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ASYMPTOTIC SOFT FUZZY CONTRACTIVE MAPPINGS IN SOFT FUZZY METRIC SPACE

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Abstract. The idea of asymptotic fuzzy contractive maps is extended in this work within framework of soft fuzzy metric spaces. Some fixed point theorems are derived for asymptotic soft fuzzy ψ-contractive mappings defined on proposed setup equipped with compactness. A few additional illustrations are also furnished to validate the relevance of the presented results.

Keywords: soft fuzzy metric spaces; fixed points; contraction mapping.

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1. Introduction

Research on fixed point theory (FPT) within fuzzy metric spaces (FMS) has garnered considerable interest due to its significant implementations in analysis, optimization and decision sciences. The asymptotic contractions (AC), originally presented by Kirk and Suzuki [10], has been played a pivot role in the development of iterative methods and stability analysis.

On the other hand, soft set theory (SST), proposed by Molodtsov [14] in 1999, offers a powerful mathematical framework for dealing with parameterized uncertainty, where traditional and fuzzy models often fall short with some limitations. The integration of SST with FMS

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has led to the formulation of soft fuzzy metric spaces (SFMS), which provide a more refined setting for exploring fixed point results that incorporate both fuzziness and parameterization. For related literature, one can also refer to ([2], [3], [4], [5], [8], [11]).

This paper extends the study of asymptotic contractive mappings (ACM) to soft fuzzy metric spaces (SMS). Firstly, the asymptotic soft contractions (ASC) are introduced and some FPT are established which generalize known results from metric spaces (MS), probabilistic MS and FMS. Overall, the study bridges classical asymptotic fixed point theory (AFPT) with the parameterized structure of SFM, thereby offering new pathways for applications in optimization and decision-making under uncertainty.

2. Preliminaries

Definition 2.1. [12] A map $\star : [0,1]\times[0,1]\to[0,1]$ is said to be continuous triangular t-norm if it satisfies following properties:

(i) $\check{x}\star\check{y}=\check{y}\star\check{x}$, and $\check{x}\star(\check{y}\star\check{z})=(\check{x}\star\check{y})\star\check{z}$;

(ii) $\star$ is continous;

(iii) $1\star\check{x}=\check{x},$;

(iv) $\check{x}\star\check{y}\leq\check{z}\star\check{w}$ if $\check{x}\leq\check{z}$ and $\check{y}\leq\check{w}(\forall\check{x},\check{y},\check{z},\check{w}\in[0,1])$.

Definition 2.2. [9] A t-norm $\star$ is known as *H-type* if the sequence $\{\star^m\check{x}\}_{m=1}^{\infty}$ is equicontinuous at $\check{x}=1$ where

$$\star^m\check{x}:=\underbrace{\check{x}\star\check{x}\star\cdots\star\check{x}}_{m\text{ times}},\quad m\in\mathbb{N}.$$

Definition 2.3. [6] Let $X\neq\Phi$, $\star$ be a continuous t-norm and $\hat{M}$ be a fuzzy set on $X\times X\times(0,\infty)$. Then, the triplet $(X,\hat{M},\star)$ is called a FMS if it satisfies the following properties, $\forall\ \check{x},\check{y},\check{z}\in X$ and $\check{r}+\check{v}>0$:

(G1): $\hat{M}(\check{x},\check{y},\check{r})>0$;

(G2): $\hat{M}(\check{x},\check{y},\check{r})=1$ if and only if $\check{x}=\check{y}$;

(G3): $\hat{M}(\check{x},\check{y},\check{r})=\hat{M}(\check{y},\check{x},\check{r})$;

(G4): $\hat{M}(\check{x},\check{z},\check{r}+\check{v})\geq\hat{M}(\check{x},\check{y},\check{r})\star\hat{M}(\check{y},\check{z},\check{v})$;

(G5): $\hat{M}(\check{x},\check{y},\cdot):(0,\infty)\to(0,1]$ is continuous.

Remark 2.4. Condition (G2) means:

$$M(\check{x},\check{x},\check{r}) = 1 \ \forall\ \check{x} \in X,\ \hat{M}(\check{x},\check{y},\check{r}) < 1,\ \check{x} \neq \check{y},\ \check{r} > 0.$$

Definition 2.5. [6] Let FMS $(X,\hat{M},\star)$ and $\{\check{x}_n\} \in X$.

- $\{\check{x}_n\}$ is a *M-Cauchy sequence* if, $\forall\ \varepsilon \in (0,1)$ and each $\check{r} > 0$, $\exists\ n_0 \in \mathbb{N}$ such that $M(\check{x}_n,\check{x}_m,\check{r}) > 1-\varepsilon, \forall\ n,m \geq n_0$.
- $\{\check{x}_n\}$ converges to $\check{x}$, if, for each $\varepsilon \in (0,1)$ and $\check{r} > 0$, $\exists\ n_0 \in \mathbb{N}$ such that $M(\check{x}_n,\check{x},\check{r}) > 1-\varepsilon, \forall\ n \geq n_0$.
- $(X,\hat{M},\star)$ is *$\hat{M}$-complete* if each $\hat{M}$-Cauchy sequence in X converges to $\check{x} \in X$.

Definition 2.6. [6] The FMS $(X,\hat{M},\star)$ is said to have fuzzy diameter zero, if for non-empty sets $\{A_i\}_{i\in I}$ in X, $R \in (0,1)$ and $\check{r} > 0$; we can find $i_{\check{r}} \in I$ such that

$$M(\check{x},\check{y},\check{r}) > 1-R \text{ for all } \check{x},\ \check{y} \in A_{i_{\check{r}}}.$$

Definition 2.7. [7] The fuzzy diameter of set A of X, with respect to $\check{r}$, is the function $\psi_A : (0,+\infty) \to [0,1]$ (also denoted by $diam(A)$), given by

$$\psi_A(\check{r}) := \inf\{M(\check{x},\check{y},\check{r}) : \check{x},\check{y} \in A\}, \quad \text{for each } \check{r} > 0.$$

Proposition 2.8. [7] *The function ψ_A is well-defined and, in addition, it satisfies the following properties:*

(i) If $\check{v} < \check{r}$, then $\psi_A(\check{v}) \leq \psi_A(\check{r})$.
(ii) If $A \subseteq B$, then $\psi_A(\check{r}) \geq \psi_B(\check{r})$.
(iii) $\psi_A(\check{r}) = 1$, for some $\check{r}$, if and only if A is unique.

We say that a contractive mapping $T : X \to X$ has a *fixed point* $\check{x}^\star$, if $\check{x}^\star = T\check{x}^\star$ and $T^n\check{x} \to \check{x}^\star$ for all $\check{x} \in X$. Similarly, we say that a mapping $T : X \to X$ has an *approximate fixed point* if $\exists\ \check{r} \in X$ such that $\check{x} \in Fix(T)$ and, given a sequence $\{\check{x}_n\}$, if $d(\check{x}_n, T\check{x}_n) \to 0$, then $\check{x}_n \to \check{x}$.

Definition 2.9. [14] Let X be the universal set, $\hat{P}$ be the collection of parameters, and $\mathscr{P}(X)$ denotes the collection of subsets of X, then $(\hat{W},\hat{P})$ is a soft set over X if $\hat{W} : \hat{P} \to \mathscr{P}(X)$.

Definition 2.10. [14] A soft set $(\hat{W},\hat{P})$ over X is called absolute if $\hat{W}(\eta) = X\ \forall\ \eta \in \hat{P}$.

An absolute soft set with parameter $\hat{P}$ is denoted by $\tilde{X}_{\hat{P}}$.

Definition 2.11. [14] The pair $(\hat{W},\hat{P})$ is a soft real set if $\hat{W} : \hat{P} \to \mathscr{B}(\mathbb{R})$, where $\mathscr{B}(\mathbb{R})$ is the collection of all non-void bounded subsets of $\mathbb{R}$.

A soft real set $(\hat{W},\hat{P})$ is soft real number (signified by $\tilde{x}$) if $\forall\ \eta \in \hat{P}$, $W(\eta)$ is a singleton member of $\mathscr{B}(\mathbb{R})$ *i.e.* $\tilde{x}(\eta) = \{k\}$ for some $k \in \mathbb{R}$,

Definition 2.12. [13] A soft set $\tilde{t}_\eta$ is a soft point if for exactly one parameter $\eta \in \hat{P}$, $\hat{W}(\eta) = \{t\}$ where $t \in X$ and $W(\mu) = \varphi\ \forall\ \mu \in \hat{P} \setminus \{\eta\}$.

Definition 2.13. [13] A map $\tilde{\star} : [0,1](\hat{P}) \times [0,1](\hat{P}) \to [0,1](\hat{P})$ (where $[0,1](\hat{P})$ denotes the collection of soft real numbers), is called a continuous soft t-norm if following conditions hold true:

(1) $\tilde{\star}$ is commutative and associative,
(2) $\tilde{\star}$ is continuous,
(3) $\tilde{x}\,\tilde{\star}\,\bar{1} = \tilde{x}\ \forall\ \tilde{t} \in [0,1](\hat{P})$,
(4) $\tilde{x}\ \leq\ \tilde{y}, \tilde{z} \leq \tilde{w} \implies \tilde{x}\,\tilde{\star}\,\tilde{z} \leq \tilde{y}\,\tilde{\star}\,\tilde{w}$.

Example: $\tilde{\lambda}\,\tilde{\star}\,\tilde{\mu} = \min\{\tilde{\lambda}, \tilde{\mu}\}$.

Definition 2.14. [13] A map $\mathscr{S}_{\mathscr{F}} : S\hat{P}(\tilde{X}_{\hat{P}}) \times S\hat{P}(\tilde{X}_{\hat{P}}) \times (0,\infty)(\hat{P}) \to [0,1](\hat{P})$ denoted by $(\tilde{X}_{\hat{P}}, \mathscr{S}_{\mathscr{F}}, \tilde{\star})$ is called a SFMS if:

SfM1: $\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \tilde{y}_{p_j}, \tilde{r}) \geq \bar{0}$,
SfM2: $\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \tilde{y}_{p_j}, \tilde{r}) = 1 \iff \tilde{x}_{p_i} = \tilde{y}_{p_j}$,
SfM3: symmetry holds,
SfM4: $(\mathscr{S}_{\mathscr{F}}\tilde{x}_{p_i}, \tilde{z}_{p_k}, \tilde{r} \oplus \tilde{c}) \geq \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \tilde{y}_{p_j}, \tilde{r})\,\tilde{\star}\,\mathscr{S}_{\mathscr{F}}(\tilde{y}_{p_j}, \tilde{z}_{p_k}, \tilde{c})$,
SfM5: $(\mathscr{S}_{\mathscr{F}}\tilde{x}_{p_i}, \tilde{y}_{p_j}, \cdot)$ is continuous in $(0,\infty)(\hat{P})$.

Definition 2.15. [14] A soft sequence $\{\tilde{x}^m_{p_i}\}$ in $(\tilde{X}_{\hat{P}}, \mathscr{S}_{\mathscr{F}}, \tilde{\star})$ converges to $\tilde{y}_{p_i}$ if

$$\lim_{m\to\infty} \mathscr{S}_{\mathscr{F}}(\tilde{x}^m_{p_i}, \tilde{y}_{p_i}, \tilde{r}) = 1, \quad \forall\ \tilde{r} > 0.$$

Definition 2.16. [14] A soft sequence $\{\tilde{x}^m_{p_i}\}$ is Cauchy in SFMS if

$$\lim_{m,t\to\infty} \mathscr{S}_{\mathscr{F}}(\tilde{x}^m_{p_i},\tilde{x}^t_{p_i},\tilde{r}) = 1, \quad \forall \tilde{r} > 0.$$

Definition 2.17. [14] The SFMS is considered complete, If each Cauchy sequence in the SFMS converges.

Definition 2.18. [14] A SFMS is compact, if every sequence admits at least one convergent subsequence.

3. Main Results

Definition 3.1. A soft t-norm $\tilde{\star}$ is known as $\tilde{H}-type$ if the sequence $\{\star^m \tilde{x}_{p_i}\}_{m=1}^{\infty}$ is equicontinuous at $\tilde{x}_{p_i} = 1$.

Definition 3.2. For a complete SFMS $(\tilde{X}_{\hat{P}}, \mathscr{S}_{\mathscr{F}}, \tilde{\star})$, a map $\breve{T} : \tilde{X} \to \tilde{X}$ is considered an *asymptotic soft fuzzy ψ-contractive mapping* (ASFψCM) if there exists a series $\psi_n : [0,1]_{(\hat{P})} \to [0,1]_{(\hat{P})}$ such that:

(ASF1): $\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i},\tilde{y}_{p_j},\tilde{r}) > 0 \;\Rightarrow\; \mathscr{S}_{\mathscr{F}}(T^n\tilde{x}_{p_i},T^n\tilde{y}_{p_j},\tilde{r}) \geq \psi_n(\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i},\tilde{y}_{p_j},\tilde{r}))$, $\forall\, \tilde{x}_{p_i},\tilde{y}_{p_j} \in \tilde{X}$ and $\forall\, \tilde{r} > 0$;

(ASF2): $\psi_n \to \psi$ uniformly on $[0,1](\hat{P})$;

(ASF3): ψ is continuous, non-decreasing, and $\psi(\tilde{r}) > \tilde{r}$.

Theorem 3.3. *Let $(\tilde{X}_{\hat{P}}, \mathscr{S}_{\mathscr{F}}, \tilde{\star})$ be a complete SFMS, where $\tilde{\star}$ denotes an H-type soft t-norm. Assume that $\breve{T} : \tilde{X} \to \tilde{X}$ is a uniformly continuous ASFψCM and $\psi : [0,1](\hat{P}) \to [0,1](\hat{P})$ such that*

$$\limsup_{s\to 0^+} \psi(s) > 0. \tag{1}$$

Then $\breve{T}$ admits a unique fixed point $\tilde{x}^\star \in \tilde{X}$. In addition, $\tilde{x}^\star$ is an approximate fixed point as well as a contractive fixed point.

There are multiple auxiliary definitions and results that will be used for the proof of Theorem 3.3.

Firstly, we introduce some notations:

Let $\{\tilde{b}_{p_{i_n}}\} \subset (0,1]\hat{P}$ be a nondecreasing sequence with the property that $\tilde{b}_{p_{i_n}} \to 1$ as $n \to \infty$ and define the collection

$$\mathscr{A}_n = \{\tilde{x}_{p_i} \in \tilde{X} : \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \breve{T}\tilde{x}_{p_i}, \tilde{r}) \geq \tilde{b}_{p_{i_n}}\},\ for\ n \in \mathbb{N}. \tag{2}$$

Lemma 3.4. *Let $\breve{T}$ be a continuous ASFψCM mapping from $\tilde{X}_{\hat{P}}$ into itself. Then,*

$$\limsup_{n\to\infty}\ \mathscr{S}_{\mathscr{F}}\left(T^n\tilde{x}_{p_i}, \breve{T}^n\tilde{y}_{p_j}, \tilde{r}\right) = 1,\ \forall\ \tilde{x}_{p_i}, \tilde{y}_{p_j} \in X\ and\ \tilde{r} > 0. \tag{3}$$

Proof. Condition (ASF3) of Definition 3.2 is trivial if $\tilde{x}_{p_i} = \tilde{y}_{p_j}$.
Consider $\tilde{x}_{p_i} \neq \tilde{y}_{p_j}$ and $0 < l$. Then,

$$\limsup_{n\to\infty}\ \mathscr{S}_{\mathscr{F}}(\breve{T}^n\tilde{x}_{p_i}, T^n\tilde{y}_{p_j}, \tilde{r}) < 1.$$

If $\exists\ k \in \mathbb{N}$ such that $\breve{T}^k\tilde{x}_{p_i} = \breve{T}^k\tilde{y}_{p_j}$, then continuity of $\breve{T}$ implies

$$\breve{T}^{k+1}\tilde{x}_{p_i} = \breve{T}(\breve{T}^k\tilde{x}_{p_i}) = \breve{T}(\breve{T}^k\tilde{y}_{p_j}) = \breve{T}^{k+1}\tilde{y}_{p_j}.$$

Hence $\breve{T}^n\tilde{x}_{p_i} = \breve{T}^n\tilde{y}_{p_j}$ for every $n \in \mathbb{N}$ which implies $\lim_{n\to\infty}\mathscr{S}_{\mathscr{F}}(T^n\tilde{x}_{p_i}, \breve{T}^n\tilde{y}_{p_j}, \tilde{r}) = 1$.

Let $\breve{T}^k\tilde{x}_{p_i} \neq \breve{T}^k\tilde{y}_{p_j}$ for any $k \in \mathbb{N}$ and $\tilde{x}_{p_i}, \tilde{y}_{p_j} \in \tilde{X}$, then, by $(ASF1)$, we get

$$\mathscr{S}_{\mathscr{F}}(\breve{T}^{n+k}\tilde{x}_{p_i}, \breve{T}^{n+k}\tilde{y}_{p_j}, \tilde{r}) \geq \psi_n(\mathscr{S}_{\mathscr{F}}(\breve{T}^k\tilde{x}_{p_i}, \breve{T}^k\tilde{y}_{p_j}, \tilde{r})),\ \text{for every } k \in \mathbb{N} \text{ and } \tilde{r} > 0,\ i.e.$$

$$\begin{aligned} l &= \limsup_{n\to\infty}\ \mathscr{S}_{\mathscr{F}}(\breve{T}^{n+k}\tilde{x}_{p_i}, \breve{T}^{n+k}\tilde{y}_{p_j}, \tilde{r}) \\ &\geq \limsup_{n\to\infty}\ \psi_n(\mathscr{S}_{\mathscr{F}}(\breve{T}^k\tilde{x}_{p_i}, \breve{T}^k\tilde{y}_{p_j}, \tilde{r})) \\ &= \psi(\mathscr{S}_{\mathscr{F}}(\breve{T}^k\tilde{x}_{p_i}, \breve{T}^k\tilde{y}_{p_j}, r)). \end{aligned}$$

Specifically, for every $\tilde{r} > 0$, $l \geq \psi(\mathscr{S}_{\mathscr{F}}(\breve{T}^{k_m}\tilde{x}_{p_i}, \breve{T}^{k_m}\tilde{y}_{p_j}, \tilde{r}))$, where $\{k_m\}$ is strictly increasing sequence of non negative integers satisfying

$$M(\breve{T}^{k_m}\tilde{x}_{p_i}, \breve{T}^{k_m}\tilde{y}_{p_j}, \tilde{r}) \to l \text{ as } m \to \infty.$$

Therefore, using ψ's continuity and by applying limit $m \to \infty$ in the previous inequality, we obtain $l \geq \psi(l)$, which is a contradiction to property $(ASF3)$.

$$\Rightarrow\ \limsup_{n\to\infty}\ \mathscr{S}_{\mathscr{F}}(\breve{T}^n\tilde{x}_{p_i}, \breve{T}^n\tilde{y}_{p_j}, \tilde{r}) = 1.$$

□

Lemma 3.5. *Let* $(\tilde{X}_{\hat{P}}, \mathscr{S}_{\mathscr{F}}, \tilde{\star})$ *be a complete SFMS and* $\check{T} : \tilde{X} \to \tilde{X}$ *be an ASFψCM such that* $\lim_{s\to 1} \psi_1(s) = 1$. *Let* $\{\tilde{x}_{p_{i_n}}\} \subseteq \tilde{X}$ *be any sequence satisfying the following condition:*

$$\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \check{T}\tilde{x}_{p_{i_n}}, \tilde{r}) \to 1 \text{ as } n \to \infty \ \forall \ \tilde{r} > 0.$$

Then, $\forall \ k \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}} T^k \tilde{x}_{p_{i_n}}, \tilde{r}) = 1, \quad \text{for every } \tilde{r} > 0. \tag{4}$$

Proof. By applying the induction principle; equation (4) is satisfied for $k = 1$.
Let equation (4) holds for some $k \in \mathbb{N}$. Then, by the ASFψ-contractivity, and the assumption on ψ_1, we obtain

$$\begin{aligned}
\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \check{T}^{k+1}\tilde{x}_{p_{i_n}}, \tilde{r}) &\geq \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \check{T}\tilde{x}_{p_{i_n}}, \tilde{r}/2) \star \mathscr{S}_{\mathscr{F}}(\check{T}\tilde{x}_{p_{i_n}}, \check{T}(\check{T}^k\tilde{x}_{p_{i_n}}), \tilde{r}/2) \\
&\geq \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \check{T}\tilde{x}_{p_{i_n}}, \tilde{r}/2) \star \psi_1(\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \check{T}^k\tilde{x}_{p_{i_n}}, \tilde{r}/2)) \\
&\to 1 \text{ as } n \to \infty.
\end{aligned}$$

Thus, $\lim_{n\to\infty} \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \check{T}^{k+1}\tilde{x}_{p_{i_n}}, \tilde{r}) = 1$, and equation (4) holds for $k+1$. □

Lemma 3.6. *Let* $(\tilde{X}_{\hat{P}}, \mathscr{S}_{\mathscr{F}}, \tilde{\star})$ *be a complete SFMS and* A_n *be defined as in equation (2). Then* $\operatorname{diam}(A_n) \to 1$ *as* $n \to \infty$ *if and only if, for any two sequences* $\{\tilde{x}_{p_{i_n}}\}$ *and* $\{\tilde{y}_{p_{j_n}}\}$ *in* A_n, *we have*

$$\begin{aligned}
&\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \check{T}\tilde{x}_{p_{i_n}}, \tilde{r}) \to 1 \text{ and} \\
&\mathscr{S}_{\mathscr{F}}(\tilde{y}_{p_{j_n}}, \check{T}\tilde{y}_{p_{j_n}}, \tilde{r}) \to 1 \text{ as } n \to \infty \\
&\Rightarrow M(\tilde{x}_{p_{i_n}}, \tilde{y}_{p_{j_n}}, \tilde{r}) \to 1 \text{ as } n \to \infty. \qquad (5)
\end{aligned}$$

Proof. The necessary condition is immediate from the triangle inequality axiom of the fuzzy metric. For sufficient condition, observe that $A_{n+1} \subseteq A_n$. Then, by Proposition 2.8, it follows that

$$\operatorname{diam}(A_{n+1}) \geq \operatorname{diam}(A_n).$$

Since we have a monotonic sequence bounded above by 1, therefore $\operatorname{diam}(A_n) \to l$ as $n \to \infty$ for some $l \in [0,1]$. Suppose that $l \neq 1$ *i.e.* $0 \leq l < 1$, then, $\exists \ 0 < s \leq l$ such that

$$0 \leq \operatorname{diam}(A_n) < 1 - s \ \forall \ n \in \mathbb{N}.$$

Hence, there are $\tilde{x}_{p_{i_n}}, \tilde{y}_{p_{j_n}} \in A_n$ such that

$$0 \leq \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \tilde{y}_{p_{j_n}}, \tilde{r}) < 1 - s.$$

so that $\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \tilde{y}_{p_{j_n}}, \tilde{r}) \not\to 1$ as $n \to \infty$. On the other hand, equation (5) ensures that both sequences

$$\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \breve{T}\tilde{x}_{p_{i_n}}, \tilde{r}) \to 1 \quad \text{and} \quad \mathscr{S}_{\mathscr{F}}(\tilde{y}_{p_{j_n}}, \breve{T}\tilde{y}_{p_{j_n}}, \tilde{r}) \to 1 \quad \text{as } n \to \infty.$$

By the hypothesis, this implies that $\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \tilde{y}_{p_{j_n}}, \tilde{r}) \to 1$, which contradicts the assumption that $l < 1$. Therefore, we must have $l = 1$. □

Lemma 3.7. *Let $\breve{T}$ be an ASFψCM such that either $\breve{T}$ is uniformly continuous or* $\lim_{s\to 1} \psi_1(s) = 1$. *Let A_n be defined as in (2). Then,* $\operatorname{diam}(A_n) \to 1$ *as $n \to \infty$.*

Proof. Following Lemma 3.6, let sequences $\{\tilde{x}_{p_{i_n}}\}$ and $\{\tilde{y}_{p_{j_n}}\}$ be such that

$$\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \breve{T}\tilde{x}_{p_{i_n}}, \tilde{r}) \to 1 \quad \text{and } \mathscr{S}_{\mathscr{F}}(\tilde{y}_{p_{j_n}}, \breve{T}\tilde{y}_{p_{j_n}}, \tilde{r}) \to 1 \textit{ as } n \to \infty \text{ for all } \tilde{r} > 0.$$

Define

$$\begin{aligned} a_n(\tilde{r}) &= \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \tilde{y}_{p_{j_n}}, \tilde{r}) \textit{ and} \\ b_{n,k}(\tilde{r}) &= \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \breve{T}^k\tilde{x}_{p_{i_n}}, \tilde{r}) \,\tilde{\star}\, \mathscr{S}_{\mathscr{F}}(\tilde{y}_{p_{j_n}}, \breve{T}^k\tilde{y}_{p_{j_n}}, \tilde{r}), \quad \forall\, k \in \mathbb{N}. \end{aligned}$$

Consider $\{\psi_n\}$ and ψ as discussed in Definition 3.2. Then, for any $\varepsilon \in (0,1)$, $\exists\, k \in \mathbb{N}$ such that

$$\psi^k(\tilde{r}) > \psi(\tilde{r}) \,\tilde{\star}\, \varepsilon, \quad \forall\, \tilde{r} > 0. \tag{6}$$

So, by (ASF1) and (6), we have

$$\begin{aligned} a_n(3\tilde{r}) &\geq \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \breve{T}^k\tilde{x}_{p_{i_n}}, \tilde{r}) \,\tilde{\star}\, \mathscr{S}_{\mathscr{F}}(\tilde{y}_{p_{j_n}}, \breve{T}^k\tilde{y}_{p_{j_n}}, \tilde{r}) \,\tilde{\star}\, \mathscr{S}_{\mathscr{F}}(\breve{T}^k\tilde{x}_{p_{i_n}}, \breve{T}^k\tilde{y}_{p_{j_n}}, \tilde{r}) \\ &\geq b_{n,k}(\tilde{r}) \,\tilde{\star}\, \psi_k(\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \tilde{y}_{p_{j_n}}, \tilde{r})) \\ &= b_{n,k}(\tilde{r}) \,\tilde{\star}\, \psi_k(a_n(\tilde{r})) \\ &\geq b_{n,k}(\tilde{r}) \,\tilde{\star}\, \psi(a_n(\tilde{r})) \,\tilde{\star}\, \varepsilon. \end{aligned} \tag{7}$$

Hence, by Lemma 3.5,

$$\lim_{n\to\infty} \mathscr{S}_{\mathscr{F}}(\breve{T}\tilde{x}_{p_{i_n}}, \breve{T}^2\tilde{x}_{p_{i_n}}, \tilde{r}) = 1, \quad \forall\, \tilde{r} > 0,$$

and, similarly,

$$\lim_{n\to\infty} \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \breve{T}^2\tilde{x}_{p_{i_n}}, \tilde{r}) = 1, \quad \forall\, \tilde{r} > 0.$$

Analogously,

$$\lim_{n\to\infty} \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_n}}, \breve{T}^k\tilde{x}_{p_{i_n}}, \tilde{r}) = 1, \quad \forall\, \tilde{r} > 0,$$

for every $k \in \mathbb{N}$, and, analogously for $\{\tilde{y}_{p_{j_n}}\}$. This shows that, for any $k \in \mathbb{N}$, $\lim_{n\to\infty} b_{n,k}(\tilde{r}) = 1$. The limit on both sides of (7) is thus obtained as

$$\lim_{n\to\infty} a_n(3\tilde{r}) \geq \varepsilon \,\tilde{\star}\, \lim_{n\to\infty} \psi(a_n(\tilde{r})). \tag{8}$$

As $\varepsilon \in (0,1)$, we get

$$\lim_{n\to\infty} a_n(3\tilde{r}) \geq \lim_{n\to\infty} \psi(a_n(\tilde{r})).$$

Assume, for the sake of contradiction, that

$$\lim_{n\to\infty} a_n(\tilde{r}) = s \neq 1.$$

Then the previous inequality implies

$$\lim_{n\to\infty} a_n(3\tilde{r}) \geq \lim_{n\to\infty} \psi(a_n(\tilde{r})),$$

which gives $s \geq \psi(s) > s$, a contradiction. Therefore,

$$\lim_{n\to\infty} a_n(\tilde{r}) = 1.$$

The conclusion then follows directly by applying Lemma 3.6.

□

Proof. (Proof of Theorem 3.3) Let A_n be defined as in equation (2). By Lemma 3.3, for every $\tilde{x}_{p_i} \in \tilde{X}$ and $\tilde{r} > 0$,

$$\lim_{n\to\infty} \mathscr{S}_{\mathscr{F}}(\breve{T}^n \tilde{x_{p_i}}, \breve{T}^{n+1} x_{p_i}, \tilde{r}) = 1,$$

which ensures that each $A_n \neq \Phi$.

Take any $\tilde{x}_{p_i}, \tilde{y}_{p_j} \in A_n$. Then

$$\begin{aligned}
\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \tilde{x}_{p_j}, \tilde{r}) &\geq \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \breve{T}\tilde{x}_{p_i}, \tilde{r}/3) \,\tilde{\star}\, \mathscr{S}_{\mathscr{F}}(\tilde{y}_{p_j}, \breve{T}\tilde{y}_{p_j}, \tilde{r}/3) \,\tilde{\star}\, \mathscr{S}_{\mathscr{F}}(\breve{T}x_{p_i}, \breve{T}y_{p_j}, \tilde{r}/3) \\
&\geq \tilde{b}_{p_{i_n}} \,\tilde{\star}\, \tilde{b}_{p_{i_n}} \,\tilde{\star}\, \psi(\tilde{b}_{p_{i_n}}) \\
&> \tilde{b}_{p_{i_n}} \,\tilde{\star}\, \tilde{b}_{p_{i_n}} = \tilde{b}_{p_{i_n}},
\end{aligned}$$

where $\{\tilde{b}_{p_{i_n}}\} \subset (0,1]$ is a sequence with $\tilde{b}_{p_{i_n}} \to 1$ and $\tilde{b}_{p_{i_n}} \,\tilde{\star}\, \tilde{b}_{p_{i_n}} = \tilde{b}_{p_{i_n}}$, as guaranteed by the continuity of the H-type t-norm $\tilde{\star}$. Hence, the family $\{A_n\}$ has fuzzy diameter zero.

It follows that

$$\bigcap_{n\in\mathbb{N}} A_n = \tilde{x}^{\star}_{p_i} = \mathrm{Fix}(\breve{T}).$$

Next, we show that $\tilde{x}^{\tilde{\star}}_{p_i}$ is an approximate fixed point.

Let $\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \breve{T}\tilde{x}_{p_i}, \tilde{r}) \to 1$ and set $\tilde{y}_{p_j} = \tilde{x}^{\tilde{\star}}_{p_i}$ $\forall n \in \mathbb{N}$. Then result of Lemma 3.6 yields $\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \tilde{x}^{\tilde{\star}}_{p_i}, \tilde{r}) \to 1$.

Finally, for any $\tilde{x} \in \tilde{X}$, let $\tilde{x_n} = \breve{T}^n\tilde{x}$. Then $\tilde{M}(x_{p_{i_n}}, \breve{T}\tilde{x}_{p_{i_n}}, \tilde{r}) \to 1$ which implies $\breve{T}^n\tilde{x}_{p_i} \to \tilde{x}^{\tilde{\star}}_{p_i}$, so that $\tilde{x}^{\tilde{\star}}_{p_i}$ is a contractive fixed point. □

Proposition 3.8. *Let $\psi(\tilde{r}) = 1$ for every $\tilde{r} \in (0,1)$. Then $\breve{T}$ is an ASFψC if and only if the fuzzy diameter of $\breve{T}^n(\tilde{X}) = 0$.*

Proof. Let $\varepsilon > 0$. Then, from (ASF3), we obtain $\psi(\tilde{r}) > \tilde{r}$, $\forall\, m \in \mathbb{N}$ such that $\psi_m(\tilde{r}) > 1-\varepsilon$. Let $\tilde{x}_{p_i}, \tilde{y}_{p_j} \in \tilde{X}$. As $\breve{T}$ is an ASFψC, we get

$$\mathscr{S}_{\mathscr{F}}(T^n\tilde{x}_{p_i}, T^n\tilde{y}_{p_j}, s) \geq \psi^n(\mathscr{S}_{\mathscr{F}}(x_{p_i}, y_{p_j}, s)) > 1-\varepsilon, \quad \forall\, n \geq m, \forall\, s > 0.$$

Hence $\psi(T^n(\tilde{X})) > 1-\varepsilon \Rightarrow T^n(\tilde{X}) = 0$.

For the sufficiency, it is enough to set

$$\psi_n(s) = \mathrm{diam}(\breve{T}^n(\tilde{X})), \quad \forall\, n \in \mathbb{N}.$$

□

Proposition 3.9. *Let $(\tilde{X}, \mathscr{S}_{\mathscr{F}}, \tilde{\star})$ be a complete SFMS and T be an ASFψ-contraction. Then $\breve{T}$ is surjective $\Longleftrightarrow \tilde{X}$ is a unique.*

Proof. The "if" part is trivial. Suppose conversely that $\breve{T}$ is surjective and $\tilde{X}$ is not a singleton set. Then the fuzzy diameter of $\tilde{X} = g < 1$ *i.e.*

$$\psi_{\tilde{X}}(\tilde{r}) := \inf\{\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \tilde{y}_{p_j}, \tilde{r}) : \tilde{x}_{p_i}, \tilde{y}_{p_j} \in \tilde{X}\} = g < 1, \quad \forall\, \tilde{r} > 0.$$

Since $\breve{T}$ is an ASFψ-contraction, we have

$$\psi(\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \tilde{y}_{p_j}, \tilde{r})) \geq \psi(g) > g, \quad \forall\, \tilde{x}_{p_i}, \tilde{y}_{p_j} \in \tilde{X},\ \tilde{r} > 0.$$

Let $\varepsilon = \frac{g}{\psi(g)} \in (0,1)$. By the definition of ASF ψ-contraction, $\exists\, k \in \mathbb{N}$ such that

$$\psi_k(\tilde{r}) > \psi(\tilde{r}) \cdot \varepsilon, \quad \forall\, \tilde{r} \in (0,1).$$

Thus, for any $\tilde{x}_{p_i}, \tilde{y}_{p_j} \in \tilde{X}$ and $s > 0$,

$$\begin{aligned} SF(\breve{T}^k \tilde{x}_{p_i}, \breve{T}^k \tilde{y}_{p_j}, s) &\geq \psi_k(M(\tilde{x}_{p_i}, \tilde{y}_{p_j}, s)) \\ &> \psi(\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, y_{p_j}, s)) \cdot \varepsilon \\ &= g. \end{aligned}$$

This means $\psi_{\breve{T}^k(\tilde{X})}(s) > g$, *i.e.* the fuzzy diameter of $\breve{T}^k(\tilde{X})$ is strictly greater than g.

But since $\breve{T}$ is surjective, so is $\breve{T}^k$ and hence $\breve{T}^k(\tilde{X}) = \tilde{X}$. Thus, the fuzzy diameter of $\breve{T}^k(\tilde{X})$ equals the fuzzy diameter of $\tilde{X}$, which is g.
This contradiction shows that $\tilde{X}$ is unique. □

4. Asymptotic Soft Contraction on Compact Spaces

The following result provides equivalent characterizations of asymptotic soft fuzzy ψ-contractions when the underlying space is compact.

Theorem 4.1. *Let $(\tilde{X}, \mathscr{S}_{\mathscr{F}}, \tilde{\star})$ be a compact SFMS and $\breve{T} : \tilde{X} \to \tilde{X}$ be a continuous map. Then, the following conditions are equivalent:*

(i) $\breve{T}$ is an ASFψ-contraction;

(ii) the core set $\tilde{Y} = \bigcap_{n\in\mathbb{N}} \breve{T}^n(\tilde{X})$ consists of a single point;

(iii) $\breve{T}$ is an ASFψ-contraction with $\psi_1(\tilde{r}) = 1 \;\forall\; \tilde{r} \in (0,1)$.

Proof. **(i) $\Rightarrow$ (iii):** Following [[11], Proposition 2], $\breve{T}$ maps $\tilde{Y}$ onto itself, and the restriction $\breve{T}|_{\tilde{Y}}$ is also an ASFC, it follows that $\tilde{Y}$ is a singleton.

(ii) $\Rightarrow$ (iii): Again, by [[11], Proposition 2], the fuzzy diameter of $\breve{T}^n(\tilde{X})$ converges to the fuzzy diameter of $\tilde{Y}$. Since $\tilde{Y}$ is a singleton, this limit equals 1. Then, by Proposition 3.8, $\breve{T}$ is an ASFψC with $\psi_1(\tilde{r}) \equiv 1$.

(iii) $\Rightarrow$ (i): By Proposition 3.8, the implication is immediately applicable. ☐

Example 4.2. Assume $\tilde{X} = \{\{\tilde{x}_{p_{i_n}}\} : n \in \mathbb{N}\} \cup \{1\}$, where $\{\tilde{x_{p_{i_n}}}\} \in (0,1)\hat{P}$ is an increasing sequence such that $\tilde{x}_{p_{i_n}} < \tilde{x}_{p_{i_{n+1}}}$ and $\lim_{n\to\infty} \tilde{x}_{p_{i_n}} = 1$.

Define $\mathscr{S}_{\mathscr{F}} : \tilde{X} \times \tilde{X} \times (0,\infty)\hat{P} \to [0,1]\hat{P}$ by

$$\mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_i}, \tilde{y}_{p_j}, \tilde{r}) = \begin{cases} 1, & \tilde{x}_{p_i} = \tilde{y}_{p_j}, \\ \min\{\tilde{x}_{p_i}, \tilde{y}_{p_j}\}, & \tilde{x}_{p_i} \neq \tilde{y}_{p_j}, \end{cases} \quad \forall\, \tilde{x}_{p_i}, \tilde{y}_{p_j} \in \tilde{X},\ \tilde{r} > 0.$$

Then, $(\tilde{X}, \mathscr{S}_{\mathscr{F}}, \tilde{\star}_m)$ is a complete SFMS. Define $\breve{T} : \tilde{X} \to \tilde{X}$ as follows:
Let $k_n = n(n+1)/2$ and set

$$\breve{T}\tilde{x}_{p_{i_0}} = \tilde{x}_{p_{i_0}}, \quad \tilde{x}_{p_{i_0}} = 1,$$
$$\breve{T}\tilde{x}_{p_{i_{k_n}}} = \tilde{x}_{p_{i_{k_{n+2-1}}}}, \quad n \in \mathbb{N},$$
$$\breve{T}\tilde{x}_{p_{i_l}} = \tilde{x}_{p_{i_{l-1}}}, \quad l \in \mathbb{N} \setminus \{k_n : n \in \mathbb{N}\}.$$

Clearly, $\breve{T}$ is a continuous map. Hence

$$\breve{T}^{k_n+1-2}(\tilde{X}) \subset [\tilde{x}_{p_{i_{k_n}}}, \tilde{x_{p_{i_0}}}], \quad n \in \mathbb{N}.$$

Since $\{\breve{T}^n(\tilde{X})\}_{n=1}^{\infty}$ is increasing, we obtain

$$\bigcap_{n\in\mathbb{N}} \breve{T}^n(\tilde{X}) = \bigcap_{n\in\mathbb{N}} \breve{T}^{k_n+1-2}(X) \subseteq \bigcap_{n\in\mathbb{N}} [x_{k_n}, x_{p_{i_0}}] = \{x_{p_{i_0}}\}.$$

Thus, $\bigcap_{n\in\mathbb{N}} \check{T}^n(\tilde{X}) = \{x_{p_{i_0}}\}$, and hence by Theorem 4.1, T is an ASFψC.
However, note that

$$\begin{aligned}
\mathscr{S}_{\mathscr{F}}(\check{T}\tilde{x}_{p_{i_2}}, \check{T}\tilde{x}_{p_{i_0}}, \tilde{r}) &= \min\{\tilde{x}_{p_{i_1}}, \tilde{x}_{p_{i_0}}\} \\
&= \tilde{x}_{p_{i_1}} \\
&< \tilde{x}_{p_{i_2}} \\
&= \min\{\tilde{x}_{p_{i_2}}, \tilde{x}_{p_{i_0}}\} \\
&= \mathscr{S}_{\mathscr{F}}(\tilde{x}_{p_{i_2}}, \tilde{x}_{p_{i_0}}, \tilde{r}),
\end{aligned}$$

which shows that $\check{T}$ is not fuzzy non-expansive. Therefore, the collection of all asymptotic soft fuzzy contractions cannot be identified with any previously known class of fuzzy contractive mappings.

5. Conclusion

Fixed Point Theory in SFMS extends classical and fuzzy results by incorporating vagueness and parameter-dependence. Soft sets handle multi-parameter uncertainty, while fuzzy metrics address imprecise distances, providing a richer framework for fixed point theorems. This work establishes results for asymptotic soft fuzzy ψ-contractions , offering genuine extensions beyond classical theory. The framework also enables the generalization of other contraction types, such as Caristi-type and Suzuki-type mappings, to the soft fuzzy setting. Overall, SFMS provides a fertile ground for novel fixed point principles with applications in optimization, decision-making, and modeling under uncertainty.

Conflict of Interests

The authors declare that there is no conflict of interests.

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