Available online at http://scik.org
Adv. Fixed Point Theory, 2026, 16:2
https://doi.org/10.28919/afpt/9683
ISSN: 1927-6303

EXISTENCE OF APPROXIMATE BEST PROXIMITY POINT RESULTS ON EXTENDED b-METRIC SPACES AND ITS APPLICATIONS

A. MARY PRIYA DHARSINI[1], K. MAHESHWARAN[2], T. C. MUJEEBURAHMAN[2], P. JOSELIN LAVINO[1], R. THEIVARAMAN[3,*]

[1]PG and Research Department of Mathematics, Holy Cross College (Autonomous), Tiruchirappalli- 620 002, Tamilnadu, India

[2]Department of Mathematics, School of Engineering and Technology, Dhanalakshmi Srinivasan University, Perambalur-621 212, Tamilnadu, India

[3]Department of Nautical Science, Academy of Maritime Education and Training (AMET) University, Kanathur-603 112, Chennai, Tamilnadu, India

Abstract. The aim of this paper is to extend the concept of determining an approximate fixed points (AFP) to determine approximate best proximity points ($ABPP$) using several contraction mappings, such as weak contraction, Zamfirescu contraction, Ciric-Reich-Rus contraction and the related consequences on b-metric spaces. In particular, we study the existence (qualitative results) and the diameter (quantitative results) of $ABPP$ on b-metric spaces. Moreover, a few examples are provided to illustrate our results. Furthermore, a suitable applications of the main findings are discussed in the domain of differential equations.

Keywords: b-metric space; weak contraction; Zamfirescu contraction; Ciric-Reich-Rus contraction; approximate best proximity point.

2020 AMS Subject Classification: 47H10, 54H25.

[*]Corresponding author

E-mail addresses: deiva@ametuniv.ac.in; deivaraman@gmail.com

Received November 06, 2025

1. Introduction

In recent decades, nonlinear functional analysis-particularly best proximity point (BPP) theory-has been extended to a variety of abstract spaces. It has been widely applied to numerical scientific problems, bridging both applied and pure mathematical methods, especially in relation to computational complexities. Further, BPP theory has played a typical role in modeling and analyzing a broad range of real-world phenomena and applications, including the study of integral and differential equations, as well as in physics, economics, social sciences, biology, and the field of engineerings.

In 1922, one of the most influential and fundamental results in this area, known as Banach contraction principle $[BCP]$ was established by Stefan Banach [1]. Since then, the field of fixed point $[FP]$ theory has witnessed significant developments and wide-ranging applications. Naturally, the conditions for $FP's$ existence are very strict. As a result, there is no assurance that $FP's$ will always exists. In the absence of exact FP, the BPP may be used because the FP methods have overly strict limitations. This is the primary reason for attempting to locate BPP on metric spaces. When a direct solution is not feasible, especially for non-self mappings, BPP theory, a generalization of FP theory, are essential for locating the best approximation solutions to the equation $\nabla a = a$, where $\nabla : \nabla^{\star} \to \nabla^{⁂}$ are disjoint subsets of a metric space. In the similar way, the AFP, was also developed and the approximate solutions (only ε-differences) are determined in many complicated situations. In this way, AFP and $ABPP$ theory has been researched in various metric spaces over the decades by several researchers.

$$FP's \Longrightarrow AFP's \Longrightarrow BPP's \Longrightarrow ABPP's$$

Notably, the author Berinde [7] proved existence and the diameter of AFP results using various contraction-type operators (Kannan, Chatterjea, Zamfirescu, and weak contractions) on metric spaces. Later, Dey and Saha [8] extended these results, and they found the diameter of the AFP for the Reich operator tends to zero when ε approaches zero. In the same manner, S. A. M. Mohsenialhosseini (see, [9],[10],[11]) derived some new AFP results for cyclical contraction mappings and extended these results to a family of contraction mappings and found a common FP for the Mohseni-Saheli contraction mapping. Following that, Theivaraman et al. extended the concept and determined more fruitful AFP and $ABPP$ results on metric spaces ,

b-metric spaces , G-metric spaces and the related subsequences using several contraction mappings (refer, [13],[14],[15],[16],[17]).

The concept of a b-metric space was initially introduced by Bakhtin [2] in 1989, providing a natural generalization of the classical metric space by relaxing the standard triangle inequality. This generalization has opened new directions in the study of FP theory, particularly where the traditional framework of metric spaces proves too restrictive. Following Bakhtin's work, several authors have contributed to the development of FP results in b-metric spaces ([3],[5],[6],[18],[19]). These contributions have significantly expanded the theoretical foundation and applicability of FP theorems beyond conventional settings.

In a notable advancement, Czerwik [4] formally discussed the notion of a b-metric space in 1993, with the explicit aim of generalizing the well-known *BCP*. Also, this formulation provided a more flexible analytical framework, which has since attracted considerable attention from researchers seeking to extend classical results. Subsequent studies have produced various generalizations and applications of FP theorems in b-metric spaces. In particular, there has been interest in extending the Banach FP theorem to address the convergence of measurable functions with respect to measure, thereby enhancing the relevance of FP theory in the broader context of analysis and applied mathematics.

The structure of the paper is organized as follows. Section 1 provides a general introduction and motivation for the study. In Section 2, we revisit fundamental concepts and relevant results from the existing literature that form the basis for our work. Section 3 is devoted to the presentation of our main contributions, where we establish new results concerning *ABPP* results in the setting of b-metric spaces. These results are derived using various types of contraction mappings, including Zamfirescu contractions, weak contractions, and others. A particular focus is placed on analyzing the diameter of $ABPP's$ for a given pair $(\mathfrak{I},\mathfrak{R})$, drawing upon and extending the frameworks provided in [16]. In Section 4, we illustrate the applicability of our main results by exploring their implications in the context of differential equations. Finally, Section 5 concludes the paper with a summary of findings and potential directions for future research.

2. PRELIMINARIES

This section reviews essential definitions and lemmas from earlier studies, which serve as foundational tools for the main results presented in the remainder of this manuscript.

Definition 2.1. [3] *Let $\eth$ be a non-empty set and $b \geq 1$ be a given real number. A function $d : \eth \times \eth \longrightarrow \mathbb{R}_+$ is called a b-metric provided that for all $\ell, \iota, \varsigma \in \nabla$ satisfies the following conditions.*

(i) $d(\ell,\iota) = 0$ *iff* $\ell = \iota$;

(ii) $d(\ell,\iota) = d(\iota,\ell)$;

(iii) $d(\ell,\iota) \leq b[d(\ell,\varsigma) + (\varsigma,\iota)]$

The pair $(\eth, d)$ is called a b-metric space. Immediately from the notion of b-metric space we have the result every metric space is a b-metric space with $b = 1$. But the converse does not hold.

Example 2.2. [3] *Let* $\eth = \{0,1,2\}$ *and* $d(2,0) = d(0,2) = \varkappa \geq 2$
$d(0,1) = d(1,2) = d(1,0) = d(2,1) = 1$ *and*
$d(0,0) = d(1,1) = d(2,2) = 0$
Then, $d(\ell,\iota) \leq \frac{\varkappa}{2}[d(\ell,\varsigma) + d(\varsigma,\iota)]$ *for all* $\ell, \iota, \varsigma \in \eth$.
if $\varkappa > 2$ *then the triangle inequality does not hold.*

Definition 2.3. [9],[11] *Let $\Im$ and $\Re$ be two nonempty subsets of a b-metric space $\eth$ and $\nabla : \Im \cup \Re \to \Re \cup \Im$ such that $\nabla(\Im) \subseteq \Re$ and $\nabla(\Re) \subseteq \Im$. Then ℓ is said to be an approximate best proximity point of the pair $(\Im, \Re)$, if*

$$d(\ell, \nabla\ell) \leq d(\Im, \Re) + \varepsilon.$$

Remark 2.4. [9],[11] *Let*

$$BPP^{\varepsilon}(\Im, \Re) = \{\ell \in BPP^{\varepsilon}(\Im, \Re) : d(\ell, \nabla\ell) < d(\Im, \Re) + \varepsilon, \text{ for some } \varepsilon > 0\}$$

be denotes the set of all approximate best proximity pairs of pair $(\Im, \Re)$ for a given $\varepsilon > 0$. Also the pair $(\Im, \Re)$ is said to be an approximate best proximity pair property, if

$$d(\ell, \nabla\ell) \leq d(\Im, \Re) \neq 0.$$

Example 2.5. *Let us take* $\eth = \mathbb{R}^2$ *and* $\Im = \{(\ell,\imath) \in \eth : (\ell-\imath)^2+\imath^2 \leq 1\}$ *and* $\Re = \{(\ell,\imath) \in \eth : (\ell+\imath)^2+\imath^2 \leq 1\}$ *with* $B(\ell,\imath) = (-\ell,\imath)$ *for* $(\ell,\imath) \in \eth$. *Then*

$$d((\ell,\imath),B(\ell,\imath)) \leq d(\Im,\Re)+\varepsilon \text{ for some } \varepsilon > 0.$$

Hence,

$$BPP^{\varepsilon}(\Im,\Re) \neq \emptyset.$$

Theorem 2.6. [9],[11] *Let* $\Im$ *and* $\Re$ *be two nonempty subsets of a b-metric space* $(\eth,d)$. *Suppose that the mapping* $\nabla : \Im \cup \Re \to \Im \cup \Re$ *satisfying* $\nabla(\Im) \subseteq \Re$ *and* $\nabla(\Re) \subseteq \Im$ *and*

$$\lim_{n\to\infty} d(\nabla^n \Im, \nabla^{n+1}\Im) = d(\Im,\Re), \text{ for some } \ell \in (\Im \cup \Re).$$

Then the pair $(\Im,\Re)$ *is called an approximate best proximity pair.*

Definition 2.7. [9],[11] *Let* $\nabla : \Im \cup \Re \to \Im \cup \Re$ *be a continuous map such that* $\nabla(\Im) \subseteq \Re, \nabla(\Re) \subseteq \Im$ *and* $\varepsilon > 0$. *Then, we define the diameter* $\Delta(BPP^{\varepsilon}(\Im,\Re))$*, i.e.,*

$$\Delta(BPP^{\varepsilon}(\Im,\Re)) = sup\{d(\ell,\imath) : \ell,\imath \in BPP^{\varepsilon}(\Im,\Re)\}.$$

Theorem 2.8. [9],[11] *Let* $\Im$ *and* $\Re$ *be two non-empty subsets of a metric space* $(\eth,d)$. *Suppose that a mapping* $\nabla : \Im \cup \Re \to \Im \cup \Re$ *satisfying* $\nabla(\Im) \subseteq \Re, \nabla(\Re) \subseteq \Im$ *is a* α*-contraction and* $\varepsilon > 0$. *Suppose that:*

- *(i)* $BPP^{\varepsilon}(\Im,\Re) \neq \emptyset$;
- *(ii) for every* $\varphi > 0$, *there exists* $\theta(\varphi) > 0$ *such that* $d(\ell,\imath) - d(\nabla\ell,\nabla\imath) \leq \varphi$ *implies that* $d(\ell,\imath) \leq \theta(\varphi)$, *for every* $\ell,\imath \in BPP^{\varepsilon}(\Im,\Re) \neq \emptyset$.

Then,

$$\Delta(BPP^{\varepsilon}(\Im,\Re)) \leq \theta(2d(\Im,\Re)+\varepsilon).$$

Definition 2.9. [7] *Let* $\Im$ *and* $\Re$ *be two non-empty subsets of a b-metric space* $(\eth,d)$. *A selfmap* $\nabla : \Im \cup \Re \to \Im \cup \Re$ *satisfying* $\nabla(\Im) \subseteq \Re, \nabla(\Re) \subseteq \Im$ *is a weak contraction if there exists* $\alpha \in (0,\frac{1}{2})$ *and* $k \geq 0$ *with* $k\alpha < 1$ *such that*

$$d(\nabla\ell,\nabla\imath) \leq \alpha d(\ell,\imath) + k d(\imath,\nabla\ell), \text{ for all } \ell,\imath \in \Im \cup \Re.$$

Definition 2.10. [7] *Let $\mathfrak{I}$ and $\mathfrak{R}$ be two non-empty subsets of a b-metric space $(\eth, d)$. A selfmap $\nabla : \mathfrak{I} \cup \mathfrak{R} \to \mathfrak{I} \cup \mathfrak{R}$ satisfying $\nabla(\mathfrak{I}) \subseteq \mathfrak{R}, \nabla(\mathfrak{R}) \subseteq \mathfrak{I}$ is said to be a Zamfirescu contraction mapping if there exists $\alpha \in [0,1), \beta \in [0, \frac{1}{2})$, $\gamma \in [0, \frac{1}{2})$ with $\alpha b < 1, \beta(1+b) < 1, \gamma b(b+1) < 1$ and for all $\ell, \imath \in \mathfrak{I} \cup \mathfrak{R}$ such that one of the following is true:*

(i) $d(\nabla\ell, \nabla\imath) \leq \alpha d(\ell, \imath)$;

(ii) $d(\nabla\ell, \nabla\imath) \leq \beta[d(\ell, \nabla\ell) + d(\imath, \nabla\imath)]$;

(iii) $d(\nabla\ell, \nabla\imath) \leq \gamma[d(\ell, \nabla\imath) + d(\imath, \nabla\ell)]$.

Definition 2.11. [7] *Let $\mathfrak{I}$ and $\mathfrak{R}$ be two non-empty subsets of a b-metric space $(\eth, d)$. A selfmap $\nabla : \mathfrak{I} \cup \mathfrak{R} \to \mathfrak{I} \cup \mathfrak{R}$ satisfying $\nabla(\mathfrak{I}) \subseteq \mathfrak{R}, \nabla(\mathfrak{R}) \subseteq \mathfrak{I}$ is said to be a Ciric-Reich-Rus contraction if there exists $\alpha, \beta \in (0,1)$ with $b\alpha + (b+1)\beta < 1$ such that*

$$d(\nabla\ell, \nabla\imath) \leq \alpha d(\ell, \imath) + \beta[d(\ell, \nabla\ell) + d(\imath, \nabla\imath)], \textit{ for all } \ell, \imath \in \mathfrak{I} \cup \mathfrak{R}.$$

3. Main Result

In this section, we establish several *ABPP* theorems within the framework of *b*-metric spaces, utilizing a variety of contraction mappings such as weak contractions, Zamfirescu contractions, and Ćirić–Reich–Rus contractions, along with their associated consequences. The proofs of these theorems are structured in two parts: the first focuses on qualitative aspects, while the second addresses quantitative estimates. Both components are centered around the behavior of $ABPP's$ for the pair $(\mathfrak{I}, \mathfrak{R})$ in *b*-metric spaces.

Theorem 3.1. *Let $\mathfrak{I}$ and $\mathfrak{R}$ are two non-empty subsets of a b-metric space $(\eth, d)$. Suppose that a mapping $\nabla : \mathfrak{I} \cup \mathfrak{R} \longrightarrow \mathfrak{I} \cup \mathfrak{R}$ satisfying $\nabla(\mathfrak{I}) \subseteq \mathfrak{R}$ and $\nabla(\mathfrak{R}) \subseteq \mathfrak{I}$ is a weak contraction then for every $\varepsilon > 0$, $BPP^{\varepsilon}(\mathfrak{I}, \mathfrak{R}) \neq \emptyset$ and the diameter,*

$$\Delta(BPP^{\varepsilon}(\mathfrak{I}, \mathfrak{R})) \leq \frac{(\c{k}b+2)d(\mathfrak{I}, \mathfrak{R}) + (\c{k}b+1)\varepsilon}{1-\alpha-\c{k}b}, \textit{ for all } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$ and $\ell \in \mathfrak{I} \cup \mathfrak{R}$. Consider,

$$\begin{aligned} d(\nabla^n\ell, \nabla^{n+1}\ell) &= d(\nabla(\nabla^{n-1}\ell), \nabla(\nabla^n\ell)) \\ &\leq \alpha d(\nabla^{n-1}\ell, \nabla^n\ell) + \c{k} d(\nabla^n\ell, \nabla^n\ell) \\ &\leq \alpha d(\nabla^{n-1}\ell, \nabla^n\ell) \end{aligned}$$

Since $\alpha \in (0,1)$ implies that $\lim_{n\to\infty} d(\nabla^n \ell, \nabla^{n+1}\ell) = 0,$ for all $\ell \in \mathfrak{I} \cup \mathfrak{R}$. Then, by Theorem 2.8, it follows that

$$BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R}) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

For diameter, to show condition (ii) of Theorem 2.8 holds. For that, take $\varphi > 0$ and $\ell, \iota \in BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})$. Also, $d(\ell,\iota) - d(\nabla\ell,\nabla\iota) \leq \varphi$ implies that $d(\ell,\iota) \leq d(\nabla\ell,\nabla\iota) + \varphi$. Since $\ell, \iota \in BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})$ implies that

$$d(\ell,\nabla\ell) \leq d(\mathfrak{I},\mathfrak{R}) + \varepsilon_1$$

And

$$d(\iota,\nabla\iota) \leq d(\mathfrak{I},\mathfrak{R}) + \varepsilon_2$$

Now, choose $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$. Therefore,

$$\begin{aligned}
d(\ell,\iota) &\leq d(\nabla\ell,\nabla\iota) + \varphi \\
&\leq \alpha d(\ell,\iota) + ķb d(\iota,\ell) + ķb d(\ell,\nabla\ell) + \varphi \\
&= \alpha d(\ell,\iota) + ķb d(\ell,\iota) + ķb[d(\mathfrak{I},\mathfrak{R}) + \varepsilon] + \varphi \\
&= (\alpha + ķb) d(\ell,\iota) + ķb d(\mathfrak{I},\mathfrak{R}) + bķ\varepsilon + \varphi \\
&= \frac{bķ d(\mathfrak{I},\mathfrak{R}) + bķ\varepsilon + \varphi}{1 - \alpha - ķb} \\
&= \theta(\varphi)
\end{aligned}$$

Thus, for every $\varphi > 0$ there exists $\theta(\varphi) > 0$ such that $d(\ell,\iota) - d(\nabla\ell,\nabla\iota) \leq \varphi$ implies that $d(\ell,\iota) \leq \theta(\varphi)$. Then, by Theorem 2.8, the diameter

$$\Delta(BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})) \leq \theta(2d(\mathfrak{I},\mathfrak{R}) + \varepsilon), \text{ for all } \varepsilon > 0.$$

This means exactly,

$$\Delta(BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})) \leq \frac{bķ d(\mathfrak{I},\mathfrak{R}) + bķ\varepsilon + 2d(\mathfrak{I},\mathfrak{R}) + \varepsilon}{1 - \alpha - ķb}$$

Hence,

$$\Delta(BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})) \leq \frac{(ķb + 2) d(\mathfrak{I},\mathfrak{R}) + (ķb + 1)\varepsilon}{1 - \alpha - ķb}, \text{ for all } \varepsilon > 0.$$

□

Theorem 3.2. *Let $\Im$ and $\Re$ are two non-empty subsets of a b-metric space $(\eth, d)$. Suppose that a mapping $\nabla : \Im \cup \Re \longrightarrow \Im \cup \Re$ satisfying $\nabla(\Im) \subseteq \Re$ and $\nabla(\Re) \subseteq \Im$ is a Zamfirescu contraction then for every $\varepsilon > 0$, $BPP^{\varepsilon}(\Im, \Re) \neq \emptyset$ and the diameter,*

$$\Delta(BPP^{\varepsilon}(\Im, \Re)) \leq \frac{2(\gamma b + 1)d(\Im, \Re) + 3\varepsilon}{1 - 2\gamma b}, \text{ for all } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$ and $\ell \in \Im \cup \Re$. Consider,

$$d(\nabla^n \ell, \nabla^{n+1} \ell) = d(\nabla(\nabla^{n-1} \ell), \nabla(\nabla^n \ell)).$$

Case 1. *Suppose (i) of Definition 2.9 holds. Then,*

$$d(\nabla^n \ell, \nabla^{n+1} \ell) \leq \alpha d(\nabla(\nabla^{n-1} \ell), \nabla(\nabla^n \ell)) \tag{3.1}$$

Case 2. *Suppose (ii) of Definition 2.9 holds. Then,*

$$d(\nabla^n \ell, \nabla^{n+1} \ell) \leq \beta [d(\nabla(\nabla^{n-1} \ell), \nabla(\nabla^n \ell) + d(\nabla(\nabla^n \ell), \nabla(\nabla^{n+1} \ell))]$$

That is,

$$d(\nabla^n \ell, \nabla^{n+1} \ell) \leq \left(\frac{\beta}{1 - \beta} \right) d(\nabla(\nabla^{n-1} \ell), \nabla(\nabla^n \ell) \tag{3.2}$$

Case 3. *Suppose (iii) of Definition 2.9 holds. Then,*

$$\begin{aligned} d(\nabla^n \ell, \nabla^{n+1} \ell) &\leq \gamma [d(\nabla^{n-1} \ell, \nabla^{n+1} \ell) + d(\nabla^n \ell, \nabla^n \ell)] \\ &= \gamma b d(\nabla^{n-1} \ell, \nabla^n \ell) + d(\nabla^n \ell, \nabla^{n+1} \ell) \end{aligned}$$

That is,

$$d(\nabla^n \ell, \nabla^{n+1} \ell) \leq \left(\frac{\gamma b}{1 - \gamma b} \right) d(\nabla(\nabla^{n-1} \ell), \nabla(\nabla^n \ell) \tag{3.3}$$

From the equations $(3.1), (3.2)$ and (3.3), choose

$$\lambda = \max\{\alpha, \frac{\beta}{1 - \beta}, \frac{\gamma b}{1 - \gamma b}\}.$$

Since $\alpha, \beta, \gamma \in (0, 1)$ and $b \geq 1$ implies that $\lambda \in (0, 1)$. Therefore,

$$\lim_{n \to \infty} d(\nabla^n \ell, \nabla^{n+1} \ell) = 0, \text{ for all } \ell \in \Im \cup \Re.$$

Then by Theorem 2.8, it follows that

$$BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R}) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

For diameter, to show condition (ii) of Theorem 2.8 holds. For that, take $\varphi > 0$ and $\ell, \imath \in BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})$. Also, $d(\ell,\imath) - d(\nabla\ell,\nabla\imath) \leq \varphi$ implies that $d(\ell,\imath) \leq d(\nabla\ell,\nabla\imath) + \varphi$. Since $\ell, \imath \in BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})$ implies that

$$d(\ell,\nabla\ell) \leq d(\mathfrak{I},\mathfrak{R}) + \varepsilon_1$$

And

$$d(\imath,\nabla\imath) \leq d(\mathfrak{I},\mathfrak{R}) + \varepsilon_2$$

Now, choose $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$. Therefore, by equation (3.3), we get

$$\begin{aligned}
d(\ell,\imath) &= d(\nabla\ell,\nabla\imath) + \varphi \\
&= \gamma[d(\ell,\nabla\imath) + d(\imath,\nabla\ell)] + \varphi \\
&= \gamma b[d(\ell,\imath) + d(\imath,\nabla\imath) + d(\imath,\ell) + d(\ell,\nabla\ell)] + \varphi \\
&= 2\gamma b d(\ell,\imath) + 2\gamma b d(\mathfrak{I},\mathfrak{R}) + 2\varepsilon + \varphi \\
&= \frac{2\gamma b d(\mathfrak{I},\mathfrak{R}) + 2\varepsilon + \varphi}{1 - 2\gamma b} \\
&= \theta(\varphi)
\end{aligned}$$

Thus, for every $\varphi > 0$ there exists $\theta(\varphi) > 0$ such that $d(\ell,\imath) - d(\nabla\ell,\nabla\imath) \leq \varphi$ implies that $d(\ell,\imath) \leq \theta(\varphi)$. Then, by Theorem 2.8, the diameter

$$\Delta(BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})) \leq \theta(2d(\mathfrak{I},\mathfrak{R}) + \varepsilon), \text{ for all } \varepsilon > 0.$$

This means that,

$$\Delta(BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})) \leq \frac{2\gamma b d(\mathfrak{I},\mathfrak{R}) + 2\varepsilon + 2d(\mathfrak{I},\mathfrak{R}) + \varepsilon}{1 - 2\gamma b}, \text{ for all } \varepsilon > 0.$$

Hence,

$$\Delta(BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})) \leq \frac{(2\gamma b + 2)d(\mathfrak{I},\mathfrak{R}) + 3\varepsilon}{1 - 2\gamma b}, \text{ for all } \varepsilon > 0.$$

□

Remark 3.3. *(1) In Theorem 3.2, only* **case 1** *holds, then it is called* α*-contraction mapping on b-metric space* $(\eth, d)$*. Then* $BPP^{\varepsilon}(\Im, \Re) \neq \emptyset$ *and the diameter,*

$$\Delta(BPP^{\varepsilon}(\Im, \Re)) \leq \frac{2d(\Im, \Re) + \varepsilon}{1-\alpha}, \text{ for all } \varepsilon > 0.$$

(2) In Theorem 3.2, only **case 2** *holds, then it is called Kannan contraction mapping on b-metric space* $(\eth, d)$*. Then* $BPP^{\varepsilon}(\Im, \Re) \neq \emptyset$ *and the diameter,*

$$\Delta(BPP^{\varepsilon}(\Im, \Re)) \leq 2(\beta+1)d(\Im, \Re) + \varepsilon(2\beta+1), \text{ for all } \varepsilon > 0.$$

(3) In Theorem 3.2, only **case 3** *holds, then it is called Chatterjea contraction mapping on b-metric space* $(\eth, d)$*. Then* $BPP^{\varepsilon}(\Im, \Re) \neq \emptyset$ *and the diameter,*

$$\Delta(BPP^{\varepsilon}(\Im, \Re)) \leq \frac{2(\gamma b+1)d(\Im, \Re) + \varepsilon(2\gamma b+1)}{1-2\gamma b}, \text{ for all } \varepsilon > 0.$$

Theorem 3.4. *Let* $\Im$ *and* $\Re$ *are two non-empty subsets of a b-metric space* $(\eth, d)$*. Suppose that a mapping* $\nabla : \Im \cup \Re \longrightarrow \Im \cup \Re$ *satisfying* $\nabla(\Im) \subseteq \Re$ *and* $\nabla(\Re) \subseteq \Im$ *is a Ciric-Reich-Rus contraction then for every* $\varepsilon > 0$*,* $BPP^{\varepsilon}(\Im, \Re) \neq \emptyset$ *and the diameter,*

$$\Delta(BPP^{\varepsilon}(\Im, \Re)) \leq \frac{2(\beta+1)d(\Im, \Re) + \varepsilon(2\beta+1)}{1-\alpha}, \text{ for all } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$ and $\ell \in \Im \cup \Re$. Consider,

$$\begin{aligned} d(\nabla^n \ell, \nabla^{n+1} \ell) &= d(\nabla(\nabla^{n-1} \ell), \nabla(\nabla^n \ell)) \\ &\leq \alpha d(\nabla^{n-1} \ell, \nabla^n \ell) + \beta[d(\nabla^{n-1} \ell, \nabla^n \ell) + d(\nabla^n, \nabla^{n+1} \ell)] \\ &= \left(\frac{\alpha+\beta}{1-\beta}\right) d(\nabla^{n-1} \ell, \nabla^n \ell) \end{aligned}$$

Since $\alpha, \beta \in (0,1)$ implies that $\lim_{n\to\infty} d(\nabla^n \ell, \nabla^{n+1} \ell) = 0$, for all $\ell \in \Im \cup \Re$. Then, by Theorem 2.8, it follows that

$$BPP^{\varepsilon}(\Im, \Re) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

For diameter, to show condition (ii) of Theorem 2.8 holds. For that, take $\varphi > 0$ and $\ell, \imath \in BPP^{\varepsilon}(\Im, \Re)$. Also, $d(\ell, \imath) - d(\nabla \ell, \nabla \imath) \leq \varphi$ implies that $d(\ell, \imath) \leq d(\nabla \ell, \nabla \imath) + \varphi$. Since $\ell, \imath \in BPP^{\varepsilon}(\Im, \Re)$ implies that

$$d(\ell, \nabla \ell) \leq d(\Im, \Re) + \varepsilon_1$$

And

$$d(\iota,\nabla\iota)\leq d(\mathfrak{I},\mathfrak{R})+\varepsilon_2$$

Now, choose $\varepsilon = \max\{\varepsilon_1,\varepsilon_2\}$. Therefore,

$$\begin{aligned}
d(\ell,\iota) &= d(\nabla\ell,\nabla\iota)+\varphi \\
&= \alpha d(\ell,\iota)+\beta[d(\iota,\nabla\iota)+d(\ell,\nabla\ell)]+\varphi \\
&= \alpha d(\ell,\iota)+\beta[2(\mathfrak{I},\mathfrak{R})+2\varepsilon]+\varphi \\
&= \frac{2\beta d(\mathfrak{I},\mathfrak{R})+2\beta\varepsilon+\varphi}{1-\alpha} \\
&= \theta(\varphi)
\end{aligned}$$

Thus, for every $\varphi>0$ there exists $\theta(\varphi)>0$ such that $d(\ell,\iota)-d(\nabla\ell,\nabla\iota)\leq\varphi$ implies that $d(\ell,\iota)\leq\theta(\varphi)$. Then, by Theorem 2.8, the diameter

$$\Delta(BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R}))\leq\theta(2d(\mathfrak{I},\mathfrak{R})+\varepsilon),\ \text{for all}\ \varepsilon>0.$$

This means that,

$$\Delta(BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R}))\leq\frac{2\beta d(\mathfrak{I},\mathfrak{R})+2\beta\varepsilon+2d(\mathfrak{I},\mathfrak{R})+\varepsilon}{1-2\alpha},\ \text{for all}\ \varepsilon>0.$$

Hence,

$$\Delta(BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R}))\leq\frac{2(\beta+1)d(\mathfrak{I},\mathfrak{R})+\varepsilon(2\beta+1)}{1-\alpha},\ \text{for all}\ \varepsilon>0.$$

□

Theorem 3.5. *Let $\mathfrak{I}$ and $\mathfrak{R}$ are two non-empty subsets of a b-metric space $(\eth,d)$. Suppose that a mapping $\nabla:\mathfrak{I}\cup\mathfrak{R}\longrightarrow\mathfrak{I}\cup\mathfrak{R}$ satisfying $\nabla(\mathfrak{I})\subseteq\mathfrak{R}$ and $\nabla(\mathfrak{R})\subseteq\mathfrak{I}$ is a Ciric contraction then for every $\varepsilon>0$, $BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})\neq\emptyset$ and the diameter,*

$$\Delta(BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R}))\leq\frac{(\beta+\gamma+2\delta b+2)d(\mathfrak{I},\mathfrak{R})+(\beta+\gamma+2\delta b+1)\varepsilon}{1-\alpha-2\delta b},\ \textit{for all}\ \varepsilon>0.$$

Proof. For diameter, to show condition (ii) of Theorem 2.8 holds. For that, take $\varphi>0$ and $\ell,\iota\in BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})$. Also, $d(\ell,\iota)-d(\nabla\ell,\nabla\iota)\leq\varphi$ implies that $d(\ell,\iota)\leq d(\nabla\ell,\nabla\iota)+\varphi$. Since $\ell,\iota\in BPP^{\varepsilon}(\mathfrak{I},\mathfrak{R})$ implies that

$$d(\ell,\nabla\ell)\leq d(\mathfrak{I},\mathfrak{R})+\varepsilon_1$$

And

$$d(\imath,\nabla\imath)\leq d(\Im,\Re)+\varepsilon_2$$

Now, choose $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}$. Therefore,

$$\begin{aligned}
d(\ell,\imath) &= d(\nabla\ell,\nabla\imath)+\varphi \\
&\leq \alpha d(\ell,\imath)+\beta d(\ell,\nabla\ell)+\gamma d(\imath,\nabla\imath)+\delta[d(\ell,\nabla\imath)+d(\imath,\nabla\ell)]+\varphi \\
&= (\alpha+2\delta\beta)d(\ell,\imath)+(\beta+\gamma+2\delta b)d(\Im,\Re)+(\beta+\gamma+2\delta b)\varepsilon+\varphi \\
&= \frac{(\beta+\gamma+2\delta b)d(\Im,\Re)+(\beta+\gamma+2\delta b)\varepsilon+\varphi}{1-\alpha-2\delta b} \\
&= \theta(\varphi)
\end{aligned}$$

Thus, for every $\varphi>0$ there exists $\theta(\varphi)>0$ such that $d(\ell,\imath)-d(\nabla\ell,\nabla\imath)\leq\varphi$ implies that $d(\ell,\imath)\leq\theta(\varphi)$. Then, by Theorem 2.8, the diameter

$$\Delta(BPP^{\varepsilon}(\Im,\Re))\leq\theta(2d(\Im,\Re)+\varepsilon),\ \text{for all } \varepsilon>0.$$

Hence,

$$\Delta(BPP^{\varepsilon}(\Im,\Re))\leq\frac{(\beta+\gamma+2\delta b+2)d(\Im,\Re)+(\beta+\gamma+2\delta b+1)\varepsilon}{1-\alpha-2\delta b},\ \text{for all } \varepsilon>0.$$

□

Example 3.6. *Let* $\eth=[0,1]$ *and consider the closed subsets* $\nabla_1=[0,3/6],\nabla_2=[2/6,3/6]$ *and* $\nabla_3=[5/6,1]$ *of a b-metric space* $(\eth,d)$ *and* $\nabla:\nabla_1\cup\nabla_2\cup\nabla_3\to\nabla_1\cup\nabla_2\cup\nabla_3$ *is defined by:*

$$\nabla\ell=\begin{cases}\frac{2}{6}+\ell & when\ \ell\in\left[0,\frac{3}{6}\right]\\ \\ \frac{3}{6}+\ell & when\ \ell\in\left[\frac{2}{6},\frac{3}{6}\right]\\ \\ 1-\frac{3}{6} & when\ \ell\in\left[\frac{5}{6},1\right]\end{cases}$$

This clearly shows that $\nabla(\nabla_1)\subseteq\nabla_2,\nabla(\nabla_2)\subseteq\nabla_3$ *and* $\nabla(\nabla_3)\subseteq\nabla_1$. *Also for every* $\ell,\imath\in\nabla_1\cup\nabla_2\cup\nabla_3\subseteq\nabla$ *satisfies the Definitions 2.9 2.10 and 2.11. Hence,* ∇ *satisfies all the conditions of the Theorems 3.1, 3.2 and 3.4.*

In the similar manner, we have proved many *ABPP* results by using various operators on b-metric spaces. The diameters of several contraction operators are shown in the table below.

S. No	Operator(s)	Diameter, for every $\varepsilon > 0$, $\Delta(BPP^{\varepsilon}(\Im,\Re))$
1	Contraction	$\leq \dfrac{2d(\Im,\Re)+\varepsilon}{1-\alpha}$
2	Kannan	$\leq 2(\beta+1)d(\Im,\Re)+\varepsilon(2\beta+1)$
3	Chatterjea	$\leq \dfrac{2(\gamma b+1)d(\Im,\Re)+\varepsilon(2\gamma b+1)}{1-2\gamma b}$
4	B-contraction	$\leq \dfrac{2(\alpha+b\gamma+1)d(\Im,\Re)+\varepsilon(2\alpha+2b\gamma+1)}{1-\beta-2b\gamma}$
5	Bianchini	$\leq (\alpha+2)d(\Im,\Re)+\varepsilon(\alpha+1)$
6	Hardy-Rogers	$\leq \dfrac{(\beta+\gamma+\delta b+\nu b+2)d(\Im,\Re)+\varepsilon(\beta+\gamma+\delta b+\nu b+1)}{1-\alpha-\delta b-\nu b}$
7	Ćirić	$\leq \dfrac{(\beta+\gamma+2\delta b+2)d(\Im,\Re)+(\beta+\gamma+2\delta b+1)\varepsilon}{1-\alpha-2\delta b}$
8	Ćirić-Reich-Rus	$\leq \dfrac{2(\beta+1)d(\Im,\Re)+\varepsilon(2\beta+1)}{1-\alpha}$
9	Reich	$\leq \dfrac{(\beta+\gamma+2)d(\Im,\Re)+(\beta+\gamma+1)\varepsilon}{1-\alpha}$
10	Zamfirescu	$\leq \dfrac{(2\gamma b+2)d(\Im,\Re)+3\varepsilon}{1-2\gamma b}$
11	Mohseni-saheli	$\leq \dfrac{(2\alpha b+2)d(\Im,\Re)+\varepsilon(2\alpha b+1)}{1-\alpha-\alpha b}$
12	Mohseni-semi	$\leq \dfrac{(\alpha+2)d(\Im,\Re)+\varepsilon(\alpha+1)}{1-\alpha}$
13	Weak contraction	$\leq \dfrac{(ķb+2)d(\Im,\Re)+(ķb+1)\varepsilon}{1-\alpha-ķb}$

4. Applications

The *ABPP* covers a wide range of applications in the domain of mathematics, particularly in differential equations, Fourier series, numerical analysis, and so on. By reading [12] and references therein, one can find a variety of applications involving *ABPP* results in differential equations. The examples below demonstrate how to apply *ABPP* results in differential equations.

Example 4.1. *Let* $\eth = C([0,1],\mathbb{R})$ *and* χ *is b-metric space with* $b \geq 1$ *defined by* $d(\ell,\imath) = \sup_{\tau\in[0,1]} |\ell,\imath|^2$. *Also, consider* $\varkappa''(\tau) = 3\varkappa^2(\tau)/2$, $0 \leq \tau \leq 1$ *and the initial conditions* $\varkappa(0) = 4$, $\varkappa(1) = 1$. *Here, the exact solution is* $\varkappa(\tau) = 4/(1+\tau)^2$. *We have,* $\varkappa_0(\tau) = c_1\tau + c_2$. *By* *using the initial conditions, we get* $\varkappa_0(\tau) = 4 - 3\tau$. *Now, define the integral operator,*

$$\Im(\varkappa) = \varkappa + \int_0^1 G(\tau,\omega)[\varkappa'' - f(\omega,\varkappa,\varkappa')]d\omega \tag{4.1}$$

where

$$G(\tau,\omega) = \begin{cases} \omega(1-\tau) & 0 \le \omega \le \tau \\ \tau(1-\tau) & \tau \le \omega \le 1 \end{cases}$$

Then, the equation (4.1) *becomes*

$$\begin{aligned}\Im(\varkappa) &= \varkappa(\tau) + \int_0^1 G(\tau,\omega)\varkappa''(s)ds - \int_0^1 G(\tau,\omega)f(\omega,\varkappa,\varkappa')ds \\ &= (4-3\tau) - \int_0^1 G(\tau,\omega)\big[-\frac{3\varkappa^2(\omega)}{2}\big]d\omega \\ &= 4 - 3\tau + \frac{3}{2}\left\{\int_0^1 G(\tau,\omega)\varkappa^2(\omega)d\omega\right\}\end{aligned}$$

So, we have

$$\begin{aligned}d(\varkappa\ell,\varkappa\imath) &= \sup_{\tau\in[0,1]} |\varkappa\ell - \varkappa\imath|^2 \\ &= \sup_{\tau\in[0,1]} \left|\frac{3}{2}\int_0^1 G(\tau,\omega)\ell^2(\omega)d\omega - \frac{3}{2}\int_0^1 G(\tau,\omega)\imath^2(\omega)d\omega\right|^2 \\ &\le (2.25)\left(\int_0^1 |G(\tau,\omega)|^2 ds\right)\left(\int_0^1 |\ell^2(\omega) - \imath^2(\omega)|^2 d\omega\right) \\ &\le (0.75)\frac{\tau^2(1-\tau)^2}{3}\int_0^1 \left|\ell^2(\omega) - \imath^2(\omega)\right|^2 d\omega \\ &\le (0.046875)\int_0^1 \left|\ell^2(\omega) - \imath^2(\omega)\right|^2 d\omega \\ &\le (0.046875)\sup_{\tau\in[0,1]} |\ell(\omega) - \imath(\omega)|^2 \\ &\le (0.046875)d(\ell,\imath)\end{aligned}$$

Hence, it satisfies all the conditions of Theorem 3.1 and Theorem 3.2. Also, by Theorem 2.8, $\Im$ *has ABPP in* $\eth = C([0,1],\mathbb{R})$. *Therefore, the given bounded value problem has ABPP in* $\eth$.

5. Conclusion

In this paper, some *ABPP* results are established on b-metric spaces by utilizing various types of contraction mappings. It is worth observing that in the limiting case $\varepsilon \longrightarrow 0$, all the results established in the present paper produces more restricted $ABPP's$. Furthermore, $ABPP's$ are consequently not less important than $BPP's$. As various future results can be demonstrated in a smaller setting to ensure the existence of the $ABPP's$.

Acknowledgement

We thank the Department of Science and Technology, Government of India, for providing support through the Fund for Improvement of S&T Infrastructure in Universities and Higher Educational Institutions (FIST) program (Grant No. SR/FIST/College-/2020/943). Further, we would like to thank the anonymous referee(s) of the paper for their valuable recommendations.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] S. Banach, Sur les Opérations dans les Ensembles Abstraits et Leur Application aux Équations Intégrales, Fundam. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[2] I.A. Bakhtin, The Contraction Mapping Principle in Almost Metric Spaces, Funct. Anal. 30 (1989), 26–37.

[3] M. Boriceanu, Fixed Point Theory for Multivalued Generalized Contraction on a Set With Two b-Metric, Stud. Univ. Babeş–Bolyai, Math. 3 (2009), 1–14.

[4] S. Czerwik, Contraction Mappings in b-Metric Spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11. http://dml.cz/dmlcz/120469.

[5] S. Agrawal, K. Qureshi, J. Nema, A Fixed Point Theorem For b-Metric Space, Int. J. Pure Appl. Math. Sci. 9 (2016), 45–50.

[6] M. Pacurar, Sequences of Almost Contractions and Fixed Points in b-Metric Spaces, An. Univ. Vest Timiş. Ser. Mat.-Inform. 48 (2010), 125–137.

[7] M. Berinde, Approximate Fixed Point Theorems, Stud. Univ. ”Babes-Bolyai”, Mathematica LI (2006), 11–25.

[8] D. Dey, M. Saha, Approximate Fixed Point of Reich Operator, Acta Math. Univ. Comenianae LXXXII (2013), 119–123.

[9] S.A.M. Mohsenalhosseini, Approximate Best Proximity Pairs in Metric Space for Contraction Maps, Adv. Fixed Point Theory, 4 (2014), 310–324.

[10] S.A.M. Mohsenalhosseini, M. Saheli, Some of Family of Contractive Type Maps and Approximate Common Fixed Point, J. Fixed Point Theory, 2021 (2021), 2.

[11] S.A.M. Mohsenalhosseini, H. Mazaheri, M.A. Dehghan, Approximate Best Proximity Pairs in Metric Space, Abstr. Appl. Anal. 2011 (2011), 596971. https://doi.org/10.1155/2011/596971.

[12] S. Khuri, I. Louhichi, A Novel Ishikawa–Green's Fixed Point Scheme for the Solution of BVPs, Appl. Math. Lett. 82 (2018), 50–57. https://doi.org/10.1016/j.aml.2018.02.016.

[13] R. Theivaraman, P. S. Srinivasan, S. Thenmozhi, S. Radenovic, Some Approximate Fixed Point Results for Various Contraction Type Mappings, Adv. Fixed Point Theory 13 (2023), 9. https://doi.org/10.28919/afpt/8080.

[14] R. Theivaraman, P.S. Srinivasan, M. Marudai, S. Thenmozhi, A. Herminau Jothy, G-Metric Spaces and the Related Approximate Fixed Point Results, Adv. Fixed Point Theory 13 (2023), 17. https://doi.org/10.28919/afpt/8178.

[15] R. Theivaraman, P.s. Srinivasan, S. Radenovic, C. Park, New Approximate Fixed Point Results for Various Cyclic Contraction Operators on E-Metric Spaces, J. Korean Soc. Ind. Appl. Math. 27 (2023), 160–179. https://doi.org/10.12941/JKSIAM.2023.27.160.

[16] R. Theivaraman, P.S. Srinivasan, A.H. Jothy, Approximate Best Proximity Pair Results on Metric Spaces Using Contraction Operators, Korean J. Math. 31 (2023), 373–383. https://doi.org/10.11568/KJM.2023.31.3.373.

[17] D. Sujatha, S.R. Ananthalakshmi, R. Theivaraman, Existence of Approximate Best Proximity Point of the Triplets (Q_1, Q_2, Q_3) on G-Metric Spaces and Its Applications, Adv. Fixed Point Theory 15 (2025), 48. https://doi.org/10.28919/afpt/9548.

[18] P.J. Lavino, A.M.P. Dharsini, A Fixed Point Theorem for Non-Self Mappings of Rational Type in b-Multiplicative Metric Spaces with Application to Integral Equations, Indian J. Sci. Technol. 17 (2024), 3743–3754. https://doi.org/10.17485/ijst/v17i36.2362.

[19] A. Antony L, M.P. Dharsini A, Some Fixed Point Theorems for F Contraction on b-Multiplicative Metric Spaces, Sci. Temper 16 (2025), 4206–4214. https://doi.org/10.58414/scientifictemper.2025.16.5.05.