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MATHEMATICAL MODELING AND ANALYSIS OF THE DYNAMICS OF A NONLINEAR FINANCE SYSTEM VIA HATTAF FRACTIONAL DERIVATIVE AND FIXED POINT THEORY

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Abstract. In this paper, we propose a new mathematical model that describes the dynamics of a nonlinear finance system. The proposed model is formulated by fractional differential equations (FDEs) involving the generalized Hattaf fractional (GHF) derivative. Firstly, we prove that our fractional model is mathematically and financially well-posed by means of fixed point theory. Additionally, we study the existence of equilibria. Furthermore, the stability analysis of the financial model is carefully studied. Finally, numerical simulations are presented to illustrate our theoretical results.

Keywords: finance; Hattaf fractional derivative; mathematical modeling; fixed point theory; stability.

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1. Introduction

Finance is the application of economic principles to decision-making that involves the allocation of money under conditions of uncertainty [1]. Investors allocate their capital across

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various types of financial assets to achieve their specific objectives. In parallel, companies and governments obtain financing by issuing debt securities in their own names, which then allow them to finance their activities and projects.

Various mathematical models have been proposed and developed to understand the dynamics of financial systems. In 2001, Ma and Chen [2, 3] established the foundations of mathematical modeling in finance using ordinary differential equations (ODEs), revealing the existence of nonlinear behaviors in the interaction of interest rate, investment demand and price index. Further, Chen [4] investigated the complex behaviors of a financial system with time-delayed feedback through numerical modeling. In 2011, Yu et al. [5] constructed a new hyperchaotic finance system by adding an additional state variable. Subsequently, Tacha et al. [6] oriented their research towards practical applications with adaptive control and electronic simulation.

The memory effect is an inherent characteristic of the nonlinear dynamics system in finance. It refers to the collective and historical knowledge, experiences, and information that society has accumulated over time. This includes knowledge of past political events, financial practices, and sociocultural norms that shape financial behavior. However, the classical calculus based on ODEs used in the above models [2, 3, 4, 5, 6] does not allow modeling this memory effect. Recently, fractional calculus based on FDEs represents a powerful tool for modeling the memory effect and the hereditary properties that exist in many phenomena arising from various fields of science, industry and engineering. In finance, He et al. [7] used the Caputo derivative [8] in the dynamic investigation and fixed-time synchronization of a fractional-order financial system. Chauhan et al. [9] studied the fractional order financial model by using Caputo-Fabrizio derivative [10]. Liping et al. [11] analyzed a new financial chaotic model using the Atangana-Baleanu operator [12].

More recently, Hattaf [13] introduced a new generalized fractional derivative that covered the Caputo-Fabrizio and Atangana-Baleanu fractional derivatives. This fractional derivative namely generalized Hattaf fractional (GHF) derivative, which used by many authors. In biology, Hajhouji et al. [14] proposed a mathematical model that takes into account immunological memory to describe the dynamics of HIV-1 infection in the presence of highly therapy. In ecology, Assadiki et al. [15] formulated a fractional prey-predator model with the GHF. In economics, Lasfar et al. [16] developed a new fractional business cycle model with general investment and variable depreciation rate taking into consideration the memory effect.

The main objective of this study is to propose a mathematical model formulated by FDEs involving the GHF derivative in order to describe the interaction between interest rate, investment demand and price index. To that end, the paper is organized as follows. Section 2 provides the formulation of the model and some interesting concepts needed to the elaboration of this study. Section 3 presents the existence and uniqueness of solution by means of Banach contraction, and it discuses the existence of equilibrium points. Section 4 focuses on the stability analysis of financial equilibrium. Finally, some numerical simulations are given in Section 5.

2. Basic Concepts and Model Formulation

In this section, we first recall the definition of the GHF derivative and its proprieties necessary for the elaboration of this study. After, we present our fractional finance model.

Definition 2.1. [13] *Let $p \in [0,1)$, $q,\gamma > 0$ and $f \in H^1(t_0,T)$. The GHF derivative of order p in the Caputo sense of the function $f(t)$ with respect to the weight function $\omega(t)$ is defined as follows:*

$$D^{p,q,\gamma}_{t_0,t,\omega} f(t) = \frac{N(p)}{1-p}\frac{1}{\omega(t)}\int_{t_0}^{t} E_q[-\mu_p(t-\tau)^{\gamma}]\frac{d}{d\tau}(\omega f)(\tau)d\tau, \tag{1}$$

where $\omega \in C^1(t_0,T)$, $\omega > 0$ on $[t_0,T]$, $N(p)$ is a normalization function such that $N(0) = N(1) = 1$, $\mu_p = \frac{p}{1-p}$ and $E_q(t) = \sum\limits_{k=0}^{+\infty} \frac{t^k}{\Gamma(qk+1)}$ is the Mittag-Leffler function of parameter q.

The GHF derivative introduced in the above definition generalizes and extends many special cases existing in the literature. In the fact, when $\omega(t) = 1$ and $q = \gamma = 1$, we get the Caputo-Fabrizio fractional derivative [10], which is given by

$$^{C}D^{p,1,1}_{t_0,t,1} f(t) = \frac{N(p)}{1-p}\int_{t_0}^{t} \exp[-\mu_p(t-\tau)]f'(\tau)d\tau.$$

We obtain the Atangana-Baleanu fractional derivative [12] when $\omega(t) = 1$ and $q = \gamma = p$, which is given by

$$^{C}D^{p,p,p}_{t_0,t,1} f(t) = \frac{N(p)}{1-p}\int_{t_0}^{t} E_p[-\mu_p(t-\tau)^{p}]f'(\tau)d\tau.$$

For $q=\gamma=p$, we get the weighted Atangana-Baleanu fractional derivative [17], which is given by

$$^{C}D^{p,p,p}_{t_0,t,\omega}f(t)=\frac{N(p)}{1-p}\frac{1}{\omega(t)}\int_{t_0}^{t}E_p[-\mu_p(t-\tau)^p]\frac{d}{d\tau}(\omega f)(\tau)d\tau.$$

For simplicity, we denote $^{C}D^{p,q,q}_{t_0,t,\omega}$ by $\mathscr{D}^{p,q}_{t_0,\omega}$. According to [13], the generalized Hattaf fractional integral operator associated to $\mathscr{D}^{p,q}_{t_0,\omega}$ is defined by

$$\mathscr{I}^{p,q}_{t_0,\omega}f(t)=\frac{1-p}{N(p)}f(t)+\frac{p}{N(p)}\ {}^{RL}\mathscr{I}^{q}_{t_0,\omega}f(t), \tag{2}$$

where $^{RL}\mathscr{I}^{q}_{t_0,\omega}$ is the standard weighted Riemann-Liouville fractional integral of order q defined by

$$^{RL}\mathscr{I}^{q}_{t_0,\omega}f(t)=\frac{1}{\Gamma(q)}\frac{1}{\omega(t)}\int_{t_0}^{t}(t-\tau)^{q-1}\omega(\tau)f(\tau)d\tau. \tag{3}$$

Theorem 2.2. [13] *Let $p\in[0,1)$, $q>0$ and $f\in H^1(t_0,T)$. Then we have the following property:*

$$\mathscr{I}^{p,q}_{t_0,\omega}(\mathscr{D}^{p,q}_{t_0,\omega}f)(t)=f(t)-\frac{\omega(t_0)f(t_0)}{\omega(t)}. \tag{4}$$

Theorem 2.3. [13] *The Laplace transform of $\omega(t)\mathscr{D}^{p,q}_{0,\omega}$ is given by*

$$\mathscr{L}\{\omega(t)\mathscr{D}^{p,q}_{0,\omega}f(t)\}(s)=\frac{N(p)}{1-p}\frac{s^q\mathscr{L}\{\omega(t)f(t)\}(s)-s^{q-1}\omega(0)f(0)}{s^q+\mu_p}.$$

Lemma 2.4. [18] *Let $q>0$, $x(t)$, $u(t)$ be nonnegative functions and $v(t)=M\geq 0$ with $N(p)-(1-p)M>0$. Assume that*

$$x(t)\leq u(t)+M\mathscr{I}^{p,q}_{0,\omega}x(t).$$

Then

$$x(t)\leq\frac{N(p)}{N(p)-(1-p)M}\left[u(t)+\int_0^t\sum_{n=1}^{+\infty}\frac{(pM)^n(t-\tau)^{nq-1}u(\tau)}{\Gamma(nq)\,[N(p)-(1-p)M]^n}d\tau\right].$$

Furthermore, if in addition $u(t)$ is a nondecreasing function on $[0,T]$, we have

$$x(t)\leq\frac{N(p)u(t)}{N(p)-(1-p)M}E_q\left(\frac{pMT^q}{N(p)-(1-p)M}\right).$$

According to Krasnoselskii's fixed point theorem [19, 20], we have the following lemma.

Lemma 2.5. [19, 20] *Let M be a nonempty closed convex subset of a Banach space $(C, ||.||)$. Suppose that A and B map M into C such that*

(i): $A\psi_1 + B\psi_2 \in M$, *for all* $\psi_1, \psi_2 \in M$;

(ii): *A is a contraction mapping;*

(iii): *B is continuous and $B(M)$ is contained in a compact subset of C.*

Then $A+B$ has a fixed point $\psi \in M$.

Now, we propose the following financial model involving GHF derivative:

$$\begin{cases} \mathscr{D}_{0,\omega}^{p,q} x(t) = z(t) + (y(t) - a)x(t), \\ \mathscr{D}_{0,\omega}^{p,q} y(t) = 2 - by(t) - x^2(t), \\ \mathscr{D}_{0,\omega}^{p,q} z(t) = x(t)y(t) - x(t) - cz(t), \end{cases} \tag{5}$$

where $x(t)$, $y(t)$ and $z(t)$ denote the interest rate, the investment demand and the price index at time t, respectively. Further, the positive parameters a, b and c are, respectively, the saving amount, the cost per investment and elasticity of the demand of the commercial markets.

It is important to note that our financial model presented by system (5) improves and generalizes numerous financial models existing in the literature. For instance,

- When $q = p$ and $w(t) = 1$, we get the fractional model proposed by Liping et al. [11].
- When $p = q = 1$ and $w(t) = 1$, we obtain the model of Ma and Chen [2, 3].

3. The Existence of Solution and Equilibrium Points

In this section, we first investigate the existence and uniqueness of solutions of system (5) by means of fixed point theory.

Let $\mathscr{C} = C([0,T], \mathbb{R}^3)$ be the Banach space of continuous functions g from $[0,T]$ into $\mathbb{R}^3$ equipped with the sup-norm

$$\|g\| = \sup_{t\in[0,T]} |g(t)|.$$

The system (5) can be rewritten as follows:

$$\begin{cases} \mathscr{D}_{0,\omega}^{p,q} u(t) = F(t, u(t)), \\ u(0) = u_0, \end{cases} \tag{6}$$

where $u(t)=(x(t),y(t),z(t))^T$, $u_0=(x(0),y(0),z(0))^T$ and the vector function F is given by

$$F=\begin{pmatrix}F_1\\ F_2\\ F_3\end{pmatrix}=\begin{pmatrix}z+(y-a)x\\ 2-by+x^2\\ xy-x-cz\end{pmatrix}.$$

Applying the Hattaf fractional integral to both sides of (6), we get

$$u(t)=\frac{\omega(0)u(0)}{\omega(t)}+\mathscr{I}^{p,q}_{0,\omega}F(t,u(t)) \tag{7}$$

$$=\frac{\omega(0)u_0}{\omega(t)}+\frac{1-p}{N(p)}F(t,u(t))+\frac{p}{N(p)}\frac{1}{\Gamma(q)}\frac{1}{\omega(t)}\int_0^t(t-\tau)^{q-1}\omega(\tau)F(\tau,u(\tau))d\tau.$$

Next, we will demonstrate that F satisfies a Lipschitz condition in its second variable, from which the following lemma follows.

Lemma 3.1. *The vector F is Lipschtiz in its second variable.*

Proof. We have

$$\begin{aligned}|F(t,u_1)-F(t,u_2)| &= |F_1(t,u_1(t))-F_1(t,u_2(t))|+|F_2(t,u_1(t))-F_2(t,u_2(t))|\\
&\quad+|F_3(t,u_3(t))-F_3(t,u_3(t))|\\
&=|z_1(t)+(y_1(t)-a)x_1(t)-z_2(t)-(y_2(t)-a)x_2(t))|\\
&\quad+|2-by_1(t)+x_1^2(t)-2+by_2(t)-x_2^2(t)|\\
&\quad+|x_1(t)y_1(t)-x_1(t)-cz_1(t)-x_2(t)y_2(t)+x_2(t)+cz_2(t)|\\
&\le |z_1(t)-z_2(t)|+2|y_1(t)x_1(t)-y_2(t)x_2(t)|+(a+1)|x_1(t)\\
&\quad-x_2(t)|+b|y_1(t)-y_2(t)|+(|x_1(t)|+|x_2(t)|)|x_1(t)-x_2(t)|\\
&\quad+c|z_1(t)-z_2(t)|\\
&\le (a+1+2m_1+2m_2)(|x_1(t)-x_2(t)|+(2m_2+b)|y_1(t)-y_2(t)|\\
&\quad+(c+1)|z_1(t)-z_2(t)|,\end{aligned}$$

where $m_1=\sup_{t\in[0,T]}(|x_1(t)|,|x_2(t)|)$ and $m_2=\sup_{t\in[0,T]}(|y_1(t)|,|y_2(t)|)$. Hence, the Lipschitz condition holds and F satisfies

$$|F(t,u_1)-F(t,u_2)|\le L|u_1-u_2|, \tag{8}$$

where $L = \max\{a+1+2m_1+2m_2, b+2m_2, c+1\}$. □

Now, we consider the following hypothesis:

(H_0) : There exist positive constants ϕ_1 and ϕ_2 such that

$|F(t,Z(t))| \leq \phi_1||Z||+\phi_2$, for all $t \in [0,T]$.

In addition, we define the operators A and B such that:

$$Au(t) = \frac{\omega(0)u_0}{\omega(t)} + \frac{1-p}{N(p)}F(t,u(t))$$

,

$$Bu(t) = \frac{p}{N(p)}\frac{1}{\Gamma(q)}\frac{1}{\omega(t)}\int_0^t (t-\tau)^{q-1}\omega(\tau)F(\tau,u(\tau))d\tau.$$

Moreover, we set $\beta_1 = \left(\frac{1-p}{N(p)} + \frac{pT^q}{N(p)\Gamma(q+1)}\right)\phi_1$. Thus, we obtain the following result.

Theorem 3.2. *If (H_0) is satisfied, then our FDE model (5) has at least one solution when $\beta_1 < 1$ and $\frac{L(1-p)}{N(p)} < 1$.*

Proof. Let $M_m = \{u \in \mathscr{C} : \|u\| \leq m\}$ be a closed and convex set, where $m \geq \frac{\beta_2}{1-\beta_1}$ and $\beta_2 = |u_0| + \left(\frac{1-p}{N(p)} + \frac{pT^q}{N(p)\Gamma(q+1)}\right)\phi_2$.

First, we show that $A\psi_1 + B\psi_2 \in E_m$ for all $\psi_1, \psi_2 \in M_m$. Using hypothesis (H_0), we have

$$\begin{aligned}
\|A\psi_1 + B\psi_2\| &= \max_{t\in[0,T]}\left|\frac{\omega(0)u_0}{\omega(t)} + \frac{1-p}{N(p)}F(t,\psi_1(t))\right.\\
&\quad \left. + \frac{p}{N(p)\Gamma(q)\omega(t)}\int_0^t (t-\tau)^{q-1}\omega(\tau)F(\tau,\psi_2(\tau))\,d\tau\right|.\\
&\leq \max_{t\in[0,T]}\left\{\left|\frac{\omega(0)u_0}{\omega(t)}\right| + \frac{1-p}{N(p)}|F(t,\psi_1(t))|\right.\\
&\quad \left. + \frac{p}{N(p)\Gamma(q)\,\omega(t)}\int_0^t (t-\tau)^{q-1}\omega(\tau)\,|F(\tau,\psi_2(\tau))|\,d\tau\right\}.
\end{aligned}$$

It follows from $\omega(0) < \omega(t)$, for all $t \geq 0$, that

$$\begin{aligned}
\|A\psi_1 + B\psi_2\| &\leq |u_0| + \frac{1-p}{N(p)}(\phi_1\|\psi_1\|+\phi_2) + \frac{pT^q}{N(p)\Gamma(q+1)}(\phi_1\|\psi_2\|+\phi_2)\\
&= |u_0| + \left(\frac{1-p}{N(p)} + \frac{pT^q}{N(p)\Gamma(q+1)}\right)\phi_2 + \left(\frac{1-p}{N(p)} + \frac{pT^q}{N(p)\Gamma(q+1)}\right)\phi_1 m\\
&= \beta_2 + \beta_1 m \leq m.
\end{aligned}$$

Therefore, $A\psi_1 + B\psi_2 \in M_m$. This verifies the condition (i) of Lemma 2.5.

For the condition (ii) of Lemma 2.5, we have for all $u_1, u_2 \in E_m$ that

$$\begin{aligned}\|Au_1 - Au_2\| &= \max_{t\in[0,T]} \frac{1-p}{N(p)} |F(t,u_1(t)) - F(t,u_2(t))| \\ &\leq \frac{(1-p)}{N(p)} L \|u_1 - u_2\|.\end{aligned}$$

As $\frac{L(1-p)}{N(p)} < 1$, we deduce that A is a contraction mapping.

Finally, we verify that condition (iii) of Lemma 2.5 is satisfied. To this end, we show that B is continuous, uniformly bounded and equicontinuous. Clearly, the operator B is continuous due to the continuity of F. Let $u \in M_m$, we have

$$\begin{aligned}\|Bu\| &= \max_{t\in[0,T]} \left| \frac{p}{N(p)\Gamma(q)\,\omega(t)} \int_0^t (t-\tau)^{q-1}\,\omega(\tau)F(\tau,u(\tau))\,d\tau \right| \\ &\leq \frac{pT^q}{N(p)\Gamma(q+1)} \left[\phi_1\|u\| + \phi_2\right] \\ &\leq \frac{pT^q}{N(p)\Gamma(q+1)} \left(\phi_1 m + \phi_2\right).\end{aligned}$$

Therefore, B is uniformly bounded on M_m. To show the equicontinuity of B, let $u \in M_m$ and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then

$$\begin{aligned}|Bu(t_2) - Bu(t_1)| = \frac{p}{N(p)\Gamma(q)} &\left| \int_0^{t_2} (t_2-\tau)^{q-1} \frac{\omega(\tau)}{\omega(t_2)} F(\tau,u(\tau))\,d\tau \right. \\ &\left. - \int_0^{t_1} (t_1-\tau)^{q-1} \frac{\omega(\tau)}{\omega(t_1)} F(\tau,u(\tau))\,d\tau \right| \\ = \frac{p}{N(p)\Gamma(q)} &\left| \int_0^{t_1} \left[(t_2-\tau)^{q-1}\frac{1}{\omega(t_2)} - (t_1-\tau)^{q-1}\frac{1}{\omega(t_1)}\right] \omega(\tau)F(\tau,u(\tau))\,d\tau \right. \\ &\left. + \int_{t_1}^{t_2} (t_2-\tau)^{q-1} \frac{\omega(\tau)}{\omega(t_2)} F(\tau,u(\tau))\,d\tau \right| \\ \leq \frac{p}{N(p)\Gamma(q)} &\int_0^{t_1} \left| (t_2-\tau)^{q-1}\frac{1}{\omega(t_2)} - (t_1-\tau)^{q-1}\frac{1}{\omega(t_1)} \right| \omega(\tau)|F(\tau,u(\tau))|\,d\tau \\ &+ \frac{p}{N(p)\Gamma(q)} \int_{t_1}^{t_2} (t_2-\tau)^{q-1} \frac{\omega(\tau)}{\omega(t_2)} |F(\tau,u(\tau))|\,d\tau.\end{aligned}$$

Hence,

$$|Bu(t_2) - Bu(t_1)| \leq \frac{p}{N(p)\Gamma(q+1)} (\phi_1\|Z\| + \phi_2) \left[(t_2-t_1)^q \frac{1}{\omega(t_2)} - t_2^q \frac{1}{\omega(t_2)} + t_1^q \frac{1}{\omega(t_1)}\right]$$

$$+\frac{p}{N(p)\Gamma(q+1)}(\phi_1\|Z\|+\phi_2)\left[(t_2-t_1)^q\frac{1}{\omega(t_2)}\right]$$
$$\leq \frac{2p}{N(p)\Gamma(q+1)}(\phi_1 m+\phi_2)\left[(t_2-t_1)^q\frac{1}{\omega(t_2)}\right].$$

Since $t_1 \to t_2$, the right-hand side of the above inequality approaches zero. Therefore, B is equicontinuous. By the Arzelà-Ascoli theorem, it follows that B is relatively compact and hence completely continuous. Consequently, condition (iii) of Lemma 2.5 is satisfied, which allows us to conclude that FDE model (5) has at least one solution. □

Theorem 3.3. *Assume that $L < \frac{N(p)}{1-p}$. If u and v are two solutions of (5), then $u = v$. This implies the uniqueness of solution.*

Proof. Let u and v are two solutions of (5). We have

$$u(t)-v(t) = \mathscr{I}_{0,\omega}^{p,q}\left(F(t,u(t)-F(t,v(t))\right).$$

According to Lemma 3.1, we deduce that

$$|u(t)-v(t)| \leq L\mathscr{I}_{0,\omega}^{p,q}|u(t)-v(t)|.$$

Using Lemma 2.4, we have

$$|u(t)-v(t)| \leq \frac{N(p)\times 0}{N(p)-(1-p)L}E_q\left(\frac{pLT^q}{N(p)-(1-p)L}\right),$$

which leads that $u(t) = v(t)$, for all $t \in [0,T]$. □

Theorem 3.4. *If $L(\frac{1-p}{N(p)}+\frac{T^q}{\Gamma(q+1)}) < 1$, then system (5) has a unique solution for any initial condition.*

Proof. We consider the operator $\Phi : \mathscr{C} \to \mathscr{C}$ as follows

$$(\Phi u)(t) = \frac{\omega(0)u(0)}{\omega(t)} + \mathscr{I}_{0,\omega}^{p,q}F(t,u(t)),\ t \in [0,T].$$

It suffices to prove that the operator Φ has a unique fixed point. We first prove that Φ is well defined. We have

$$|(\Phi u)(t)| = |\frac{\omega(0)u_0}{w(t)} + \mathscr{I}_{0,\omega}^{p,q}F(t,u(t))|$$
$$\leq |u_0|\frac{\omega(0)}{\omega(t)} + \mathscr{I}_{0,\omega}^{p,q}|F(t,u(t))|.$$

Since $\omega(0) < \omega(t)$, for all $t \geq 0$, F is Lipschitz continuous and $t \leq T$, we deduce that $|F(u(t))|$ is bounded by a constant D and

$$\begin{aligned} |(\Phi u)(t)| &\leq |u_0| + D\mathscr{I}_{0,w}^{p;q}(1) \\ &\leq |u_0| + D\left(\frac{1-p}{N(p)} + \frac{pT^q}{N(p)\Gamma(q+1)}\right). \end{aligned}$$

Hence, the operator Φ is well defined. On the other hand, we have for all $u_1, u_2 \in \mathscr{C}$ and $t \in [0,T]$ that

$$\begin{aligned} |(\Phi u_1)(t) - (\Phi u_2)(t)| &= |\mathscr{I}_{0,w}^{p;q} F(t,u_1(t)) - F(t,u_2(t))| \\ &\leq |\frac{1-p}{N(p)}(F(t,u_1(t)) - F(t,u_2(t))) + \frac{p}{N(p)}{}^{RL}\mathscr{I}_{0,w}^{q}(F(t,u_1(t)) - F(t,u_2(t)))| \\ &\leq \frac{1-p}{N(p)} L|u_1 - u_2| + \frac{p}{N(p)} L||u_1 - u_2||_{\mathscr{C}} \frac{T^q}{\Gamma(q+1)}. \end{aligned}$$

Then

$$||\Phi u_1 - \Phi u_2||_{\mathscr{C}} \leq L\left(\frac{1-p}{N(p)} + \frac{pT^q}{N(p)\Gamma(q+1)}\right)||u_1 - u_2||.$$

Since $L\left(\frac{1-p}{N(p)} + \frac{pT^q}{N(p)\Gamma(q+1)}\right) < 1$, we deduce that Φ is a contraction mapping. Hence, system (5) has a unique solution. □

In the following, we establish the equilibrium points of our model.

Theorem 3.5.

(i) *If* $2+2c-b-abc \leq 0$, *then system (5) has only a trivial equilibrium of the form*

$$E_1\left(0, \frac{2}{b}, 0\right).$$

(ii) *If* $2+2c-b-abc > 0$, *then system (5) has besides to* E_1 *two other equilibrium points*

$$E_2\left(\sqrt{\frac{2+2c-b-abc}{c+1}}, \frac{ac+1}{c+1}, \frac{a-1}{c+1}\sqrt{\frac{2+2c-b-abc}{c+1}}\right),$$

and

$$E_3\left(-\sqrt{\frac{2+2c-b-abc}{c+1}}, \frac{ac+1}{c+1}, \frac{-(a-1)}{c+1}\sqrt{\frac{2+2c-b-abc}{c+1}}\right).$$

Proof. Any equilibrium point of system (5) clearly satisfies the subsequent algebraic equations:

$$z+(y-a)x=0, \tag{9}$$

$$2-by-x^2=0, \tag{10}$$

$$xy-x-cz=0. \tag{11}$$

From (9), we get $z=-(y-a)x$ and replacing it in (11), we have $x=0$ or $y=\dfrac{1+ac}{1+c}$. So, we discuss two cases.

- If $x=0$, then $y=\dfrac{2}{b}$ and $z=0$. Hence, the first financial equilibrium is $E_1(0,\frac{2}{b},0)$.
- If $x\neq 0$, then $y=\dfrac{1+ac}{1+c}$. It follows from (10) that if $2+2c-b-abc>0$, we find two equilibrium points $E_2\left(\sqrt{\frac{2+2c-b(1+ac)}{1+c}},\dfrac{1+ac}{1+c},z^*\right)$ and $E_3\left(-\sqrt{\frac{2+2c-b(1+ac)}{1+c}},\dfrac{1+ac}{1+c},-z^*\right)$, where $z^*=\dfrac{a-1}{c+1}\sqrt{\frac{2+2c-b(1+ac)}{1+c}}$.

□

4. Stability Analysis

In this section, we focus on the local stability of the three equilibrium points of our FDE model.

For any equilibrium point $E^*(x^*,y^*,z^*)$, let $X=x-x^*$, $Y=y-y^*$ and $Z=z-z^*$. By substituting X, Y and Z into system (5) and linearizing, we get the following system

$$\begin{cases} \mathscr{D}^{p,q}_{0,\omega}X(t)=Z(t)+X^*Y(t)+Y^*X(t)-aX(t), \\ \mathscr{D}^{p,q}_{0,\omega}Y(t)=-bY(t)-2X^*X(t), \\ \mathscr{D}^{p,q}_{0,\omega}Z(t)=Y^*X(t)+X^*Y(t)-X(t)-cZ(t). \end{cases} \tag{12}$$

By applying the Laplace transform to system (12), we obtain

$$\Delta(s).\begin{pmatrix}\tilde{X}(s)\\ \tilde{Y}(s)\\ \tilde{Z}(s)\end{pmatrix}=\begin{pmatrix}b_1(s)\\ b_2(s)\\ b_3(s)\end{pmatrix},$$

where $\tilde{X}(s)=\mathscr{L}\{\omega(t)X(t)\}$, $\tilde{Y}(s)=\mathscr{L}\{\omega(t)Y(t)\}$, $\tilde{Z}(s)=\mathscr{L}\{\omega(t)Z(t)\}$,

$$\begin{cases} b_1(s) = s^{q-1}N(p)\omega(0)X(0), \\ b_2(s) = s^{q-1}N(p)\omega(0)Y(0), \\ b_3(s) = s^{q-1}N(p)\omega(0)Z(0), \end{cases}$$

and

$$\Delta(s) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & 0 \\ x_7 & x_8 & x_9 \end{pmatrix},$$

with

$$\begin{aligned}
x_1 &= s^q N(p) - (y^* - a)(1-p)(s^q + \mu_p), \\
x_2 &= -x^*(1-p)(s^q + \mu_p), \\
x_3 &= -(1-p)(s^q + \mu_p), \\
x_4 &= 2x^*(1-p)(s^q + \mu_p), \\
x_5 &= s^q N(p) + b(1-p)(s^q + \mu_p), \\
x_6 &= 0, \\
x_7 &= -(1-p)(s^q + \mu_p)(y^* - 1), \\
x_8 &= -(1-p)(s^q + \mu_p)x^*, \\
x_9 &= s^q N(p) + c(1-p)(s^q + \mu_p).
\end{aligned}$$

Thus, the characteristic equation about E^* is given by

$$s^{3q} + a_1 s^{2q} + a_2 s^q + a_3 = 0, \tag{13}$$

where

$$\begin{aligned}
a_1 = {} & \frac{3\mu_p(1-p)^3[2X^{*^2}(1+c) + b(1-Y^*) + bc(a-Y^*)] + 2\mu_p N(p)(1-p)^2[2X^{*^2} + (1-Y^*)}{\xi_1} \\
& + \frac{c(a+b-Y^*) + b(a-Y^*)] + N^2(p)(1-p)\mu_p(a+b+c-Y^*)}{\xi_1}, \\
a_2 = {} & \frac{3\mu_p{}^2(1-p)^3(2X^{*^2}(1+c) + b(1-Y^*) + bc(a-Y^*)) + \mu_p{}^2 N(p)(1-p)^2}{\xi_1}
\end{aligned}$$

$$+\frac{(2X^{*^2}+(1-Y^*)+b(a-Y^*)+c(a+b-Y^*))}{\xi_1},$$

$$a_3=\frac{\mu_p^3(1-p)^3(2X^{*^2}(1+c)+b(1-Y^*)+bc(a-Y^*)}{\xi_1},$$

with $\xi_1=(1-p)^3[2X^{*^2}(1+c)+b(1-Y^*)+bc(a-Y^*)]+N(p)(1-p)^2[2X^{*^2}+(1-Y^*)+c(a+b-Y^*)+b(a-Y^*)]+N^2(p)(1-p)(a+b+c-Y^*)+N^3(p)$.

Let $s^q=\lambda$. Hence, (13) becomes

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0. \tag{14}$$

In the order to investigate the stability of the first equilibrium E_1 of (5), we consider the following hypotheses:

(H_1) : $N(p)(1-p)^2[ab+ac+bc-\frac{2+2c}{b}-1]+N(p)^2(1-p)(a+b+c-\frac{2}{b})+N^3(p)>(1-p)^3(2+2c-b-abc)$.

(H_2) : $N(p)^2(1-p)(a+b+c-\frac{2}{b})+2N(p)(1-p)^2(ab+ac+bc-\frac{2+2c}{b}-1)>3(1-p)^3(2+2c-b-abc)$.

(H_3) : $\dfrac{N(p)^3(ab+ac+bc-\frac{2+2c}{b}-1)(a+b+c-\frac{2}{b})}{-2-2c+b+abc}>N(p)^2-(1-p)^3(9-\mu_p)(-2-2c+b+abc)-N(p)(1-p)^2(9-2\mu_p)(ab+ac+bc-\frac{2+2c}{b}-1)-2(1-p)N(p)^2(a+b+c-\frac{2}{b})$.

Theorem 4.1.

(i) *If 2+2c-b-abc <0 and (H_1) - (H_3) hold, then the trivial equilibrium $E_1(0,\frac{2}{b},0)$ is locally asymptotically stable.*

(ii) *If 2+2c-b-abc >0 and (H_1) holds, then the trivial equilibrium $E_1(0,\frac{2}{b},0)$ is unstable.*

Proof. For $E_1(0,\frac{2}{b},0)$, (14) becomes

$$\lambda^3+p_1\lambda^2+p_2\lambda+p_3=0, \tag{15}$$

where

$$p_1=\frac{3b\mu_p(1-p)^3(-2-2c+b+abc)+2\mu_pN(p)(1-p)^2(ab^2+abc+b^2c-2-2c-b)}{\xi_2}$$
$$+\frac{\mu_pN^2(p)(1-p)(ab+b^2+bc-2)}{\xi_2},$$

$$p_2=\frac{3b{\mu_p}^2(1-p)^3(-2-2c+b+abc)+N(p)(1-p)^2{\mu_p}^2(ab^2+abc+b^2c-2-2c-b)}{\xi_2},$$

$$p_3=\frac{b\mu_p^3(1-p)^3(-2-2c+b+abc)}{\xi_2},$$

and $\xi_2=(1-p)^3b(-2-2c+b+abc)+N(p)(1-p)^2[ab^2+abc+b^2c-2-2c-b]+N^2(p)(1-p)(ab+b^2+bc-2)+bN^3(p)$.

For (i), we have $2+2c-b-abc<0$ and (H_1) - (H_2) hold. Then

$$p_1>0,\ \ p_3>0,\ \ and\ \ p_1p_2-p_3>0.$$

Based on Routh-Hurwitz criterion, all the roots of equation (14) have negative real parts.Therefore, the financial equilibrium E_1 is locally asymptotically stable.

For (ii), we consider the following function:

$$f(\lambda)=\lambda^3+p_1\lambda^2+p_1\lambda+p_3.$$

We have $\lim_{\lambda\to+\infty} f(\lambda)=+\infty$ and $f(0)=p_3<0$. Thus, the financial equilibrium E_1 is unstable. □

Next, we analyze the stability of the two remaining equilibrium points. In this case, we consider the following hypotheses:

(A_1) : $N(p)(1-p)[4-3b\frac{1+ac}{1+c}+b(a+c)]+N(p)^2(\frac{a-1}{1+c}+b+c)+N(p)>-2(1-p)^2(2+2c-b-abc)$.

(A_2) : $2N(p)(1-p)(4-3b\frac{1+ac}{1+c}+b(a+c))+N(p)^2(\frac{a-1}{1+c}+b+c)>-6(1-p)(2+2c-b-abc)$.

(A_3) : $N(p)^3(4+b(a+c))(\frac{a-1}{1+c}+b+c)+2N(p)^2(1-p)(4+b(a+c)-3b\frac{1+ac}{1+c})^2>2(2+2c-b-abc)[(N(p)^3+\mu_pN(p)(5p-8)(1-p)^4(4+b(a+c)-3b\frac{1+ac}{1+c})+2N(p)^2(p-1)(\frac{a-1}{1+c}+b+c)-2(1-p)^2(8+p)(2+2c-b-abc)]$.

Theorem 4.2. *If 2+2c-b-abc >0 and (A_1) - (A_3) hold, then the financial equilibrium points E_2 and E_3 are locally asymptotically stable.*

Proof. For E_2 and E_3, (14) becomes

$$\lambda^3+q_1\lambda^2+q_2\lambda+q_3=0, \tag{16}$$

where

$$q_1 = \frac{6\mu_p(1-p)^2(2+2c-b-abc)(1+c)+2\mu_p N(p)(1-p)^2((4+b(a+c))}{\xi_3}$$
$$+\frac{(1+c)-3b(1+ac))+N(p)^2(1-p)\mu_p((a-1)+(b+c)(1+c))}{\xi_3},$$
$$q_2 = \frac{6\mu_p^2(1-p)^3(2+2c-b-abc)(1+c)+\mu_p^2(1-p)^2N(p)((4+b(a+c))}{\xi_3}$$
$$+\frac{(1+c)-3b(1+ac))}{\xi_3},$$
$$q_3 = \frac{2\mu_p^3(1-p)^3(2+2c-b-abc)(1+c)}{\xi_3},$$

and $\xi_3 = 2(1-p)^3(1+c)(2+2c-b-abc)+N(p)(1-p)^2((4+b(a+c))(1+c)-3b(1+ac))+N(p)^2(1-p)((a-1)+(b+c)(1+c))+N(p)^3(1+c)$.

For $2+2c-b-abc>0$ and the conditions (A_1) - (A_3) hold, we have

$$q_1 > 0, \;\; q_3 > 0 \;\; and \;\; q_1q_2 - q_3 > 0.$$

According to the Routh-Hurwitz criterion, we deduce that all the roots of equation (16) have negative real parts. Therefore, the financial equilibrium E_2 and E_3 are locally asymptotically stable. □

5. Numerical Simulations

This section presents some numerical simulations to illustrate our analytical results. The discretization of the continuous model (5) is based on the numerical method introduced in [21].

Firstly, we choose $a=3$, $b=2$ and $c=1$. In this case, we have $2+2c-b-abc=-4<0$. Then system (5) has the unique equilibrium point $E_1(0,1,0)$, which is locally asymptotically stable. Figure 1 demonstrates this result.

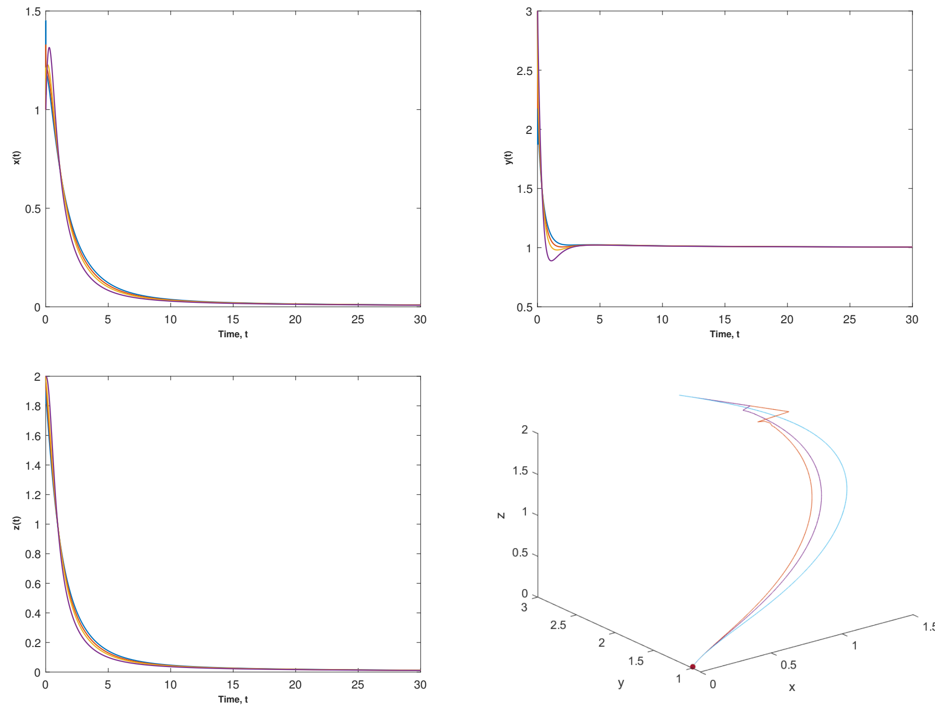

FIGURE 1. Stability of the trivial equilibrium E_1.

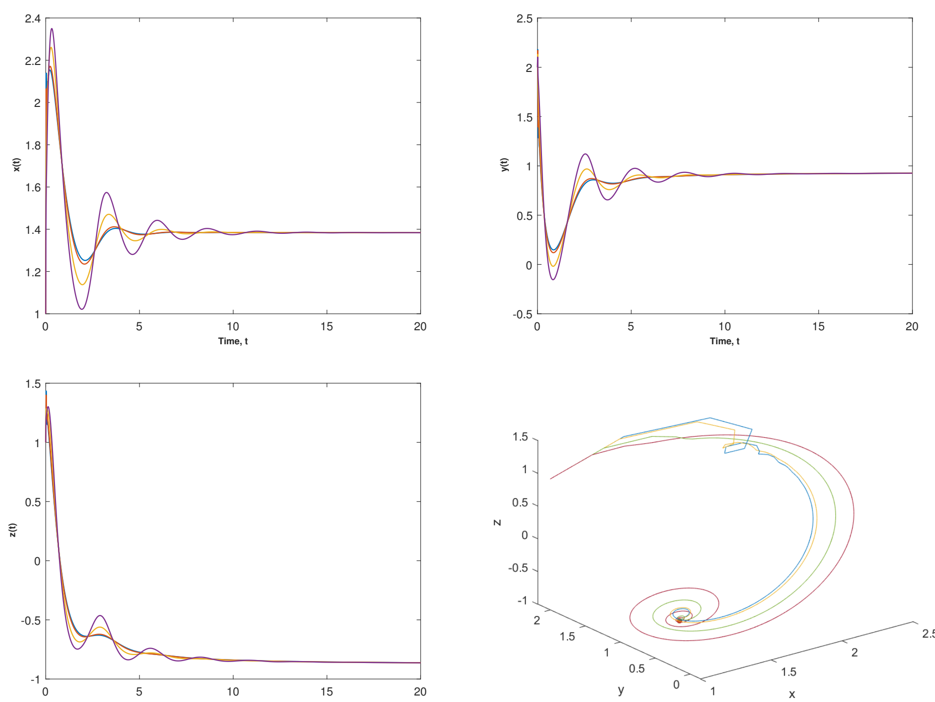

FIGURE 2. Stability of the equilibrium E_3.

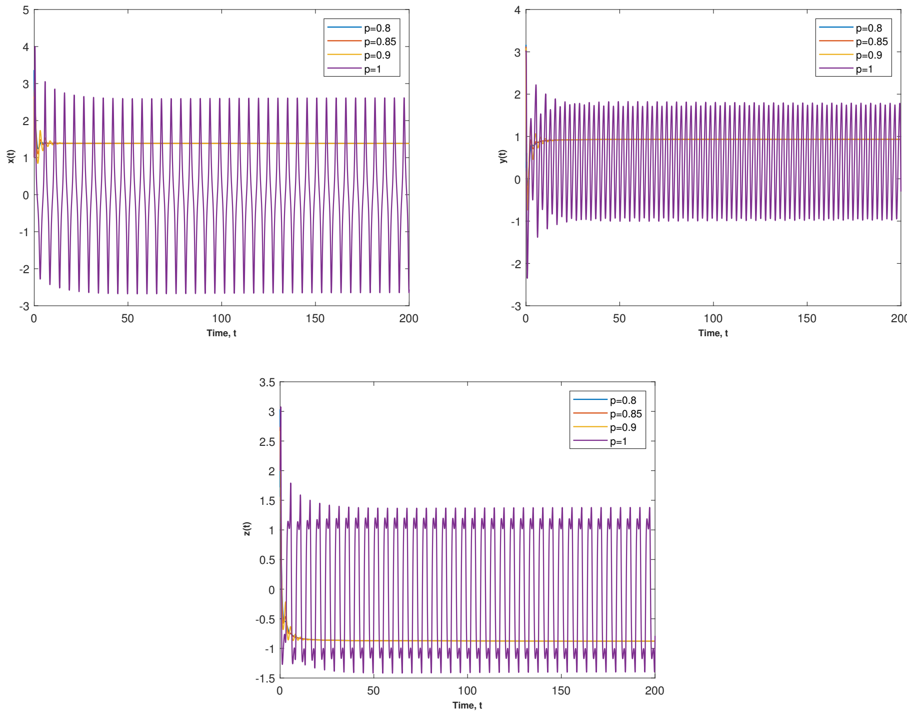

FIGURE 3. Behavior of the financial model (5) under varying fractional order p.

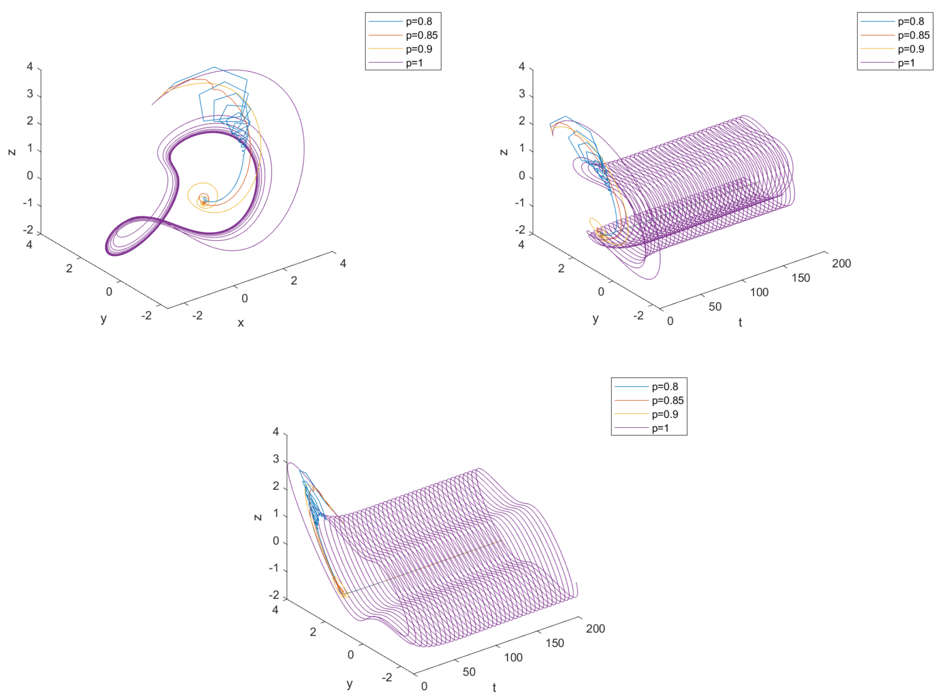

FIGURE 4. Phase portraits for different values of p.

Secondly, we choose $a = 0.3$, $b = 0.1$, $c = 0.1$. In this case, the equilibrium point $E_3(1.38071, 0.936364, -0.878634)$ is asymptotically stable. Figure 2 illustrates this result.

Finally, Figures 3 and 4 illustrate the dynamical behaviors of our FDE model (5) for different values of fractional order p.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] P.P. Drake, F.J. Fabozzi, The Basics of Finance: An Introduction to Financial Markets, Business Finance, and Portfolio Management, Wiley, 2010. https://doi.org/10.1002/9781118267790.

[2] J. Ma, Y. Chen, Study for the Bifurcation Topological Structure and the Global Complicated Character of a Kind of Nonlinear Finance System (I), Appl. Math. Mech. 22 (2001), 1240–1251. https://doi.org/10.1007/bf02437847.

[3] J. Ma, Y. Chen, Study for the Bifurcation Topological Structure and the Global Complicated Character of a Kind of Nonlinear Finance System (II), Appl. Math. Mech. 22 (2001), 1375–1382. https://doi.org/10.1023/a:1022806003937.

[4] W. Chen, Dynamics and Control of a Financial System with Time-Delayed Feedbacks, Chaos Solitons Fractals 37 (2008), 1198–1207. https://doi.org/10.1016/j.chaos.2006.10.016.

[5] H. Yu, G. Cai, Y. Li, Dynamic Analysis and Control of a New Hyperchaotic Finance System, Nonlinear Dyn. 67 (2011), 2171–2182. https://doi.org/10.1007/s11071-011-0137-9.

[6] O. Tacha, C.K. Volos, I. Kyprianidis, I. Stouboulos, S. Vaidyanathan, et al., Analysis, Adaptive Control and Circuit Simulation of a Novel Nonlinear Finance System, Appl. Math. Comput. 276 (2016), 200–217. https://doi.org/10.1016/j.amc.2015.12.015.

[7] Y. He, J. Peng, S. Zheng, Fractional-Order Financial System and Fixed-Time Synchronization, Fractal Fract. 6 (2022), 507. https://doi.org/10.3390/fractalfract6090507.

[8] M. Caputo, Linear Models of Dissipation Whose Q Is Almost Frequency Independent–II, Geophys. J. Int. 13 (1967), 529–539. https://doi.org/10.1111/j.1365-246x.1967.tb02303.x.

[9] R. Chauhan, S. Kumar, B.S.T. Alkahtani, S.S. Alzaid, A Study on Fractional Order Financial Model by Using Caputo–Fabrizio Derivative, Results Phys. 57 (2024), 107335. https://doi.org/10.1016/j.rinp.2024.107335.

[10] M. Caputo, M. Fabrizio, A new Definition of Fractional Derivative without Singular Kernel, Progr. Fract. Differ. Appl. 1 (2015), 73–85.

[11] C. Liping, M.A. Khan, A. Atangana, S. Kumar, A New Financial Chaotic Model in Atangana-Baleanu Stochastic Fractional Differential Equations, Alex. Eng. J. 60 (2021), 5193–5204. https://doi.org/10.1016/j.aej.2021.04.023.

[12] A. Atangana, D. Baleanu, New Fractional Derivatives with Nonlocal and Non-Singular Kernel: Theory and Application to Heat Transfer Model, Therm. Sci. 20 (2016), 763–769. https://doi.org/10.2298/tsci160111018a.

[13] K. Hattaf, A New Generalized Definition of Fractional Derivative with Non-Singular Kernel, Computation 8 (2020), 49. https://doi.org/10.3390/computation8020049.

[14] Z. Hajhouji, K. Hattaf, N. Yousfi, A Generalized Fractional HIV-1 Infection Model with Humoral Immunity and Highly Active Antiretroviral Therapy, J. Math. Comput. Sci. 32 (2023), 160–174. https://doi.org/10.22436/jmcs.032.02.06.

[15] F. Assadiki, M. El Younoussi, K. Hattaf, N. Yousfi, Dynamics of an Ecological Prey–Predator Model Based on the Generalized Hattaf Fractional Derivative, Math. Model. Comput. 11 (2024), 166–177. https://doi.org/10.23939/mmc2024.01.166.

[16] S. Lasfar, M. Elkarmouchi, K. Hattaf, N. Yousfi, Mathematical Analysis of the Dynamics of a Fractional Economic Cycle Model and the Existence of Solutions by Means of Fixed Point Theory, Adv. Fixed Point Theory 14 (2024), 5. https://doi.org/10.28919/afpt/8350.

[17] M. Al-Refai, On Weighted Atangana–Baleanu Fractional Operators, Adv. Differ. Equ. 2020 (2020), 3. https://doi.org/10.1186/s13662-019-2471-z.

[18] K. Hattaf, A.A. Mohsen, H.F. Al-Husseiny, Gronwall Inequality and Existence of Solutions for Differential Equations with Generalized Hattaf Fractional Derivative, J. Math. Comput. Sci. 27 (2022), 18–27. https://doi.org/10.22436/jmcs.027.01.02.

[19] T. Burton, A Fixed-Point Theorem of Krasnoselskii, Appl. Math. Lett. 11 (1998), 85–88. https://doi.org/10.1016/s0893-9659(97)00138-9.

[20] M.A. Krasnoselskii, Some Problems of Nonlinear Analysis, Amer. Math. Soc. Transl. (2) 10 (1958), 345–409.

[21] K. Hattaf, On the Stability and Numerical Scheme of Fractional Differential Equations with Application to Biology, Computation 10 (2022), 97. https://doi.org/10.3390/computation10060097.