,m(u)), K(u,v,n(u))\}\, du + kv \;\geq\; \int_0^1 \Gamma(u,v) \min\{m(u), n(u)\}\, du + v,$$

$$\int_0^1 \max\{K(u,v,m(u)), K(u,v,n(u))\}\, du + kv \;\geq\; \int_0^1 \Gamma(u,v) \max\{m(u), n(u)\}\, du + v.$$

Both inequalities hold since they reduce to $(\beta + k)v \geq v$, with $\beta + k = 1$.

The numerical computation of

$$\int_0^1 G(v,u,P(u))\, du$$

matches exactly with the closed form $P(v)$. The plot below illustrates the agreement.

v	$LHS_{\min}$	$RHS_{\min}$	$LHS_{\max}$	$RHS_{\max}$
0.0	2.000	1.000	3.500	2.000
0.2	2.100	1.200	3.600	2.200
0.4	2.200	1.400	3.700	2.400
0.6	2.300	1.600	3.800	2.600
0.8	2.400	1.800	3.900	2.800
1.0	2.500	2.000	4.000	3.000

Conflict of Interests

The authors declare that there is no conflict of interests.

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