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BEST PROXIMITY POINTS FOR GENERALIZED PROXIMITY CONTRACTION IN COMPLETE METRIC SPACES

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Abstract. In this paper, we define a new type of generalized proximal cyclic contraction mapping and give some theorems on existence of the best proximity point of generalized proximal cyclic contraction in the complete metric spaces.

Keywords: Metric space, Generalized cyclic contraction, Fixed point theorem.

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1. Introduction

The Banach Contraction Principle states that, if a self-mapping T of a complete metric space X is a contraction mapping, then T has a unique fixed point. This principle has been extended in several ways such as [4] and [6]. In 2003 Kirk-Srinavasan-Veeramani [16] introduced the notion of cyclic contraction mapping and proved some fixed point theorems for the operators in the class of cyclic contraction. In 2005, Eldred, Kirk and Veeramani [8] proved the existence of a best proximity point for relatively nonexpansive mappings by using the notion of proximal normal structure. In 2006, Eldred and Veeramani [9] introduced the notion of cyclic contraction and gave sufficient condition for the existence

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of a best proximity point for a cyclic contraction mapping T on a uniformly convex Banach space.

Fixed point theory plays an important role in furnishing a uniform treatment to solve various equations of the form $Tx = x$ for self-mappings T defined on subsets of metric spaces. Given two nonempty subsets A and B of a metric space, consider a non-self mapping T from A to B. Because T is not a self-mapping, the equation $Tx = x$ is unlikely to have a solution. Therefore, it is of primary importance to seek an element x that in some sense is closest to Tx. That is, when the equation $Tx = x$ has no solution, one tries to determine an approximate solution x subject to the condition that the distance between x and Tx is minimal. Best approximation theorems and best proximity point theorems are relevant in this perspective. A classical best approximation theorem, due to Fan [10], states that if A is a nonempty compact and convex subset of a Hausdorff locally convex topological vector space X and $T : A \to X$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. There have been many subsequent extensions and variants of Fans Theorem, see [20, 21, 27, 28, 31] and references therein. On the otherhand, though best approximation theorems ensure the existence of approximate solutions, such results need not yield optimal solutions. But, best proximity point theorems furnish sufficient conditions that assure the existence of approximate solutions which are optimal as well. Indeed, if there is no exact solution to the fixed point equation $Tx = x$ for a non-self mapping $T : A \to B$, then a best proximity theorem offers sufficient conditions for the existence of an optimal approximate solution x, called a best proximity point of the mapping T , satisfying the condition that $d(x, Tx) = d(A, B)$. A best proximity point theorem for non-self proximal contractions has been investigated in [23]. Further [2, 9, 7, 14] examine several variants of contractions for the existence of a best proximity point. Anuradha and Veeramani [3] derived a best proximity point theorem for proximal pointwise contractions. Eldred, Kirk, and Veeramani [8] obtained a best proximity point theorem for relatively nonexpansive mappings. A best proximity point theorem for contractive non-self-mappings has been established in [23]. Further, best proximity point theorems for set-valued mappings have been elicited in

[15, 25, 26, 29]. This paper presents a best proximity point theorem for generalized cyclic Kannan type contractions in the setting of complete metric spaces.

2. Preliminaries

Let (X, d) be a complete metric space and A, B be nonempty subsets of X. A mapping $T : X \to X$ is a contraction if and only if for each $x, y \in X$ there exists a constant $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$. Let $T : A \cup B \to A \cup B$ such that $T(A) \subset B$ and $T(B) \subset A$ we say that

(i) T is cyclic contraction if

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B), \tag{2.1}$$

for some $\alpha \in (0, 1)$ and for all $x \in A$ and $y \in B$ where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$

(ii) $x \in A \cup B$ is a best proximity point for T if $d(x, Tx) = d(A, B)$.

3. Main results

In this section, we introduced the definition of generalized proximal cyclic contraction and establish existence of the best proximity point by considering some sequences which converge to that best proximity point.

Definition 3.1. Let A and B be nonempty subsets of a complete metric space (X, d). A map $T : A \cup B \to A \cup B$ is called a generalized cyclic proximity contraction if the following conditions hold:

(1) $T(A) \subset B$ and $T(B) \subset A$;

(2) There exists $a_1, a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 < 1$ such that

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + (1 - (a_1 + a_2 + a_3))d(A, B), \tag{3.1}$$

for all $x \in A, y \in B$.

Lemma 3.1. Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose the mapping $T : A \cup B \to A \cup B$ be a generalized cyclic proximity contraction.

Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

Proof. Suppose $x_0 \in A \cup B$. Define an iterative sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in N$. Now from **(3.1)**, we get

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \\ &\le a_1 d(x_0, x_1) + a_2 d(x_0, Tx_0) + a_3 d(x_1, Tx_1) + [1 - (a_1 + a_2 + a_3)]d(A, B) \\ &\le a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + (1 - a_1 - a_2 - a_3)d(A, B) \\ &\le \frac{a_1 + a_2}{1 - a_3} d(x_0, x_1) + \left(1 - \frac{a_1 + a_2}{1 - a_3}\right) d(A, B) \end{aligned}$$

which implies that

$$\begin{aligned} d(x_1, x_2) - d(A, B) &\le \frac{a_1 + a_2}{1 - a_3}\Big[d(x_0, x_1) - d(A, B)\Big] \\ &\le \gamma[d(x_0, x_1) - d(A, B)], \end{aligned} \tag{3.2}$$

where $\gamma = \frac{a_1+a_2}{1-a_3} < 1$.

and

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \\ &\le a_1 d(x_1, x_2) + a_2 d(x_1, Tx_1) + a_3 d(x_2, Tx_2) + [1 - (a_1 + a_2 + a_3)]d(A, B) \\ &\le a_1 d(x_1, x_2) + a_2 d(x_1, x_2) + a_3 d(x_2, x_3) + (1 - a_1 - a_2 - a_3)d(A, B) \\ &\le \frac{a_1 + a_2}{1 - a_3} d(x_1, x_2) + \left(1 - \frac{a_1 + a_2}{1 - a_3}\right) d(A, B) \end{aligned}$$

which implies that

$$\begin{aligned} d(x_2, x_3) - d(A, B) &\le \frac{a_1 + a_2}{1 - a_3}\Big[d(x_1, x_2) - d(A, B)\Big] \\ &= \gamma[d(x_1, x_2) - d(A, B)], \end{aligned} \tag{3.3}$$

Inductively, we obtain

$$d(x_n, x_{n+1}) - d(A, B) \le \gamma^n[d(x_n, x_{n-1}) - d(A, B)].$$

Then we obtain

$$d(x_n, x_{n+1}) - d(A, B) = d(x_n, x_{n-1}) - d(A, B),$$

which implies that $d(x_n, x_{n+1}) < d(x_n, x_{n-1})$ for all $n \in N$. Therefore, the sequence $\{d(x_n, x_{n+1})\}$ is strictly decreasing so the sequence $\{x_n\}$ is a Cauchy sequence. Now from **(3.2)** and **(3.3)**, we have

$$d(x_1, x_2) - d(A, B) = \gamma[d(x_0, x_1) - d(A, B)]$$

and

$$\begin{aligned} d(x_2, x_3) - d(A, B) &= \gamma[d(x_1, x_2) - d(A, B)] \\ &\leq \gamma[\gamma d(x_0, x_1) - \gamma d(A, B)] \\ &=\leq \gamma^2[d(x_0, x_1) - d(A, B)]. \end{aligned}$$

Repeating this process, it follows that

$$\begin{aligned} d(x_n, x_{n+1}) - d(A, B) &= \gamma(d(x_n, x_{n-1}) - d(A, B)) \\ &\leq \gamma^2[d(x_{n-1}, x_{n-2}) - d(A, B)] \\ &\leq \ldots\ldots \\ &\leq \gamma^n[d(x_0, x_1) - d(A, B)]. \end{aligned}$$

Since $\gamma \in [0, 1)$, we have $\lim_{n\to\infty} \gamma^n = 0$ and so

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

This completes the proof.

Lemma 3.2. Let A and B be nonempty closed subsets of a metric space (X, d). Suppose the mapping $T : A \cup B \to A \cup B$ be a generalized cyclic proximity contraction between A and B and $x_{2n} = Tx_{2n-1}$. Then the sequence $\{x_n\}$ is bounded.

Proof. It follows from **Lemma 3.1** that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

Since $\{d(x_{2n-1}, x_{2n})\}$ is subsequence of $\{d(x_n, x_{n+1})\}$. We obtain

$$\lim_{n\to\infty} d(x_{2n-1}, x_{2n}) = d(A, B).$$

Hence $\{d(x_{2n-1}, x_{2n})\}$ is bounded, so there exists $L > 0$ such that

$$d(x_{2n-1}, x_{2n}) \leq L,$$

for all $n \in N$. For each $n \in N$, we have

$$\begin{aligned}
d(x_{2n}, Tx_0) &= d(Tx_{2n-1}, Tx_0) \\
&\le a_1 d(x_{2n-1}, x_0) + a_2 d(x_{2n-1}, Tx_{2n-1}) + a_3 d(x_0, Tx_0) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\le a_1 d(x_{2n-1}, x_0) + a_2 d(x_{2n-1}, x_{2n}) + a_3 d(x_0, Tx_0) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\le a_1 d(x_{2n-1}, x_{2n}) + a_1 d(x_{2n}, x_0) + a_2 d(x_{2n-1}, x_{2n}) + a_3 d(x_0, Tx_0) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\le (a_1 + a_2) d(x_{2n-1}, x_{2n}) + a_1 d(x_{2n}, x_0) + a_3 (d(x_{2n}, Tx_0) + d(x_{2n}, x_0)) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B)
\end{aligned}$$

which implies that

$$\begin{aligned}
d(x_{2n}, Tx_0) &= d(Tx_{2n-1}, Tx_0) \\
&\le a_1 d(x_{2n-1}, x_0) + a_2 d(x_{2n-1}, Tx_{2n-1}) + a_3 d(x_0, Tx_0) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\le a_1 [d(x_0, Tx_0) + d(Tx_0, x_{2n}) + d(x_{2n}, x_{2n-1})] + a_2 d(x_{2n-1}, x_{2n}) \\
&\quad + a_3 d(x_0, Tx_0) + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\le (a_1 + a_2) d(x_{2n-1}, x_{2n}) + (a_1 + a_3) d(x_0, Tx_0) \\
&\quad + a_1 d(x_{2n}, Tx_0) + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\le \frac{a_1 + a_2}{1 - a_1} d(x_{2n-1}, x_{2n}) + \frac{a_1 + a_3}{1 - a_1} d(x_0, Tx_0) \\
&\quad + \left(1 - \frac{a_2 + a_3}{1 - a_1}\right) d(A, B).
\end{aligned}$$

Suppose $\mu = \frac{a_1+a_2}{1-a_1} d(x_{2n-1}, x_{2n}) + \frac{a_1+a_3}{1-a_1} d(x_0, Tx_0) + \left(1 - \frac{a_2+a_3}{1-a_1}\right) d(A, B)$. Therefore, $x_{2n} \in \overline{B}(Tx_0, \mu)$ for all $n \in N$. For each $n \in N$, since

$$d(x_{2n+1}, Tx_0) \le d(x_{2n}, x_{2n+1}) + d(x_{2n}, Tx_0) \le L + \mu,$$

we obtain $x_{2n+1} \in \overline{B}(Tx_0, L+\mu)$ for all $n \in N$. On the other hand, since

$$x_{2n} \in \overline{B}(Tx_0, \mu) \subset \overline{B}(Tx_0, L+\mu)$$

for all $n \in N$, we also have $x_{2n} \in \overline{B}(Tx_0, \mu)$ for all $n \in N$. by above, we get

$$x_{2n} \in \overline{B}(Tx_0, L+\mu)$$

for all $n \in N$. So the sequence $\{x_n\}$ is bounded. This complete the proof.

Theorem 3.1. Let A and B be nonempty closed subsets of a metric space (X, d). Suppose the mapping $T : A \cup B \to A \cup B$ be a generalized cyclic proximity contraction between A and B. For fixed element x_0 in A and suppose $x_{2n} = Tx_{2n-1}$. If the sequence $\{x_{2n_k}\}$ has a subsequence converging to some element x in A. Then x is a best proximity point of T.

Proof. Since $T : A \cup B \to A \cup B$ a generalized cyclic proximity contraction, we have that for $2n_k \in N$ with $2n_k \geq n_0 + 1$. Suppose that a subsequence $\{x_{2n_k}\}$ converges to x in $A \cup B$. Then, it follows from **Lemma 3.1** that $d(x_{2n_k-1}, x_{2n_k}) \to d(A, B)$.
Further, we get

$$\begin{aligned}
d(x_{2n_k}, Tx) &= d(Tx_{2n_k-1}, Tx) \\
&\leq a_1 d(x_{2n_k-1}, x) + a_2 d(x_{2n_k-1}, Tx_{2n_k-1}) + a_3 d(x, Tx) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\leq a_1 d(x_{2n_k-1}, x) + a_2 d(x_{2n_k-1}, x_{2n_k}) + a_3 (d(x, x_{2n}) + d(x_{2n}, Tx)) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\leq a_1 (d(x_{2n_k-1}, x_{2n}) + d(x_{2n_k}, x)) + a_2 d(x_{2n_k-1}, x_{2n_k}) \\
&\quad + a_3 (d(x, x_{2n_k}) + d(x_{2n_k}, Tx)) + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\leq \frac{a_1 + a_2}{1 - a_3} d(x_{2n_k-1}, x_{2n}) + \frac{a_1 + a_3}{1 - a_3} d(x, x_{2n_k}) + \left(1 - \frac{a_1 + a_2}{1 - a_3}\right) d(A, B),
\end{aligned}$$

which implies that

$$d(x_{2n_k}, Tx) \leq \frac{a_1 + a_3}{1 - a_3} d(x, x_{2n_k}) + d(A, B).$$

Since $x_{2n_k} \to x$, then, we conclude that $d(x_{2n_K}, Tx) \to d(A, B)$. Therefore $d(x, Tx) = d(A, B)$. So that x is a best proximity point of T.

Example. Consider the complete metric space X = R with the usual metrics. Suppose that $A = [0, 1/2]$, $B = [1, 1/2]$ and $T : A \cup B \to A \cup B$ is defined by

$$Tx = \begin{cases} 1 & x \in [0, 1/2] \\ 0 & y \in [1, 1/2) \end{cases}$$

for all $x \in A$ and $y \in B$. If $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{3}$ and $a_3 = \frac{1}{9}$. Then it is easy to show that T is a generalized cyclic proximity contraction map.

4. TS-cyclic Proximity Contraction

In this section, we introduced the definition of generalized TS-cyclic proximity contraction and we shall state and prove some existence results of the best proximity point.

Definition 4.1.[24] A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form a k-cyclic mapping between A and B if there exists a nonnegative real number $k < 1/2$ such that

$$d(Tx, Sy) \leq k[d(x, Tx) + d(y, Sy)] + (1 - 2k)d(A, B), \tag{4.1}$$

for $x \in A$ and $y \in B$.

Definition 4.2. Let A and B be nonempty subsets of a complete metric space (X, d). A pair of mappings $S, T : A \cup B \to A \cup B$ is called a TS-cyclic proximity contraction if the following conditions hold:

(1) $T(A) \subset B$ and $T(B) \subset A$;

(2) there exists $a_1, a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 < 1$ such that

$$d(Tx, Sy) \leq a_1 d(x, y) + a_2(x, Tx) + a_3 d(y, Sy) + (1 - (a_1 + a_2 + a_3))d(A, B), \tag{4.2}$$

for all $x \in A, y \in B$.

Lemma 4.1. Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose the mapping $T, S : A \cup B \to A \cup B$ be a TS-cyclic proximity contraction. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A, B).$$

Proof. Suppose $x_0 \in A \cup B$ be given. Define an iterative sequence $\{x_n\}$ by $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n+1}$ for all $n \in N$. Since mappings T and S satisfying TS-cyclic proximity contraction. So from **(4.2)**, we get

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(Tx_{2n}, Sx_{2n+1}) \\
&\leq a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, Tx_{2n}) + a_3 d(x_{2n+1}, Sx_{2n+1}) \\
&\quad + [1 - (a_1 + a_2 + a_3)] d(A, B) \\
&\leq a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, x_{2n+1}) + a_3 d(x_{2n+1}, x_{2n+2}) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\leq \frac{a_1 + a_2}{1 - a_3} d(x_{2n}, x_{2n+1}) + \Big(1 - \frac{a_1 + a_2}{1 - a_3}\Big) d(A, B)
\end{aligned}$$

Now if $\gamma = \frac{a_1+a_2}{1-a_3}$, then, by inductively, we get

$$d(x_{2n+1}, x_{2n+2}) \leq \gamma d(x_{2n}, x_{2n+1}) + (1 - \gamma) d(A, B)$$

also

$$d(x_{2n+2}, x_{2n+3}) \leq \gamma^2 d(x_{2n}, x_{2n+1}) + (1 - \gamma^2) d(A, B)$$

Therefore,

$$d(x_n, x_{n+1}) \leq \gamma^n d(x_n, x_{n-1}) + (1 - \gamma^n) d(A, B).$$

Letting $n \to \infty$, then, we have $\lim_{n\to\infty} \gamma^n = 0$. Hence, the last inequality implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

This completes the proof.

Lemma 4.2. Let A and B be nonempty closed subsets of a metric space (X, d). Suppose the mapping $T, S : A \cup B \to A \cup B$ be a TS-cyclic proximity contraction between A and B. Suppose $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Then the sequence $\{x_n\}$ is bounded.

Proof. It follows from by **Lemma 4.1**, that $\{d(x_{2n-1}, x_{2n})\}$ is convergent and hence it is bounded, we get

$$\begin{aligned}
d(x_{2n}, Tx_0) &= d(Sx_{2n-1}, Tx_0) = d(Tx_0, Sx_{2n-1}) \\
&\leq a_1 d(x_0, x_{2n-1}) + a_2 d(x_0, Tx_0) + a_3 d(x_{2n-1}, Sx_{2n-1}) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\leq a_1 d(x_0, x_{2n-1}) + a_2 d(x_0, Tx_0) + a_3 d(x_{2n-1}, x_{2n}) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\leq a_1 d(x_0, x_{2n-1}) + a_2 (d(x_0, x_{2n} + d(x_{2n}, Tx_0)) \\
&\quad + a_3 d(x_{2n-1}, x_{2n}) + (1 - a_1 - a_2 - a_3) d(A, B)
\end{aligned}$$

which implies that

$$\begin{aligned}
d(x_{2n}, Tx_0) \leq &\frac{a_1 + a_2}{1 - a_2} d(x_0, x_{2n-1}) + \frac{a_1 + a_3}{1 - a_2} d(x_{2n}, x_{2n-1}) \\
&+ \left(1 - \frac{a_1 + a_2}{1 - a_3}\right) d(A, B)
\end{aligned}$$

Therefore, the subsequence $\{x_{2n}\}$ is bounded. Similarly, it can be shown that $\{x_{2n+1}\}$ is also bounded. So, this completes the proof.

Therem 4.1. Let A and B be nonempty closed subsets of a metric space (X, d). Suppose the mapping $T, S : A \cup B \to A \cup B$ be a TS-cyclic proximity contraction between A and B. For fixed element x_0 in A and suppose $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Assume that the sequence $\{x_{2n_k}\}$ has a subsequence converging to some element x in $A \cup B$. Then x is a best proximity point of T and S.

Proof. Assume that a subsequence $\{x_{2n_k}\}$ converges to x in $A \cup B$. It follows from **Lemma 4.1** that $d(x_{2n_k-1}, x_{2n_k})$ converges to $d(A, B)$. Since $T, S : A \cup B \to A \cup B$ a

TS-cyclic proximity contraction between A and B. Further, we get

$$\begin{aligned}
d(x_{2n_k}, Tx) &= d(Tx_{2n_k-1}, Tx) \\
&\le a_1 d(x_{2n_k-1}, x) + a_2 d(x_{2n_k-1}, Tx_{2n_k-1}) + a_3 d(x, Tx) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\le a_1 d(x_{2n_k-1}, x) + a_2 d(x_{2n_k-1}, x_{2n_k}) + a_3 (d(x, x_{2n}) + d(x_{2n}, Tx)) \\
&\quad + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\le a_1 (d(x_{2n_k-1}, x_{2n}) + d(x_{2n_k}, x)) + a_2 d(x_{2n_k-1}, x_{2n_k}) \\
&\quad + a_3 (d(x, x_{2n_k}) + d(x_{2n_k}, Tx)) + (1 - a_1 - a_2 - a_3) d(A, B) \\
&\le \frac{a_1 + a_2}{1 - a_3} (d(x_{2n_k-1}, x_{2n}) + \frac{a_1 + a_3}{1 - a_3} d(x, x_{2n_k}) + \Big(1 - \frac{a_1 + a_2}{1 - a_3}\Big) d(A, B) \\
&\le \frac{a_1 + a_2}{1 - a_3} (d(x_{2n_k-1}, x_{2n}) + \Big(1 - \frac{a_1 + a_2}{1 - a_3}\Big) d(A, B).
\end{aligned}$$

which implies that

$$d(x_{2n_k}, Tx) \le k(d(x_{2n_k-1}, x_{2n}) + (1 - k) d(A, B),$$

where $k = \frac{a_1+a_2}{1-a_3} < 1$, and letting $k \to \infty$, then we conclude that

$$d(A, B) \le d(x, Tx) \le k d(A, B) + (1 - k) d(A, B).$$

Hence $d(x, Tx) = d(A, B)$, that is, x is best proximity point of T. This completes the proof.

Remark 4.1. For $\alpha_1 = k$ and $\alpha_2 = \alpha_3 = 0$, then the cyclic contraction **(3.1)** reduces to cyclic contraction **(2.1)**. If we take $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 = k$ then, the cyclic contraction **(3.1)** reduces to the so called weak cyclic Kannan contraction [18] if there exists $k \in (0, 1/2)$, while for $\alpha_1 = \alpha_2 = \alpha_3 = k$ we obtain the Reich type cyclic contraction if there exists $k \in (0, 1/3)$. Obviously, for $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 = k$, the cyclic contraction **(4.2)** reduces to **(4.1)**.

As a direct consequence of **Theorem 4.1**, if we take $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 = k$, where $0 \le k < 0.5$, we obtain following corollaries:

Corollary 4.1. *Let A and B be two nonempty subsets of a metric space. Suppose that the mappings $T : A \to B$ and $S : B \to A$ form a TS-cyclic proximity contraction between A and B. For a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Then $d(x_n, x_{n+1}) \to d(A, B)$.*

Corollary 4.2. *Let A and B be two nonempty subsets of a metric space. Suppose that the mappings $T : A \to B$ and $S : B \to A$ form a TS-cyclic proximity contraction between A and B. For a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Then the sequence $\{x_n\}$ is bounded.*

Corollary 4.3. *Let A and B be two nonempty subsets of a metric space. Suppose that the mappings $T : A \to B$ and $S : B \to A$ form a TS-cyclic proximity contraction between A and B. For a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then x is a best proximity point of T.*

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