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STRONGLY CONVERGENT DOUBLY ACCELERATED FORWARD-BACKWARD ALGORITHM FOR MONOTONE INCLUSION AND FIXED POINT PROBLEMS

AUSTINE EFUT OFEM*, SEITHUTI PHILEMON MOSHOKOA, MALESELA CLIFFORD KEKANA

Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria, South Africa

Abstract. In this paper, we study the development of efficient iterative algorithm for finding a common solution to monotone inclusion problems and fixed point problems in real Hilbert spaces. These problems arise naturally in optimization, equilibrium modeling, signal processing, and other applied fields. We propose a doubly accelerated forward-backward algorithm with a viscosity term, combining inertial techniques and extragradient steps to enhance convergence speed and stability. The proposed algorithm relaxes restrictive assumptions on the underlying operators and guarantees strong convergence under mild conditions. Extensive numerical experiments illustrate the effectiveness and superiority of our method compared to existing approaches.

Keywords: monotone inclusion problem; fixed point problem; Tseng method; strong convergence; viscosity; double inertial algorithm.

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1. Introduction

In this paper, we are concerned with the development of efficient iterative algorithms for solving the *common solution of monotone inclusion problems (MIP) and fixed point problems (FPP)* in real Hilbert spaces. Let H be a real Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$

*Corresponding author
E-mail addresses: ofemaustine@gmail.com; ofemaeo@tut.ac.za
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and the induced norm $\|\cdot\|$.

We first consider the classical *monotone inclusion problem (MIP)*, which seeks a point $p \in H$ such that

$$0 \in (E+F)p,$$

where $E : H \to H$ is a monotone operator and $F : H \to 2^H$ is a maximal monotone operator. The solution set of this problem is denoted by $(E+F)^{-1}(0)$. This problem provides a unifying framework for a wide variety of models arising in applied mathematics, including convex minimization, variational inequalities, equilibrium problems, signal and image processing, split feasibility problems, and DC programming; see, for example, [5, 4]. Due to its broad applicability, the monotone inclusion problem has been extensively studied, and numerous iterative methods have been proposed for its solution.

One of the most classical methods for solving MIP is the *forward-backward algorithm (FBA)*, which generates a sequence $\{v_k\}$ by

$$v_{k+1} = (I+\lambda_k F)^{-1}(I-\lambda_k E)v_k,$$

where I is the identity operator and $\lambda_k > 0$ is a step size. The operator $(I-\lambda_k E)$ is the forward step, while $(I+\lambda_k F)^{-1}$ is the backward (resolvent) step. Although widely studied [6, 7, 8], the classical FBA requires E to be *inverse strongly monotone*, which can be restrictive in practice. This limitation motivated the development of alternative approaches, including extragradient methods, Tseng-type splitting methods, and hybrid projection algorithms [10, 11, 12, 13].

Takahashi et al. [2] proposed an iterative scheme under restrictive conditions on the control parameters:

$$v_{k+1} = \alpha_k u + (1-\alpha_k) J_{\mu_k B}\big(v_k - \mu_k A v_k\big), \quad \forall k \geq 1, \tag{1}$$

where A is a k-inverse strongly monotone mapping. They proved strong convergence assuming sequences $\{\alpha_k\}$ and $\{\mu_k\}$ satisfy classical summability and boundedness conditions.

Later, Tseng [9] relaxed the strong cocoercivity requirement by introducing the *Tseng splitting method*:

$$(2)\qquad \begin{cases} v_1 \in H, \\ y_k = J_{\lambda_k B}(v_k - \lambda_k A v_k), \\ v_{k+1} = y_k - \lambda_k (A y_k - A v_k), \end{cases}$$

where A is monotone and Lipschitz continuous. Despite its significance, this method suffers from weak convergence and step-size dependence on the Lipschitz constant.

In the Hilbert space setting, Thong and Vinh [26] proposed a *modified inertial forward-backward splitting with a viscosity term* for approximating a common solution of MIP and the fixed point problem of a nonexpansive mapping T. Similarly, Olakoya et al. [1] developed a *viscosity-inertial algorithm with self-adaptive step size* for solving MVIP constrained by common fixed point problems of strict pseudocontractions. These methods combine forward-backward splitting, extragradient steps, and inertial effects to accelerate convergence while maintaining strong convergence guarantees.

Let $S : H \to H$ be a nonlinear mapping. The *fixed point problem (FPP)* seeks $p \in H$ such that $p = Sp$. Fixed point problems play a central role in nonlinear analysis, optimization, and modeling. Classical iterative schemes include Picard, Mann [14], and Ishikawa [15] iterations, along with their numerous modifications. Over the years, significant progress has been made in establishing weak and strong convergence results for nonexpansive and pseudocontractive mappings.

Recently, increasing attention has been devoted to *finding a common solution to both MIP and FPP* [16, 17, 18]:

$$p \in (A+B)^{-1}(0) \cap \mathrm{Fix}(T),$$

where $\mathrm{Fix}(T)$ denotes the set of fixed points of T. This framework unifies multiple constraints, providing a powerful modeling tool for real-world problems in optimization, signal processing, game theory, and inverse problems.

To enhance efficiency and convergence speed, *inertial techniques* inspired by Polyak's heavy-ball method [19] have been widely incorporated. Inertial algorithms introduce momentum terms of the form

$$a_k = v_k + \beta_k (v_k - v_{k-1}),$$

where $\{\beta_k\}$ is a sequence of inertial parameters, often leading to significantly faster convergence. Some recent results on inertial-type algorithms can be found in [27, 28].

Motivated by these developments, this paper proposes *new doubly accelerated forward-backward algorithms with viscosity* for approximating the common solution of MIP and FPP. Our methods relax restrictive assumptions on the underlying operators while achieving faster convergence. We provide convergence analysis and numerical experiments demonstrating the effectiveness and superiority of the proposed schemes compared to existing methods.

The remainder of the paper is organized as follows. Section 2 introduces necessary preliminaries and definitions related to monotone operators and fixed point theory. Section 3 presents the proposed algorithms along with detailed convergence analysis. Section 4 provides numerical experiments illustrating the performance of the algorithms. Finally, Section 5 concludes the paper and discusses possible directions for future research.

2. PRELIMINARIES

In this section, we present some lemmas and definitions that will be useful in this research. The strong convergence and weak convergence of sequences are denoted by $\to$ and $\rightharpoonup$, respectively.

Definition 2.1. Let H be a real Hilbert space. Then the operator $S : H \to H$ is said to be:

(i) L-Lipschitz continuous on H, if a constant $L > 0$ exists such that

$$\|Su - Sv\| \leq L\|u - v\|, \forall u, v \in H;$$

(ii) α-co-coercive (or inverse strongly monotone), if there exists $\alpha > 0$ such that

$$\langle Su - Sv, u - v\rangle \geq \alpha\|Su - Sv\|, \forall u, v \in H;$$

(iii) α-strongly monotone, if there exists $\alpha > 0$ such that

$$\langle Su - Sv, u - v\rangle \geq \alpha\|a - b\|^2, \forall u, v \in H;$$

(iv) monotone, if

$$\langle Su - Sv, u - v\rangle \geq 0, \forall u, v \in H.$$

Definition 2.2. Let C be a nonempty subset of a real Hilbert space H and $F : H \to 2^H$ be a multivalued mapping. Then the graph and domain of the mapping F are defined by Graph(F)=$\{(a,b) : a \in H, b \in Fa\}$ and Dom(F)=$\{a \in H : Fa \neq \emptyset\}$, respectively. The mapping F is called

(i) monotone, if

$$\langle a-b, u-v\rangle \geq 0, \forall a,b \in H, u \in Fa, v \in Fb;$$

(ii) maximally monotone, if F is monotone and for any $(a,u) \in H \times H$, we have

$$\langle a-b, u-v\rangle \geq 0, \forall (b,v) \in \text{Graph}(F) \implies (a,u) \in \text{Graph}(F).$$

The resolvent mapping of a maximally monotone operator F, $J_\delta^F : H \to H$ is defined by $J_\delta^F = (I+\delta F)^{-1}$, where $\delta > 0$ and I stand for the identity operator on H.

It is well known that for each $u \in F$, a unique closest point $P_C u$ in C exists such that

$$\|u - P_C u\| \leq \|u - v\|, \ \forall v \in C.$$

P_C is known as the metric projection of H on C and it is important to note that the mapping P_C is nonexpansive and fulfills the inequality

$$d = P_C u \iff \langle u-d, d-v\rangle \geq 0, \forall v \in C.$$

To know more about the properties of P_C, see [22].

Let H be a real Hilbert space and $f : H \to (-\infty, +\infty)$ be a function, the proximal mapping of f is defined by

$$\text{prox}_f(a) = \text{argmin}_{v\in H}\{f(v) + \frac{1}{2}\|v-a\|^2\}, \forall a \in H.$$

We define the subdifferential of f at a as

$$\partial f(a) = \{z \in H : f(b) - f(a) \geq \langle z, b-a\rangle, \forall b \in H\}.$$

The cone $N_C(a)$ of C at a is given as

$$N_C(a) = \begin{cases} \{b \in H : \langle b, d-a\rangle \leq 0, \forall d \in C\}, & \text{if } a \in C \\ \emptyset & \text{otherwise.} \end{cases} \tag{3}$$

The indicator function $\partial_C(a)$ of C at a is defined by

$$\partial_C(a) = \begin{cases} 0, & \text{if } a \in C \\ \infty & \text{otherwise.} \end{cases} \tag{4}$$

Lemma 2.1. [23] *If H is a real Hilbert space and the operator $M : H \to H$ is monotone and continuous on H. Then M is a maximally monotone operator.*

Lemma 2.2. [23] *If H is a real Hilbert space and the operators $M, F : H \to H$ are maximally monotone H with $Dom(F) = H$. Then $M+F$ is also maximally monotone.*

Lemma 2.3. *Let H be a real Hilbert space. Then for every $u, v \in H$ and $c \in \mathbb{R}$, we have*
(i) $\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle$;
(ii) $\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2$;
(iii) $\|cu+(1-c)v\|^2 = c\|u\|^2 + (1-c)\|v\|^2 - c(1-c)\|u-v\|^2$.

Lemma 2.4. [24] *Let $\{\rho_k\}$ and $\{\sigma_k\}$ be two non-negative real sequences satisfying*

$$\rho_{k+1} \leq \rho_k + \sigma_k, \forall k \geq 1.$$

Suppose $\sum_{k=1}^{\infty} \sigma_k < \infty$, then $\lim_{k\to\infty} \rho_k$ exists.

Lemma 2.5. [25] *Let $\{u_k\}$ be a sequence of non-negative real numbers such that*

$$u_{k+1} \leq (1-\nu_k)u_k + \nu_k v_k, \ \forall k \geq 1,$$

where $\{\nu_k\} \subset (0,1)$ with $\sum_{k=0}^{\infty} \nu_k = \infty$. If $\limsup\limits_{m\to\infty} v_k \leq 0$ for every subsequence $\{u_{k_j}\}$ of $\{u_k\}$, the following inequality hold:

$$\liminf_{k\to\infty}(u_{k_{j+1}} - u_{k_j}) \geq 0,$$

Then $\lim\limits_{k\to\infty} u_k = 0$.

3. Main Results

In this section, we present our method and establish its strong convergence results in real Hilbert spaces. We start with the following assumption under which our strong convergence results will be established:

Assumption 3.1.

(A_1) The operator $E:\mathscr{H}\to\mathscr{H}$ is monotone and L-Lipschitz continuous, and $F:\mathscr{H}\to 2^{\mathscr{H}}$ is maximal monotone.

(A_2) The solution set is denoted by $\Omega = Fix(S)\cap(E+F)^{-1}(0)\neq\emptyset$.

(A_3) The mapping $S:\mathscr{H}\to\mathscr{H}$ is η–demimetric such that $I-S$ is demiclosed at zero.

(A_4) The mapping $f:\mathscr{H}\to\mathscr{H}$ is a contraction with contraction constant $\kappa\in[0,1)$.

(A_5) Let $\{\alpha_k\}\subset(0,1)$, $\{\beta_k\}$, $\{\gamma_k\}\subset[a,b]\subset(0,1)$ such that $\alpha_k+\beta_k+\gamma_k=1$, $\lim\limits_{k\to\infty}\alpha_k=0$, $\sum\limits_{k=1}^{\infty}\alpha_k=\infty$ and $\lim\limits_{k\to\infty}\frac{\varepsilon_k}{\alpha_k}=0=\lim\limits_{k\to\infty}\frac{\xi_k}{\alpha_k}$, where $\{\varepsilon_k\}$ and $\{\xi_k\}$ are positive real sequences

(A_6) $p_k\subset[0,\infty)$ with $\sum_{k=0}^{\infty}p_k<\infty$.

Algorithm 3.2.

Initialization: Choose $\phi\geq 3$, $\psi>0$, $\lambda_1>0$ and $\mu\in(0,1)$. Let $v_0,v_1\in\mathscr{H}$ and set $k=1$.

Iterative Steps: Calculate v_{k+1} as follows:

Step 1: Given the iterates v_{k-1} and v_k $(k\geq 1)$, choose ϕ_k and ψ_k such that $\phi_k\in[0,\bar{\phi}_k]$ and $\psi_k\in[0,\bar{\psi}_k]$, where

$$\bar{\phi}_k=\begin{cases}\min\left\{\frac{k-1}{k+\phi-1},\frac{\varepsilon_k}{\|v_k-v_{k-1}\|}\right\}, & \text{if } v_k\neq v_{k-1},\\ \frac{k-1}{k+\phi-1}, & \text{otherwise.}\end{cases}\tag{5}$$

$$\bar{\psi}_k=\begin{cases}\min\left\{\psi,\frac{\xi_k}{\|v_k-v_{k-1}\|}\right\}, & \text{if } v_k\neq v_{k-1},\\ \psi, & \text{otherwise.}\end{cases}\tag{6}$$

Step 2: Set

$$a_k = v_k + \phi_k(v_k - v_{k-1}), \tag{7}$$

$$b_k = v_k + \psi_k(v_k - v_{k-1}), \tag{8}$$

and compute

$$c_k = (I + \lambda_k F)^{-1}(I - \lambda_k E)a_k. \tag{9}$$

If $c_k = a_k$, then stop (c_k is a solution to MVIP). Else, proceed to **Step 3**.

Step 3: Compute

$$v_{k+1} = \alpha_k f(b_k) + \beta_k(c_k + \lambda_k(Ea_k - Ec_k)) + \gamma_k S(c_k + \lambda_k(Ea_k - Ec_k)) \tag{10}$$

Update

$$\lambda_{k+1} = \begin{cases} \min\left\{\frac{\mu\|a_k - c_k\|}{\|Ea_k - Ec_k\|}, \lambda_k + p_k\right\}, & \text{if } Ea_k \neq Ec_k, \\ \\ \lambda_k + p_k, & \text{otherwise.} \end{cases} \tag{11}$$

Put $k := k+1$ and return to **Step 1**.

Lemma 3.3. *([3]) Let $\{\lambda_k\}$ be the sequence generated by (11). Then*

$$\lim_{k\to\infty} \lambda_k = \lambda \in \left[\min\left\{\lambda_1, \frac{\mu}{L}\right\}, \lambda_1 + p\right],$$

where

$$p = \sum_{k=1}^{\infty} p_k < \infty.$$

Moreover,

$$\|Ea_k - Ec_k\| \leq \frac{\mu}{\lambda_{k+1}}\|a_k - c_k\|.$$

Remark 3.4.

We observe from (5), (6) and Assumption (A_5) that

$$\lim_{k\to\infty} \phi_k\|v_k - v_{k-1}\| = \lim_{k\to\infty} \psi_k\|v_k - v_{k-1}\| = 0$$

and

$$\lim_{k\to\infty}\frac{\phi_k}{\alpha_k}\|v_k - v_{k-1}\| = \lim_{k\to\infty}\frac{\psi_k}{\alpha_k}\|v_k - v_{k-1}\| = 0.$$

Lemma 3.5. *Let v_k be a sequence defined by Algorithm 3.2, suppose Assumption 3.1 holds. Then, $\forall\ p \in \Omega$, we obtain the following inequality*

$$\|h_k - p\|^2 \le \|a_k - p\|^2 - (1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2})\|c_k - a_k\|^2. \tag{12}$$

Proof. From (10)*, let $h_k = c_k + \lambda_k(Ea_k - Ec_k)$. Then, we have*

$$\begin{aligned}
\|h_k - p\|^2 &= \|c_k + \lambda_k(Ea_k - Ec_k) - p\|^2\\
&= \|c_k - p\|^2 + \lambda_k^2\|Ea_k - Ec_k\|^2 + 2\lambda_k\langle c_k - p, Ea_k - Ec_k\rangle\\
&= \|c_k - a_k + a_k - p\|^2 + \lambda_k^2\|Ea_k - Ec_k\|^2 + 2\lambda_k\langle c_k - p, Ea_k - Ec_k\rangle\\
&= \|c_k - a_k\|^2 + \|a_k - p\|^2 + 2\langle c_k - a_k, a_k - p\rangle + \lambda_k^2\|Ea_k - Ec_k\|^2\\
&\quad + 2\lambda_k\langle c_k - p, Ea_k - Ec_k\rangle\\
&= \|c_k - a_k\|^2 + \|a_k - p\|^2 + \lambda_k^2\|Ea_k - Ec_k\|^2 + 2\langle c_k - a_k, c_k - p\rangle + \langle c_k - a_k, a_k - c_k\rangle\\
&\quad + 2\lambda_k\langle Ea_k - Ec_k, p - c_k\rangle\\
&= \|c_k - a_k\|^2 + \|a_k - p\|^2 + \lambda_k^2\|Ea_k - Ec_k\|^2 + 2\langle c_k - a_k, c_k - p\rangle - 2\|c_k - a_k\|^2\rangle\\
&\quad + 2\lambda_k\langle Ea_k - Ec_k, p - c_k\rangle\\
&= \|a_k - p\|^2 - \|c_k - a_k\|^2 + \lambda_k^2\|Ea_k - Ec_k\|^2 - 2\langle a_k - c_k - \lambda_k(Ea_k - Ec_k), c_k - p\rangle\\
&\le \|a_k - p\|^2 - (1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2})\|c_k - a_k\|^2 - 2\langle a_k - c_k - \lambda_k(Ea_k - Ec_k), c_k - p\rangle. \qquad (13)
\end{aligned}$$

Next, we prove that

$$\langle a_k - c_k - \lambda_k(Ea_k - Ec_k), c_k - p\rangle \ge 0. \tag{14}$$

From (9), we have $(I - \lambda_k E) \in (I + \lambda_k F)a_k$. Due to maximal monotonicity of F, there exist $g_k \in Fc_k$, such that the following,

$$(I - \lambda_k E)a_k = c_k + \lambda_k g_k$$

which means

$$g_k = \lambda_k^{-1}(a_k - c_k - \lambda_k E a_k). \tag{15}$$

We have $0 \in (E+F)p$, from the definition of p. Since $Ec_k + g_k \in (E+F)g_k$ and $(E+F)$ is maximal monotone, we obtain

$$\langle Ec_k + g_k, c_k - p\rangle \geq 0. \tag{16}$$

Combining (15) and (16), we have

$$\lambda_k^{-1}\langle a_k - c_k - \lambda_k E a_k + \lambda_k E c_k, c_k - p\rangle \geq 0.$$

This means that (14) holds. By (13) and (14), we obtain

$$\|h_k - p\|^2 \leq \|a_k - p\|^2 - (1 - \mu^2 \frac{\lambda_k^2}{\lambda_{k+1}^2})\|a_k - c_k\|^2.$$

Lemma 3.6. *Suppose that Assumption 3.1 holds and v_k is the sequence defined by Algorithm 3.2. Then, $\{v_k\}$ is bounded.*

Thus, from (12), we have

$$\|h_k - p\| \leq \|a_k - p\|. \tag{17}$$

Proof. Now, since $\lim_{k\to\infty}\left(1 - \frac{\mu^2\lambda_k^2}{\lambda_{k+1}^2}\right) = 1 - \mu^2 > 0$. Hence, there exists $k_0 \in \mathbb{N}$ such that $1 - \frac{\mu^2\lambda_k^2}{\lambda_{k+1}^2} > \frac{1-\mu^2}{2} > 0, \quad \forall k \geq k_0$.
From (7), we have

$$\begin{aligned}\|a_k - p\| &= \|v_k + \phi_k(v_k - v_{k-1}) - p\| \\ &\leq \|v_k - p\| + \phi_k\|v_k - v_{k-1}\| \\ &\leq \|v_k - p\| + \alpha_k \frac{\phi_k}{\alpha_k}\|v_k - v_{k-1}\|. \end{aligned} \tag{18}$$

From Remark 3.4, we have $\frac{\phi_k}{\alpha_k}\|v_k - v_{k-1}\| \to 0$ as $k \to \infty$. Thus, there exist a constant M_1 such that

$$\frac{\phi_k}{\alpha_k}\|v_k - v_{k-1}\| \leq M_1, \forall\, k \geq 1. \tag{19}$$

Combining (17), (18) and (19), we obtain

$$\|h_k - p\| \leq \|a_k - p\| \leq \|v_k - p\| + \alpha_k M_1,\ \forall\, k \geq k_0. \tag{20}$$

Also from (8), we obtain the following

$$\begin{aligned} \|b_k - p\| &= \|v_k + \psi_k(v_k - v_{k-1}) - p\| \\ &\leq \|v_k - p\| + \psi_k \|v_k - v_{k-1}\| \\ &\leq \|v_k - p\| + \alpha_k \frac{\psi_k}{\alpha_k}\|v_k - v_{k-1}\|. \end{aligned} \tag{21}$$

From Remark 3.4, we have $\frac{\psi_k}{\alpha_k}\|v_k - v_{k-1}\| \to 0$ as $k \to \infty$. Thus, there exist a constant M_2 such that

$$\frac{\psi_k}{\alpha_k}\|v_k - v_{k-1}\| \leq M_2,\ \forall\, k \geq 1. \tag{22}$$

Combining (21) and (22), we have

$$\|b_k - p\| \leq \|v_k - p\| + \alpha_k M_2,\ \forall\, k \geq k_0. \tag{23}$$

Since $h_k = c_k + \lambda_k(Ea_k - Ec_k)$. Then from the definition of v_{k+1}, we have

$$\begin{aligned} \|v_{k+1} - p\| &= \|\alpha_k f(b_k) + \beta_k h_k + \gamma_k S h_k - p\| \\ &= \|\alpha_k(f(b_k) - p) + \beta_k(h_k - p) + \gamma_k(Sh_k - p)\| \\ &\leq \alpha_k\|(f(b_k) - f(p) + f(p) - p)\| + \beta_k\|h_k - p\| + \gamma_k\|Sh_k - p\| \\ &\leq \alpha_k\|f(b_k) - f(p)\| + \alpha_k\|f(p) - p\| + \beta_k\|h_k - p\| + \gamma_k\|Sh_k - p\| \\ &\leq \alpha_k\kappa\|b_k - p\| + \alpha_k\|f(p) - p\| + \beta_k\|h_k - p\| + \gamma_k\|Sh_k - p\| \\ &\leq \alpha_k\kappa\|b_k - p\| + \alpha_k\|f(p) - p\| + (1 - \alpha_k)\|h_k - p\|. \end{aligned} \tag{24}$$

Substitute (23) and (20) into (24), we obtain

$$\begin{aligned} \|v_{k+1} - p\| &\leq \alpha_k\kappa(\|v_k - p\| + \alpha_k M_2) + \alpha_k\|f(p) - p\| + (1 - \alpha_k)(\|v_k - p\| + \alpha_k M_1) \\ &= (1 - (1 - \kappa)\alpha_k)\|v_k - p\| + \alpha_k^2 M_2 + \alpha_k(1 - \alpha_k)M_1 + \alpha_k\|f(p) - p\| \\ &\leq (1 - (1 - \kappa)\alpha_k)\|v_k - p\| + \alpha_k^2 M_2 + \alpha_k M_1 + \alpha_k\|f(p) - p\| \end{aligned}$$

$$
\begin{aligned}
&\leq (1-(1-\kappa)\alpha_k)\|v_k-p\| + \alpha_k M_3 + \alpha_k\|f(p)-p\| \\
&\leq (1-(1-\kappa)\alpha_k)\|v_k-p\| + \alpha_k(1-\kappa)\frac{M_3+\|f(p)-p\|}{1-\kappa} \\
&\leq \max\,\{\|v_k-p\|, \frac{M_3+\|f(p)-p\|}{1-\kappa}\} \\
&\leq \ldots \\
&\leq \max\,\{\|v_k-p\|, \frac{M_3+\|f(p)-p\|}{1-\kappa}\},\ \forall\, k > k_0,
\end{aligned}
$$

where $M_3 = M_1 + M_2$. This means that $\{v_k\}$ is bounded. It follows that $\{a_k\},\{b_k\},\{h_k\}$ and $\{c_k\}$ are also bounded.

Lemma 3.7. *Suppose $\{a_k\}$ and $\{c_k\}$ are sequences generated by Algorithm 3.2. Let $\{a_{k_n}\}$ and $\{c_{k_n}\}$ be subsequences of $\{a_k\}$ and $\{c_k\}$ respectively. If $a_{k_n} \rightharpoonup p^* \in H$ and $\lim_{k\to\infty}\|a_{k_n} - c_{k_n}\| = 0$, then $p^* \in (E+F)^{-1}(0)$.*

Proof. Suppose $(x,y) \in G(E+F)$, this means $y - Ex \in Fx$. From (9) we know that $c_{k_n} = (I+\lambda_{k_n}F)^{-1}(I-\lambda_{k_n}E)\,a_{k_n}$. This implies that $(I-\lambda_{k_n}E)a_{k_n} \in (I+\lambda_{k_n}F)c_{k_n}$. It follows that

$$\lambda_{k_n}^{-1}(a_{k_n} - c_{k_n} - \lambda_{k_n}Ea_{k_n}) \in Fc_{k_n}.$$

Since F is maximal monotone we get

$$\langle x - c_{k_n}, y - Ex - \lambda_{k_n}^{-1}(a_{k_n} - c_{k_n} - \lambda_{k_n}Ea_{k_n})\rangle \geq 0,$$

or equivalently,

$$\langle x - c_{k_n}, y\rangle - \langle x - c_{k_n}, Ex + \lambda_{k_n}^{-1}(a_{k_n} - c_{k_n} - \lambda_{k_n}Ea_{k_n})\rangle \geq 0.$$

This with the monotonicity of E yields

$$
\begin{aligned}
\langle x - c_{k_n}, y\rangle &\geq \langle x - c_{k_n}, Ex + \lambda_{k_n}^{-1}(a_{k_n} - c_{k_n} - \lambda_{k_n}Ea_{k_n})\rangle \\
&= \langle x - c_{k_n}, Ex - Ea_{k_n}\rangle + \langle x - c_{k_n}, \lambda_{k_n}^{-1}(a_{k_n} - c_{k_n})\rangle \\
&= \langle x - c_{k_n}, Ex - Ec_{k_n}\rangle + \langle x - c_{k_n}, Ec_{k_n} - Ea_{k_n}\rangle + \langle x - c_{k_n}, \lambda_{k_n}^{-1}(a_{k_n} - c_{k_n})\rangle \\
&\geq \langle x - c_{k_n}, Ec_{k_n} - Ea_{k_n}\rangle + \langle x - c_{k_n}, \lambda_{k_n}^{-1}(a_{k_n} - c_{k_n})\rangle.
\end{aligned}
$$

Due to the fact that $\lim_{n\to\infty}\|a_{k_n}-c_{k_n}\|=0$ and E is the Lipschitz continuous, we get $\lim_{k\to\infty}\|Ea_{k_n}-Ec_{k_n}\|=0$. Since $\lim_{k\to\infty}\lambda_k=\lambda>0$, we conclude that

$$\lim_{n\to\infty}\langle x-c_{k_n},y\rangle=\langle x-p^*,y\rangle\geq 0,$$

alongside with the maximal monotonicity of $E+F$ yields $p^*\in(E+F)^{-1}(0)$.

Lemma 3.8. *Suppose Assumption 3.1 holds and v_k is the sequence defined by Algorithm 3.2. Then, the following inequality holds for all $p\in\Omega$ and $k\in\mathbb{N}$:*

$$\|v_{k+1}-p\|^2\leq(1-(1-\kappa)\alpha_k\|v_k-p\|^2+(1-\kappa)\alpha_k)[\frac{2}{1-\kappa}\langle f(p)-p,v_{k+1}-p\rangle$$
$$+\frac{3M_4}{1-\kappa}.\frac{\phi_k}{\alpha_k}\|v_k-v_{k-1}+\frac{3M_5}{1-\kappa}.\frac{\psi_k}{\alpha_k}\|v_k-v_{k-1}\|]\ \forall\ k\geq k_0.$$

Proof. Indeed from (5) and (7), we obtain

$$\begin{aligned}\|a_k-p\|^2&=\|v_k+\phi_k(v_k-v_{k-1})-p\|^2\\&=\|v_k-p\|^2+\phi_k^2\|v_k-v_{k-1}\|^2+2\phi_k\langle v_k-p,v_k-v_{k-1}\rangle\\&\leq\|v_k-p\|^2+\phi_k^2\|v_k-v_{k-1}\|^2+2\phi_k\|v_k-p\|\|v_k-v_{k-1}\|\end{aligned}\tag{25}$$

Also, from (6) and (8), we have

$$\begin{aligned}\|b_k-p\|^2&=\|v_k+\psi_k(v_k-v_{k-1})-p\|^2\\&=\|v_k-p\|^2+\psi_k^2\|v_k-v_{k-1}\|^2+2\psi_k\langle v_k-p,v_k-v_{k-1}\rangle\\&\leq\|v_k-p\|^2+\psi_k^2\|v_k-v_{k-1}\|^2+2\psi_k\|v_k-p\|\|v_k-v_{k-1}\|\end{aligned}\tag{26}$$

Now, from (25), (26), (20), (10), and Lemma 2.3 we have

$$\begin{aligned}\|v_{k+1}-p\|^2&=\|\alpha_kf(b_k)+\beta_kh_k+\gamma_kSh_k-p\|^2\\&=\|\alpha_k(f(b_k)-p)+\beta_k(h_k-p)+\gamma_k(Sh_k-p)\|^2\\&=\|\alpha_k(f(b_k)-f(p))+\beta_k(h_k-p)+\gamma_k(Sh_k-p)+\alpha_k(f(p)-p)\|^2\\&\leq\|\alpha_k(f(b_k)-f(p))+\beta_k(h_k-p)+\gamma_k(Sh_k-p)\|^2+2\alpha_k\langle f(p)-p,v_{k+1}-p\rangle\\&\leq\alpha_k\|f(b_k)-f(p)\|^2+\beta_k\|h_k-p\|^2+\gamma_k\|Sh_k-p\|^2+2\alpha_k\langle f(p)-p,v_{k+1}-p\rangle\end{aligned}$$

$$\leq \alpha_k\kappa^2\|b_k-p\|^2+\beta_k\|h_k-p\|^2+\gamma_k\|h_k-p\|^2+2\alpha_k\langle f(p)-p, v_{k+1}-p\rangle$$

$$\leq \alpha_k\kappa\|b_k-p\|^2+\beta_k\|h_k-p\|^2+\gamma_k\|h_k-p\|^2+2\alpha_k\langle f(p)-p, v_{k+1}-p\rangle$$

$$= \alpha_k\kappa\|b_k-p\|^2+(1-\alpha_k)\|h_k-p\|^2+2\alpha_k\langle f(p)-p, v_{k+1}-p\rangle$$

$$\leq \alpha_k\kappa\|b_k-p\|^2+(1-\alpha_k)\|a_k-p\|^2+2\alpha_k\langle f(p)-p, v_{k+1}-p\rangle. \tag{27}$$

$$\leq \alpha_k\kappa\{\|v_k-p\|^2+2\phi_k\|v_k-p\|\|v_k-v_{k+1}\|+\phi_k^2\|v_k-v_{k-1}\|^2\}$$

$$+(1-\alpha_k)\{\|v_k-p\|^2+2\psi_k\|v_k-p\|\|v_k-v_{k+1}\|\psi_k^2\|v_k-v_{k-1}\|^2\}$$

$$2\alpha_k\langle f(p)-p, v_{k+1}-p\rangle \tag{28}$$

$$= (1-(1-\kappa)\alpha_k)\|v_k-p\|^2+(1-\kappa)\alpha_k\frac{2}{1-\kappa}\langle f(p)-p, v_{k+1}-p\rangle$$

$$+\phi_k\|v_k-v_{k-1}\|[2\|v_k-p\|+\phi_k\|v_k-v_{k-1}]$$

$$+\psi_k\|v_k-v_{k-1}\|[2\|v_k-p\|+\psi_k\|v_k-v_{k-1}]$$

$$\leq (1-(1-\kappa)\alpha_k\|v_k-p\|^2+(1-\kappa)\alpha_k)[\frac{2}{1-\kappa}\langle f(p)-p, v_{k+1}-p\rangle$$

$$+\frac{3M_4}{1-\kappa}.\frac{\phi_k}{\alpha_k}\|v_k-v_{k-1}\|+\frac{3M_5}{1-\kappa}.\frac{\psi_k}{\alpha_k}\|v_k-v_{k-1}\|]\ \forall\ k\geq k_0, \tag{29}$$

where $M_4=\sup_{k\in\mathbb{N}}\{\|v_k-p\|,\phi\|v_k-v_{k-1}\|\}$ and $M_5=\sup_{k\in\mathbb{N}}\{\|v_k-p\|,\psi\|v_k-v_{k-1}\|\}$.

Lemma 3.9. *Suppose that Assumption 3.1 is fulfilled and $\{v_k\}$ is the sequence generated by Algorithm 6. Then, the following inequality holds:*

$$(1-\alpha_k)(1-\mu\frac{\lambda_k^2}{\lambda_{k+1}^2})\|c_k-a_k\|^2+\beta_k\gamma_k\|h_k-Sh_k\|^2\leq\|v_k-p\|^2-\|v_{k+1}-p\|^2+\alpha_kM_9\ \forall\ k\geq k_0. \tag{30}$$

Proof. Indeed, from (20), we have

$$\|a_k-p\|^2\leq(\|v_k-p\|+\alpha_kM_1)^2$$

$$=\|v_k-p\|^2+\alpha_k(2M_1\|v_k-p\|+\alpha_kM_1^2). \tag{31}$$

From the boundedness of $\{v_k\}$, we infer that a constant $M_6>0$ exist, such that $2M_1\|v_k-p\|+\alpha_kM_1^2\leq M_6$. Therefore, (30) becomes

$$\|a_k-p\|^2\leq\|v_k-p\|^2+\alpha_kM_6. \tag{32}$$

Also, from (23), we have

$$
\begin{aligned}
\|b_k-p\|^2 &\le (\|v_k-p\|+\alpha_k M_2)^2 \\
&= \|v_k-p\|^2+\alpha_k(2M_2\|v_k-p\|+\alpha_k M_2^2). \qquad (33)
\end{aligned}
$$

From the boundedness of $\{v_k\}$, we infer that a constant $M_7>0$ exist, such that $2M_1\|v_k-p\|+\alpha_k M_2^2\le M_7$. Therefore, (22) becomes

$$
\|b_k-p\|^2 \le \|v_k-p\|^2+\alpha_k M_7. \qquad (34)
$$

Now, from (33) and Lemma 2.3, we have

$$
\begin{aligned}
\|v_{k+1}-p\|^2 &= \|\alpha_k f(b_k)+\beta_k h_k+\gamma_k Sh_k-p\|^2 \\
&= \|\alpha_k(f(b_k)-p)+\beta_k(h_k-p)+\gamma_k(Sh_k-p)\|^2 \\
&\le \alpha_k\|(f(b_k)-p)\|^2+\beta_k\|h_k-p\|^2 \\
&+\gamma_k\|Sh_k-p\|^2-\beta_k\gamma_k\|h_k-Sh_k\|^2 \\
&\le \alpha_k(\|f(b_k)-f(p)\|+\|f(p)-p\|)^2+\beta_k\|h_k-p\|^2 \\
&+\gamma_k\|Sh_k-p\|^2-\beta_k\gamma_k\|h_k-Sh_k\|^2 \\
&\le \alpha_k(\kappa\|b_k-p\|+f(p)-p\|)^2+\beta_k\|h_k-p\|^2 \\
&+\gamma_k\|Sh_k-p\|^2+\beta_k\gamma_k\|h_k-Sh_k\|^2 \\
&\le \alpha_k(\kappa^2\|b_k-p\|^2+2\kappa\|b_k-p\|\|f(p)-p\|+\|f(p)-p\|^2) \\
&+(1-\alpha_k)\|h_k-p\|^2-\beta_k\gamma_k\|h_k-Sh_k\|^2 \\
&\le \alpha_k\|b_k-p\|^2+2\|b_k-p\|\|f(p)-p\|+\|f(p)-p\|^2 \\
&+(1-\alpha_k)\|h_k-p\|^2-\beta_k\gamma_k\|h_k-Sh_k\|^2. \qquad (35)
\end{aligned}
$$

Since $\{b_k\}$ is bounded, it follows that a constant $M_8>0$ exist, such that $2\|b_k-p\|\|f(p)-p\|+\|f(p)-p\|^2\le M_8$. Therefore, (35) becomes

$$
\|v_{k+1}-p\|^2 \le \alpha_k\|b_k-p\|^2+(1-\alpha_k)\|h_k-p\|^2-\beta_k\gamma_k\|h_k-Sh_kp\|^2+\alpha_k M_8. \qquad (36)
$$

Substituting (12) into (36), we obtain

$$\|v_{k+1}-p\|^2 \leq \alpha_k\|b_k-p\|^2+(1-\alpha_k)\|a_k-p\|^2$$

$$-(1-a_k)(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2})\|c_k-a_k\|^2-\beta_k\gamma_k\|h_k-Sh_kp\|^2+\alpha_kM_8. \tag{37}$$

Putting (32) and (34) into (37)

$$\begin{aligned}\|v_{k+1}-p\|^2 &\leq \alpha_k(\|v_k-p\|^2+\alpha_kM_7)+(1-\alpha_k)(\|v_k-p\|^2+\alpha_kM_6)\\ &\quad-(1-a_k)(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2})\|c_k-a_k\|^2-\beta_k\gamma_k\|h_k-Sh_k\|^2+\alpha_kM_8\\ &\leq \|v_k-p\|^2-(1-\alpha_k)(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2})\|c_k-a_k\|^2-\beta_k\gamma_k\|h_k-Sh_k\|^2\\ &\quad+\alpha_kM_7+\alpha_kM_6+\alpha_kM_8 \end{aligned} \tag{38}$$

This implies that

$$(1-\alpha_k)(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2})\|c_k-a_k\|^2+\beta_k\gamma_k\|h_k-Sh_k\|^2 \leq \|v_k-p\|^2-\|v_{k+1}-p\|^2+\alpha_kM_9,\ \forall\, k\geq k_0,$$

where $M_9=M_6+M_7+M_8>0$.

Theorem 3.10. *Assume that Assumption* (3.1) *holds. Then the sequence* $\{v_k\}$ *generated by Algorithm* (3.2) *converges to p in the norm, where* $p=P_\Omega\circ f(p)$.

Proof. Now, we shows that $\{\|v_k-p\|^2\}$ converges to zero. Clearly, by Lemma 2.5, it is sufficient to show that $\limsup\limits_{n\to\infty}\langle f(p)-p,v_{k_n+1}-p\rangle\leq 0$ for every subsequence $\{\|v_{k_n}-p\|\}$ of $\{\|v_k-p\|\}$ satisfying $\liminf\limits_{k\to\infty}(\|v_{k_n+1}-p\|-\|v_{k_n-p\|}) \geq 0$. For the reason of this analysis, we assume that $\{\|v_{k_n}-p\|\}$ is a subsequence of $\{\|v_k-p\|\}$ such that $\liminf\limits_{n\to\infty}(\|v_{k_n+1}-p\|-\|v_{k_n}-p\|)\geq 0$. Then

$$\liminf_{n\to\infty}(\|v_{k_n+1}-p\|^2-\|v_{k_n}-p\|^2)=\liminf_{k\to\infty}[(\|v_{k_n+1}-p\|-\|v_{k_n}-p\|)(\|v_{k_n+1}-p\|-\|v_{k_n}-p\|)]\geq 0.$$

From (30) and the assumption (3.1) (A_5), we deduce

$$\begin{aligned}&\limsup_{n\to\infty}[(1-\alpha_{k_n})(1-\mu\frac{\lambda_{k_n}^2}{\lambda_{k_n+1}^2})\|c_{k_n}-a_{k_n}\|^2+\beta_{k_n}\gamma_{k_n}\|h_{k_n}-Sh_{k_n}\|^2]\\ &\qquad\leq \limsup_{n\to\infty}[\|v_{k_n}-p\|^2-\|v_{k_n+1}-p\|^2+\alpha_{k_n}M_9]\leq 0.\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|a_{k_n} - c_{k_n}\| = 0, \tag{39}$$

and

$$\lim_{n\to\infty} \|h_{k_n} - Sh_{k_n}\| = 0. \tag{40}$$

Since $h_{k_n} = c_{k_n} + \lambda_{k_n}(Ea_{k_n} - Ec_{k_n})$, we have

$$h_{k_n} - c_{k_n} = \lambda_{k_n}(Ea_{k_n} - Ec_{k_n}). \tag{41}$$

Now, from (11), we have

$$\|Ea_{k_n} - Ec_{k_n}\| \leq \mu \frac{\|a_{k_n} - c_{k_n}\|}{\lambda_{k_n+1}},\ \forall\, n \in \mathbb{N}. \tag{42}$$

We observe that (42) holds if $Ea_{k_n} \neq Ec_{k_n}$ Thus, combing (41) and (42), we obtain

$$\|h_{k_n} - c_{k_n}\| \leq \lambda_k \mu \frac{\|a_{k_n} - c_{k_n}\|}{\lambda_{k_n+1}}.$$

Due to (39) and (3.3), we have that

$$\|h_{k_n} - c_{k_n}\| \to 0 \ as\ n \to \infty. \tag{43}$$

Also, using (10), we have

$$\|v_{k_n+1} - h_{k_n}\| \leq \alpha_k \|f(b_{k_n}) - h_{k_n}\| + \beta_k \|h_{k_n} - h_{k_n}\| + \gamma_{k_n} \|Sh_{k_n} - h_{k_n}\|. \tag{44}$$

Since $\lim \alpha_{k_n} = 0$, then using (40), we have

$$\lim_{n\to\infty} \|v_{k_n+1} - h_{k_n}\| = 0. \tag{45}$$

Using (7) and Assumption (3.1) (A_5), we have

$$\begin{aligned} \|a_{k_n} - v_{k_n}\| &= \phi_{k_n} \|v_{k_n} - v_{k_n-1}\| \\ &= \alpha_{k_n} \frac{\phi_{k_n}}{\alpha_{k_n}} \|v_{k_n} - v_{k_n-1}\| \to 0 \ as\ n \to \infty. \end{aligned} \tag{46}$$

Next, observe that

$$\|v_{k_n+1} - v_{k_n}\| \leq \|v_{k_n+1} - h_{k_n}\| + \|h_{k_n} - c_{k_n}\| + \|c_{k_n} - a_{k_n}\| + \|a_{k_n} - v_{k_n}\|. \tag{47}$$

Using (39),(43),(45) and (46), we obtain

$$\lim_{n\to\infty} \|v_{k_n+1} - v_{k_n}\| = 0. \tag{48}$$

Now, we have

$$\|c_{k_n} - v_{k_n}\| \leq \|c_{k_n} - h_{k_n}\| + \|h_{k_n} - v_{k_n+1}\| + \|v_{k_n+1} - v_{k_n}\| \to 0 \quad \text{as} \quad n \to \infty. \tag{49}$$

Since $\{v_k\}$ is bounded, then $\omega_w(v_k)$ is nonempty. Let $p \in \omega_w(v_k)$ be an arbitrary element. Then, there exists a subsequence $\{v_{k_n}\}$ of $\{v_k\}$ such that

$$v_{k_n} \rightharpoonup p \quad \text{as } n \to \infty.$$

By (46), we have

$$a_{k_n} \rightharpoonup p \quad \text{as } n \to \infty.$$

Now, by invoking Lemma (3.7) and applying (49), we have

$$p \in (A+B)^{-1}(0). \tag{50}$$

Moreover, by (3) and (49), we have

$$c_{k_n} \rightharpoonup p \quad \text{as } n \to \infty \quad \text{and} \quad h_{k_n} \rightharpoonup p \quad \text{as } n \to \infty.$$

Since $I - S$ is demiclosed at zero, we have

$$p \in Fix(S). \tag{51}$$

Since $p \in \omega_w(v_k)$ is arbitrary, it follows from (50) and (51) that

$$\omega_w(v_k) \subset \Omega.$$

Moreover, by the boundedness of $\{v_{k_n}\}$, there exists a subsequence $\{v_{k_{n_j}}\}$ of $\{v_{k_n}\}$ such that

$$v_{k_{n_j}} \rightharpoonup v^\dagger$$

and

$$\lim_{j\to\infty} \langle f(\bar{v}) - \bar{v}, v_{k_{n_j}} - \bar{v}\rangle = \limsup_{n\to\infty} \langle f(\bar{v}) - \bar{v}, v_{k_n} - \bar{v}\rangle.$$

Since $\bar{v} = P_\Omega \circ f(\bar{v})$, it follows that

$$\limsup_{n\to\infty} \langle f(\bar{v}) - \bar{v}, v_{k_n} - \bar{v}\rangle = \lim_{j\to\infty} \langle f(\bar{v}) - \bar{v}, v_{k_{n_j}} - \bar{v}\rangle = \langle f(\bar{v}) - \bar{v}, v^\dagger - \bar{v}\rangle \leq 0. \tag{52}$$

From (47) and (52), we get

$$\limsup_{n\to\infty}\langle f(\bar{v})-\bar{v},v_{k_n+1}-\bar{v}\rangle=\limsup_{n\to\infty}\langle f(\bar{v})-\bar{v},v_{k_n}-\bar{v}\rangle=\langle f(\bar{v})-\bar{v},v^{\dagger}-\bar{v}\rangle\leq 0. \tag{53}$$

Applying Lemma 2.5 to (29), and using (53) together with the fact in Remark 3 and $\lim_{k\to\infty}\alpha_k=0$,we obtain

$$\lim_{k\to\infty}\|v_k-\bar{v}\|=0.$$

as required. □

4. Numerical Experiments

In this section, we present some numerical experiments to illustrate the performance of our method, Algorithm 3.2 (OMMM Alg), in comparison with Algorithms (1.1), Algorithm (1.2), Algorithm 1 of Thong and Vinh (TV Alg, for short), Algorithm 3.2 Alakoya et al. [1] (AOM Alg, for short). All numerical computations were carried out using MATLAB version R2019(b).

In our computations, we choose $\alpha_k=\frac{1}{2k+1},\quad \varepsilon_k=\xi_k=\frac{1}{(2k+1)^3},\quad \gamma_k=\xi_k=\frac{1}{2}(1-\alpha_k),\phi=0.88,\quad \psi=0.55,\quad \lambda_1=2.8,\quad \mu=0.56,\quad Sv=\frac{6}{5}v,\quad Tv=\frac{3}{7}v,\quad f(v)=\frac{1}{2}v.$

Example 4.1. Let $H=\mathbb{R}$, the set of all real numbers with the inner product defined by

$$\langle u,v\rangle=uv,\quad \forall u,v\in\mathbb{R},$$

and the induced norm $|\cdot|$. We define $E:H\to H$ by

$$Eu=u+\sin u$$

and $F:H\to H$ by

$$Fu=3u,\quad \forall u\in H.$$

Clearly, E is $\frac{1}{2}$-inverse strongly monotone and F is maximal monotone.

We consider different initial starting points as follows with $u=u_0$:

- **Case 1**: Take $u_0=\frac{33}{8}$ and $u_1=2$.
- **Case 2**: Take $u_0=\frac{8}{3}$ and $u_1=5$.
- **Case 3**: Take $u_0=\frac{39}{10}$ and $u_1=\frac{17}{10}$.
- **Case 4**: Take $u_0=7$ and $u_1=\frac{7}{2}$.

We compare the performance of Algorithm 3.2 (OMMM Alg) with Algorithms (1.1), Algorithms (1.2), AOM Alg and TV Alg. We plot the graphs of errors against the number of iterations in each case using $\|u_{k+1} - u_k\| < 10^{-4}$ as the stopping criterion. The numerical results are reported in Table 1 and Fig. 1.

TABLE 1. Iterations/CPU Time for Different Algorithms

Algorithm	**Case 1**		**Case 2**		**Case 3**		**Case 4**	
	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
OMMM Alg	30	0.0192	29	0.0009	28	0.0011	25	0.0009
AOM Alg	43	0.0195	41	0.0021	40	0.0012	32	0.0018
TV Alg	77	0.0206	83	0.0044	74	0.0021	80	0.0022
Algorithm (1.1)	600	0.0360	600	0.0158	600	0.0158	600	0.0153
Algorithm (1.2)	600	0.0373	600	0.0159	600	0.0159	600	0.0151

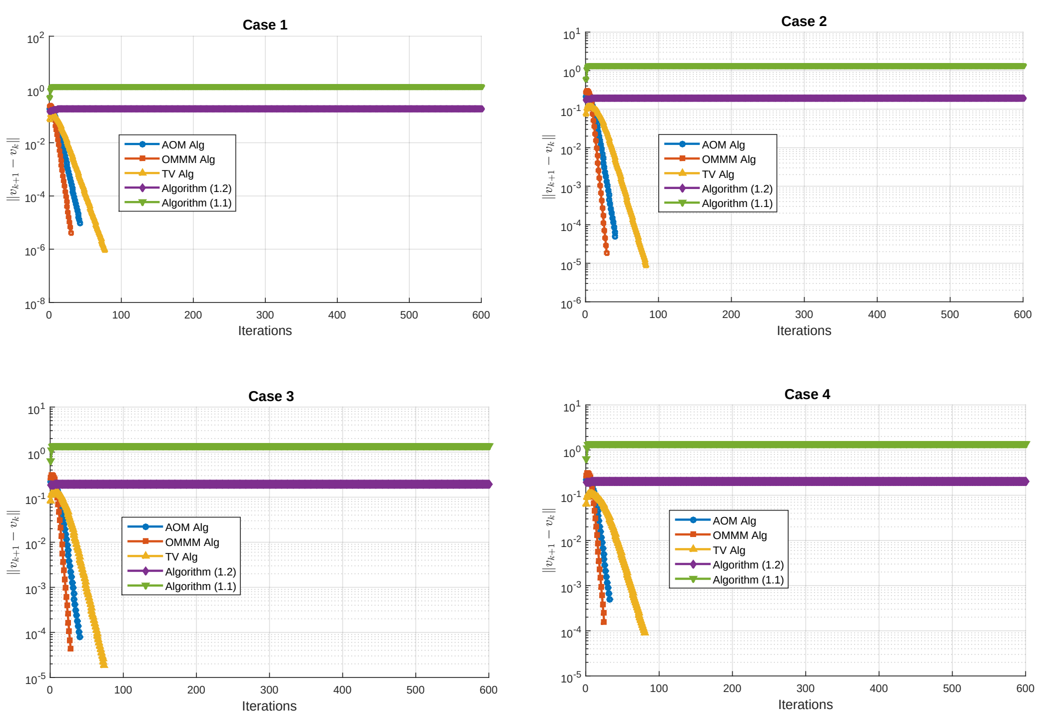

FIGURE 1. Plot of the convergence analysis for various iterative methods for four cases

5. Conclusion

In this paper, we have proposed *doubly accelerated forward-backward algorithms with a viscosity term* for approximating a common solution to monotone inclusion problems and fixed point problems in real Hilbert spaces. By combining inertial extrapolation, forward-backward splitting, and extragradient corrections, the proposed methods enhance convergence speed while relaxing restrictive assumptions on the underlying operators. We have established *strong convergence results* under mild conditions, including monotonicity, uniform continuity, and appropriate control parameter selection. The inclusion of inertial and viscosity terms proved effective in accelerating convergence and improving numerical performance. Extensive numerical experiments demonstrate that the proposed algorithms outperform existing methods in terms of iteration counts, computational time, and stability, confirming their practical efficiency for solving complex problems with multiple constraints. Future work may consider extending these algorithms to stochastic monotone inclusion problems, multi-agent optimization, and infinite-dimensional split feasibility problems. Additionally, adaptive strategies for selecting inertial and step-size parameters could further enhance convergence and robustness. In summary, the proposed algorithms provide a *powerful and flexible framework* for solving common solution problems, bridging the gap between theoretical convergence guarantees and practical computational efficiency.

Conflict of Interests

The authors declare that there is no conflict of interests.

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