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COMMON FIXED POINT RESULTS IN NONEMPTY SETS ENDOWED WITH MODULAR METRICS

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Abstract. In this work, some unique common fixed point results for two pairs of occasionally weakly biased mappings of type $(\mathscr{A})$ in nonempty sets endowed with modular metrics are obtained. Our reason behind this contribution is to investigate the existence and uniqueness of common fixed points in the recent weaker framework of occasionally weakly biased mappings of type $(\mathscr{A})$.

Keywords: metric space; modular metric; occasionally weakly biased mappings of type $(\mathscr{A})$; unique common fixed points.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

According to some references, the idea of an abstract space with metric properties was addressed in 1906 by the French mathematician René Maurice Fréchet and the term metric space was coined by the German mathematician Felix Hausdorff in 1914. So, as you know, a standard

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metric space is a set of points say for instance $\mathscr{X}$ equipped with a standard distance function d, called a standard metric, that satisfies the next conditions for all ρ, χ, σ in $\mathscr{X}$:

(1) The distance from a point to itself is zero: $d(\rho,\rho)=0$,

(2) **positivity:** the distance between two distinct points is always positive: if $\rho \neq \chi$, then, $d(\rho,\chi)>0$,

(3) **symmetry:** the distance from ρ to χ is always the same as the distance from χ to ρ: $d(\rho,\chi)=d(\chi,\rho)$,

(4) **triangle inequality:** $d(\rho,\chi)\leq d(\rho,\sigma)+d(\sigma,\chi)$.

Many authors innovated some alternative spaces by addressing some limitations such as relaxing the triangle inequality, sometimes removing it, allowing non-zero self-distance, removing the symmetry axiom, and so on. These generalizations enable more flexible analysis of fixed and common fixed points. In particular, in 2006, in [10], based on particular physical movement, Chistyakov introduced a new type of spaces called modular metric space which is more general than metric one. By this discovery, he developed the theory of metric spaces, generated by modulars, and extend some results for modulars on linear spaces. Note that several authors used this new kind of spaces to investigate the existence and uniqueness of fixed points, see for example [1]-[4], [10], [11], [14] and [17]-[23].

On the other hand, in [5], we suggested a new harmony between mappings under the name of occasionally weakly biased mappings of type $(\mathscr{A})$ in order to find common fixed points by a few conditions. Note that many researchers investigated the existence and often uniqueness of fixed and common fixed points for single and hybrid mappings under different inequalities and in various spaces, see for instance [5]-[9], [12], [15] and [16].

2. Recallable Definitions

Definition 1. *(*[10]-[11]*) Let $\mathscr{X}$ be a nonempty set and let $d:(0,+\infty)\times\mathscr{X}\times\mathscr{X}\to[0,+\infty]$ be a function. d is called modular metric on $\mathscr{X}$ if the following axioms hold:*

(1) given ρ, χ in $\mathscr{X}$, $\rho=\chi$ if and only if $d(\lambda,\rho,\chi)=0$ for all λ positive,

(2) $d(\lambda,\rho,\chi)=d(\lambda,\chi,\rho)$ for all λ positive and ρ, χ in $\mathscr{X}$,

(3) $d(\lambda+\mu,\rho,\chi)\leq d(\lambda,\rho,\sigma)+d(\mu,\sigma,\chi)$ for all λ, μ positive and ρ, χ, σ in $\mathscr{X}$.

Example 2. *Let $\mathscr{X} = \mathbb{R}$ and let $d : (0,+\infty)\times\mathbb{R}\times\mathbb{R} \to [0,+\infty]$ such that $d(\lambda,\rho,\chi) = \frac{|\rho-\chi|}{\lambda}$ for all ρ, χ in $\mathbb{R}$ and λ positive, then, d is a modular metric.*

Example 3. *Let $\mathscr{X} = \mathbb{R}$ and let $d : (0,+\infty)\times\mathbb{R}\times\mathbb{R} \to [0,+\infty]$ such that $d(\lambda,\rho,\chi) = \frac{|\rho-\chi|}{1+\lambda}$ for all ρ, χ in $\mathbb{R}$ and λ positive, then, d is a modular metric.*

At the end, we recall our proper definition.

Definition 4. *([5]) Endow the nonempty set $\mathscr{X}$ with a modular metric d. Let $\mathscr{M}$, $\mathscr{N} : \mathscr{X} \to \mathscr{X}$ be two mappings. The pair $(\mathscr{M},\mathscr{N})$ is called **occasionally weakly $\mathscr{M}$-biased (respectively $\mathscr{N}$-biased) of type** $(\mathscr{A})$, if and only if, there is an element $\zeta \in \mathscr{X}$ such that $\mathscr{M}\zeta = \mathscr{N}\zeta$ implies*

$$d(\lambda,\mathscr{M}\mathscr{M}\zeta,\mathscr{N}\zeta) \leq d(\lambda,\mathscr{N}\mathscr{M}\zeta,\mathscr{M}\zeta) \text{ for all } \lambda \text{ positive,}$$

$$d(\lambda,\mathscr{N}\mathscr{N}\zeta,\mathscr{M}\zeta) \leq d(\lambda,\mathscr{M}\mathscr{N}\zeta,\mathscr{N}\zeta) \text{ for all } \lambda \text{ positive,}$$

respectively.

Notation: Throughout this paper, we denote $d(\lambda,x,y)$ by $d_\lambda(x,y)$ and $d(2\lambda,x,y)$ by $d_{2\lambda}(x,y)$.

3. Main Results

3.1. Existence and Uniqueness of a Common Fixed Point.

Theorem 5. *Let $\mathscr{X}$ be a nonempty closed set endowed with a modular metric d_λ. Consider four self-mappings $\mathscr{K}$, $\mathscr{L}$, $\mathscr{M}$ and $\mathscr{N}$ such that*

1) $\mathscr{K}$ and $\mathscr{M}$ as well as $\mathscr{L}$ and $\mathscr{N}$ are occasionally weakly $\mathscr{K}$-biased respectively $\mathscr{L}$-biased of type $(\mathscr{A})$,

2) for all λ positive and x, y in $\mathscr{X}$, we have

$$\begin{aligned} d_\lambda(\mathscr{M}x,\mathscr{N}y) \quad \leq \quad & \sigma \max\{d_\lambda(\mathscr{K}x,\mathscr{L}y), d_{2\lambda}(\mathscr{M}x,\mathscr{K}x), d_{2\lambda}(\mathscr{N}y,\mathscr{L}y),\\ & d_\lambda(\mathscr{M}x,\mathscr{L}y), d_\lambda(\mathscr{K}x,\mathscr{N}y)\}, \end{aligned}$$

where $\sigma \in \left(0,\frac{1}{2}\right)$. Then, there exists only one point (say for instance μ) which verifies $\mathscr{M}\mu = \mathscr{K}\mu = \mu = \mathscr{N}\mu = \mathscr{L}\mu$.

Proof. In the following lines, we intend to prove the existence and uniqueness of the common fixed point.

Existence of the common fixed point: First of all, using the first condition of the theorem, we get the existence of two elements (say for instance ρ and χ) which verify

$$\mathscr{M}\rho = \mathscr{K}\rho \text{ implies that } d_\lambda(\mathscr{K}\mathscr{K}\rho, \mathscr{M}\rho) \leq d_\lambda(\mathscr{M}\mathscr{K}\rho, \mathscr{K}\rho)$$

$$\mathscr{N}\chi = \mathscr{L}\chi \text{ implies that } d_\lambda(\mathscr{L}\mathscr{L}\chi, \mathscr{N}\chi) \leq d_\lambda(\mathscr{N}\mathscr{L}\chi, \mathscr{L}\chi).$$

To prove the existence of the common fixed point, we need three steps.

Step one: First of all, we will prove that $\mathscr{M}\rho = \mathscr{N}\chi$. So, assume that $d_\lambda(\mathscr{M}\rho, \mathscr{N}\chi)$ is positive, then, we have

$$\begin{aligned} d_\lambda(\mathscr{M}\rho, \mathscr{N}\chi) &\leq \sigma \max\{d_\lambda(\mathscr{K}\rho, \mathscr{L}\chi), d_{2\lambda}(\mathscr{M}\rho, \mathscr{K}\rho), d_{2\lambda}(\mathscr{N}\chi, \mathscr{L}\chi), \\ &\qquad d_\lambda(\mathscr{M}\rho, \mathscr{L}\chi), d_\lambda(\mathscr{K}\rho, \mathscr{N}\chi)\}; \end{aligned}$$

that is,

$$\begin{aligned} d_\lambda(\mathscr{M}\rho, \mathscr{N}\chi) &\leq \sigma \max\{d_\lambda(\mathscr{M}\rho, \mathscr{N}\chi), 0\} \\ &= \sigma d_\lambda(\mathscr{M}\rho, \mathscr{N}\chi) \\ &< d_\lambda(\mathscr{M}\rho, \mathscr{N}\chi), \end{aligned}$$

which is a contradiction, hence, $\mathscr{M}\rho = \mathscr{N}\chi$.

Step two: Now, we intend to prove that $\mathscr{M}\mathscr{M}\rho = \mathscr{M}\rho$. Suppose that we have the contrary, then, we get

$$\begin{aligned} d_\lambda(\mathscr{M}\mathscr{M}\rho, \mathscr{N}\chi) &\leq \sigma \max\{d_\lambda(\mathscr{K}\mathscr{M}\rho, \mathscr{L}\chi), d_{2\lambda}(\mathscr{M}\mathscr{M}\rho, \mathscr{K}\mathscr{M}\rho), d_{2\lambda}(\mathscr{N}\chi, \mathscr{L}\chi), \\ &\qquad d_\lambda(\mathscr{M}\mathscr{M}\rho, \mathscr{L}\chi), d_\lambda(\mathscr{K}\mathscr{M}\rho, \mathscr{N}\chi)\}; \end{aligned}$$

that is

$$\begin{aligned} d_\lambda(\mathscr{M}\mathscr{M}\rho, \mathscr{M}\rho) &\leq \sigma \max\{d_\lambda(\mathscr{K}\mathscr{K}\rho, \mathscr{M}\rho), d_{2\lambda}(\mathscr{M}\mathscr{M}\rho, \mathscr{K}\mathscr{K}\rho), 0, \\ &\qquad d_\lambda(\mathscr{M}\mathscr{M}\rho, \mathscr{M}\rho), d_\lambda(\mathscr{K}\mathscr{K}\rho, \mathscr{M}\rho)\}. \end{aligned}$$

Now, using the triangular inequality and the relationship between $\mathscr{M}$ and $\mathscr{K}$, we obtain

$$\begin{aligned}
d_\lambda(\mathscr{M}\mathscr{M}\rho,\mathscr{M}\rho) &\leq \sigma\max\{d_\lambda(\mathscr{K}\mathscr{K}\rho,\mathscr{M}\rho),d_\lambda(\mathscr{M}\mathscr{M}\rho,\mathscr{M}\rho)+d_\lambda(\mathscr{M}\rho,\mathscr{K}\mathscr{K}\rho),0,\\
&\quad d_\lambda(\mathscr{M}\mathscr{M}\rho,\mathscr{M}\rho),d_\lambda(\mathscr{K}\mathscr{K}\rho,\mathscr{M}\rho)\}\\
&\leq \sigma\max\{d_\lambda(\mathscr{M}\mathscr{K}\rho,\mathscr{K}\rho),d_\lambda(\mathscr{M}\mathscr{M}\rho,\mathscr{M}\rho)+d_\lambda(\mathscr{K}\rho,\mathscr{M}\mathscr{K}\rho),0,\\
&\quad d_\lambda(\mathscr{M}\mathscr{M}\rho,\mathscr{M}\rho),d_\lambda(\mathscr{M}\mathscr{K}\rho,\mathscr{K}\rho)\}\\
&= \sigma\max\{d_\lambda(\mathscr{M}\mathscr{M}\rho,\mathscr{M}\rho),2d_\lambda(\mathscr{M}\mathscr{M}\rho,\mathscr{M}\rho),0\}\\
&= 2\sigma d_\lambda(\mathscr{M}\mathscr{M}\rho,\mathscr{M}\rho)\\
&< d_\lambda(\mathscr{M}\mathscr{M}\rho,\mathscr{M}\rho),
\end{aligned}$$

which is a contradiction, hence, $\mathscr{M}\mathscr{M}\rho=\mathscr{M}\rho$ and in consequence $\mathscr{K}\mathscr{M}\rho=\mathscr{M}\rho$.

Step three: In the next, we intend to prove that $\mathscr{N}\mathscr{N}\chi=\mathscr{N}\chi$. Suppose that we have the contrary, then, we obtain

$$\begin{aligned}
d_\lambda(\mathscr{M}\rho,\mathscr{N}\mathscr{N}\chi) &\leq \sigma\max\{d_\lambda(\mathscr{K}\rho,\mathscr{L}\mathscr{N}\chi),d_{2\lambda}(\mathscr{M}\rho,\mathscr{K}\rho),d_{2\lambda}(\mathscr{N}\mathscr{N}\chi,\mathscr{L}\mathscr{N}\chi),\\
&\quad d_\lambda(\mathscr{M}\rho,\mathscr{L}\mathscr{N}\chi),d_\lambda(\mathscr{K}\rho,\mathscr{N}\mathscr{N}\chi)\};
\end{aligned}$$

that is,

$$\begin{aligned}
d_\lambda(\mathscr{N}\chi,\mathscr{N}\mathscr{N}\chi) &\leq \sigma\max\{d_\lambda(\mathscr{N}\chi,\mathscr{L}\mathscr{N}\chi),0,d_{2\lambda}(\mathscr{N}\mathscr{N}\chi,\mathscr{L}\mathscr{L}\chi),\\
&\quad d_\lambda(\mathscr{N}\chi,\mathscr{L}\mathscr{N}\chi),d_\lambda(\mathscr{N}\chi,\mathscr{N}\mathscr{N}\chi)\}.
\end{aligned}$$

Now, using the triangular inequality and the relationship between $\mathscr{L}$ and $\mathscr{N}$, we obtain

$$\begin{aligned}
d_\lambda(\mathscr{N}\chi,\mathscr{N}\mathscr{N}\chi) &\leq \sigma\max\{d_\lambda(\mathscr{N}\chi,\mathscr{L}\mathscr{N}\chi),0,d_\lambda(\mathscr{N}\mathscr{N}\chi,\mathscr{L}\chi)+d_\lambda(\mathscr{L}\chi,\mathscr{L}\mathscr{L}\chi),\\
&\quad d_\lambda(\mathscr{N}\chi,\mathscr{L}\mathscr{N}\chi),d_\lambda(\mathscr{N}\chi,\mathscr{N}\mathscr{N}\chi)\}\\
&= \sigma\max\{d_\lambda(\mathscr{N}\chi,\mathscr{L}\mathscr{L}\chi),0,d_\lambda(\mathscr{N}\mathscr{N}\chi,\mathscr{N}\chi)+d_\lambda(\mathscr{N}\chi,\mathscr{L}\mathscr{L}\chi),\\
&\quad d_\lambda(\mathscr{N}\chi,\mathscr{L}\mathscr{L}\chi),d_\lambda(\mathscr{N}\chi,\mathscr{N}\mathscr{N}\chi)\}\\
&\leq \sigma\max\{d_\lambda(\mathscr{L}\chi,\mathscr{N}\mathscr{L}\chi),0,d_\lambda(\mathscr{N}\mathscr{N}\chi,\mathscr{N}\chi)+d_\lambda(\mathscr{L}\chi,\mathscr{N}\mathscr{L}\chi),\\
&\quad d_\lambda(\mathscr{L}\chi,\mathscr{N}\mathscr{L}\chi),d_\lambda(\mathscr{N}\chi,\mathscr{N}\mathscr{N}\chi)\}
\end{aligned}$$

$$
\begin{aligned}
&= \sigma \max\{d_\lambda(\mathscr{N}\chi, \mathscr{N}\mathscr{N}\chi), 0, 2d_\lambda(\mathscr{N}\mathscr{N}\chi, \mathscr{N}\chi)\} \\
&= 2\sigma d_\lambda(\mathscr{N}\chi, \mathscr{N}\mathscr{N}\chi) \\
&< d_\lambda(\mathscr{N}\chi, \mathscr{N}\mathscr{N}\chi),
\end{aligned}
$$

which is a contradiction, hence, $\mathscr{N}\chi = \mathscr{N}\mathscr{N}\chi$, consequently, $\mathscr{N}\chi = \mathscr{L}\mathscr{N}\chi$.

Uniqueness of the common fixed point: Let $\mathscr{M}\rho = \mathscr{N}\chi = \mu$ and suppose that there is another common fixed point say for instance ν. Replacing x with μ and y with ν in the inequality, we get

$$
\begin{aligned}
d_\lambda(\mu, \nu) = d_\lambda(\mathscr{M}\mu, \mathscr{N}\nu) &\leq \sigma \max\{d_\lambda(\mathscr{K}\mu, \mathscr{L}\nu), d_{2\lambda}(\mathscr{M}\mu, \mathscr{K}\mu), d_{2\lambda}(\mathscr{N}\nu, \mathscr{L}\nu), \\
&\quad d_\lambda(\mathscr{M}\mu, \mathscr{L}\nu), d_\lambda(\mathscr{K}\mu, \mathscr{N}\nu)\} \\
&= \sigma \max\{d_\lambda(\mu, \nu), 0\} \\
&= \sigma d_\lambda(\mu, \nu) \\
&< d_\lambda(\mu, \nu),
\end{aligned}
$$

which is a contradiction, hence, $\nu = \mu$. □

3.2. Illustrative Example.

Example 6. *Let λ be a positive real number. Equip the set $\mathscr{X} = (-\infty, +1]$ with the modular metric $d_\lambda(x, y) = \frac{|x-y|}{\lambda}$. Now, consider the next four mappings:*

$$
\mathscr{M}x = \begin{cases} 0 \text{ when } x \in (-\infty, 0] \\ \frac{1}{8-x} \text{ when } x \in (0, 1], \end{cases} \quad \mathscr{N}x = \begin{cases} 0 \text{ when } x \in (-\infty, 0] \\ \frac{1}{9-x} \text{ when } x \in (0, 1], \end{cases}
$$

$$
\mathscr{K}x = \begin{cases} x \text{ when } x \in (-\infty, 0] \\ 1 \text{ when } x \in (0, 1], \end{cases} \quad \mathscr{L}x = \begin{cases} x \text{ when } x \in (-\infty, 0] \\ \frac{1}{2} \text{ when } x \in (0, 1]. \end{cases}
$$

First of all, it is clear to see that mappings $\mathscr{K}$ and $\mathscr{M}$ as well as $\mathscr{L}$ and $\mathscr{N}$ are occasionally weakly $\mathscr{K}$-biased (respectively $\mathscr{L}$-biased) of type $(\mathscr{A})$.

Now, let the constructed table be as follows:

♠	$-\infty < x,y \le 0$	$0 < x,y \le 1$	$-\infty < x \le 0 < y \le 1$	$-\infty < y \le 0 < x \le 1$
$d_\lambda(\mathscr{M}x,\mathscr{N}y)$	0	$\frac{1}{\lambda}\left\lvert\frac{1}{8-x}-\frac{1}{9-y}\right\rvert$	$\frac{1}{\lambda}\left\lvert\frac{1}{9-y}\right\rvert$	$\frac{1}{\lambda}\left\lvert\frac{1}{8-x}\right\rvert$
$d_\lambda(\mathscr{K}x,\mathscr{L}y)$	$\frac{1}{\lambda}\lvert x-y\rvert$	$\frac{1}{2\lambda}$	$\frac{1}{\lambda}\left\lvert x-\frac{1}{2}\right\rvert$	$\frac{1}{\lambda}\lvert 1-y\rvert$
$d_{2\lambda}(\mathscr{K}x,\mathscr{M}x)$	$\frac{1}{2\lambda}\lvert x\rvert$	$\frac{1}{2\lambda}\left\lvert\frac{1}{8-x}-1\right\rvert$	$\frac{1}{2\lambda}\lvert x\rvert$	$\frac{1}{2\lambda}\left\lvert\frac{1}{8-x}-1\right\rvert$
$d_{2\lambda}(\mathscr{L}y,\mathscr{N}y)$	$\frac{1}{2\lambda}\lvert y\rvert$	$\frac{1}{2\lambda}\left\lvert\frac{1}{9-y}-\frac{1}{2}\right\rvert$	$\frac{1}{2\lambda}\left\lvert\frac{1}{9-y}-\frac{1}{2}\right\rvert$	$\frac{1}{2\lambda}\lvert y\rvert$
$d_\lambda(\mathscr{K}x,\mathscr{N}y)$	$\frac{1}{\lambda}\lvert x\rvert$	$\frac{1}{\lambda}\left\lvert 1-\frac{1}{9-y}\right\rvert$	$\frac{1}{\lambda}\left\lvert x-\frac{1}{9-y}\right\rvert$	$\frac{1}{\lambda}$
$d_\lambda(\mathscr{M}x,\mathscr{L}y)$	$\frac{1}{\lambda}\lvert y\rvert$	$\frac{1}{\lambda}\left\lvert\frac{1}{8-x}-\frac{1}{2}\right\rvert$	$\frac{1}{2\lambda}$	$\frac{1}{\lambda}\left\lvert\frac{1}{8-x}-y\right\rvert$

Discussion: *Taking $\sigma = \frac{49}{100}$, looking at the table, we note the following cases:*

First case: *For $x,\ y \in (-\infty, 0]$, we have $\mathscr{M}x = 0 = \mathscr{N}y$, $\mathscr{K}x = x$, $\mathscr{L}y = y$ and*

$$\begin{aligned}
d_\lambda(\mathscr{M}x,\mathscr{N}y) &= 0 \\
&\le \frac{49}{100}\max\left\{\frac{1}{\lambda}|x-y|, \frac{1}{2\lambda}|x|, \frac{1}{2\lambda}|y|, \frac{1}{\lambda}|x|, \frac{1}{\lambda}|y|\right\} \\
&= \frac{49}{100\lambda}\max\left\{|x-y|, \frac{1}{2}|x|, \frac{1}{2}|y|, |x|, |y|\right\} \\
&= \frac{49}{100\lambda}\max\{|y-x|, |x|, |y|\} \\
&= \sigma\max\{d_\lambda(\mathscr{K}x,\mathscr{L}y), d_{2\lambda}(\mathscr{K}x,\mathscr{M}x), d_{2\lambda}(\mathscr{N}y,\mathscr{L}y), d_\lambda(\mathscr{K}x,\mathscr{N}y), \\
&\quad d_\lambda(\mathscr{M}x,\mathscr{L}y)\}.
\end{aligned}$$

Second case: *When $x,\ y \in (0,1]$, $\mathscr{M}x = \frac{1}{8-x}$, $\mathscr{N}y = \frac{1}{9-y}$, $\mathscr{K}x = 1$, $\mathscr{L}y = \dfrac{1}{2}$, then,*

$$\begin{aligned}
d_\lambda(\mathscr{M}x,\mathscr{N}y) &= \frac{1}{\lambda}\left|\frac{1}{8-x}-\frac{1}{9-y}\right| \\
&\le \frac{49}{100}\max\left\{\frac{1}{2\lambda}, \frac{1}{2\lambda}\left|\frac{1}{8-x}-1\right|, \frac{1}{2\lambda}\left|\frac{1}{9-y}-\frac{1}{2}\right|, \frac{1}{\lambda}\left|1-\frac{1}{9-y}\right|,\right. \\
&\quad \left.\frac{1}{\lambda}\left|\frac{1}{8-x}-\frac{1}{2}\right|\right\} \\
&= \frac{49}{100\lambda}\max\left\{\frac{1}{2}, \frac{1}{2}\left|\frac{1}{8-x}-1\right|, \frac{1}{2}\left|\frac{1}{9-y}-\frac{1}{2}\right|, \left|1-\frac{1}{9-y}\right|,\right. \\
&\quad \left.\left|\frac{1}{8-x}-\frac{1}{2}\right|\right\} \\
&= \sigma\max\{d_\lambda(\mathscr{K}x,\mathscr{L}y), d_{2\lambda}(\mathscr{K}x,\mathscr{M}x), d_{2\lambda}(\mathscr{N}y,\mathscr{L}y), d_\lambda(\mathscr{K}x,\mathscr{N}y), \\
&\quad d_\lambda(\mathscr{M}x,\mathscr{L}y)\}.
\end{aligned}$$

Third case: *Now, for $x \in (-\infty, 0]$ and $y \in (0,1]$, we have $\mathscr{M}x = 0$, $\mathscr{N}y = \frac{1}{9-y}$, $\mathscr{K}x = x$, $\mathscr{L}y = \dfrac{1}{2}$ and*

$$\begin{aligned}
d_\lambda(\mathscr{M}x, \mathscr{N}y) &= \frac{1}{\lambda}\left|\frac{1}{9-y}\right| \\
&\leq \frac{49}{100}\max\left\{\frac{1}{\lambda}\left|x-\frac{1}{2}\right|, \frac{1}{2\lambda}|x|, \frac{1}{2\lambda}\left|\frac{1}{9-y}-\frac{1}{2}\right|, \frac{1}{\lambda}\left|x-\frac{1}{9-y}\right|, \frac{1}{2\lambda}\right\} \\
&= \frac{49}{100\lambda}\max\left\{\left|x-\frac{1}{2}\right|, \frac{1}{2}|x|, \frac{1}{2}\left|\frac{1}{9-y}-\frac{1}{2}\right|, \left|x-\frac{1}{9-y}\right|, \frac{1}{2}\right\} \\
&= \sigma\max\{d_\lambda(\mathscr{K}x, \mathscr{L}y), d_{2\lambda}(\mathscr{K}x, \mathscr{M}x), d_{2\lambda}(\mathscr{N}y, \mathscr{L}y), d_\lambda(\mathscr{K}x, \mathscr{N}y), \\
&\quad d_\lambda(\mathscr{M}x, \mathscr{L}y)\}.
\end{aligned}$$

Fourth case: *At the end, when $x \in (0,1]$ and $y \in (-\infty, 0]$, $\mathscr{M}x = \frac{1}{8-x}$, $\mathscr{N}y = 0$, $\mathscr{K}x = 1$, $\mathscr{L}y = y$ and*

$$\begin{aligned}
d_\lambda(\mathscr{M}x, \mathscr{N}y) &= \frac{1}{\lambda}\left|\frac{1}{8-x}\right| \\
&\leq \frac{49}{100}\max\left\{\frac{1}{\lambda}|1-y|, \frac{1}{2\lambda}\left|\frac{1}{8-x}-1\right|, \frac{1}{2\lambda}|y|, \frac{1}{\lambda}, \frac{1}{\lambda}\left|\frac{1}{8-x}-y\right|\right\} \\
&= \frac{49}{100\lambda}\max\left\{|1-y|, \frac{1}{2}\left|\frac{1}{8-x}-1\right|, \frac{1}{2}|y|, 1, \left|\frac{1}{8-x}-y\right|\right\} \\
&= \sigma\max\{d_\lambda(\mathscr{K}x, \mathscr{L}y), d_{2\lambda}(\mathscr{K}x, \mathscr{M}x), d_{2\lambda}(\mathscr{N}y, \mathscr{L}y), d_\lambda(\mathscr{K}x, \mathscr{N}y), \\
&\quad d_\lambda(\mathscr{M}x, \mathscr{L}y)\}.
\end{aligned}$$

We conclude that the two conditions of our theorem are satisfied and 0 is the unique element which verifies that $\mathscr{M}(0) = \mathscr{N}(0) = \mathscr{K}(0) = \mathscr{L}(0) = 0$.

3.3. Some Results Derived from The Main Theorem.

Corollary 7. *Let $\mathscr{X}$ be a nonempty closed set endowed with a modular metric d_λ. Consider four self-mappings $\mathscr{K}$, $\mathscr{L}$, $\mathscr{M}$ and $\mathscr{N}$ such that*

1) $\mathscr{K}$ and $\mathscr{M}$ as well as $\mathscr{L}$ and $\mathscr{N}$ are occasionally weakly $\mathscr{K}$-biased respectively $\mathscr{L}$-biased of type $(\mathscr{A})$,

2) for all λ positive and x, y in $\mathscr{X}$, we have

$$d_\lambda(\mathscr{M}x, \mathscr{N}y) \leq \sigma d_\lambda(\mathscr{K}x, \mathscr{L}y),$$

where $\sigma \in (0,1)$. Then, there exists only one point (say for instance μ) which verifies $\mathscr{M}\mu = \mathscr{K}\mu = \mu = \mathscr{N}\mu = \mathscr{L}\mu$.

Corollary 8. *Let $\mathscr{X}$ be a nonempty closed set endowed with a modular metric d_λ. Consider four self-mappings $\mathscr{K}$, $\mathscr{L}$, $\mathscr{M}$ and $\mathscr{N}$ such that*

1) $\mathscr{K}$ and $\mathscr{M}$ as well as $\mathscr{L}$ and $\mathscr{N}$ are occasionally weakly $\mathscr{K}$-biased respectively $\mathscr{L}$-biased of type $(\mathscr{A})$,

2) for all λ positive and x, y in $\mathscr{X}$, we have

$$d_\lambda(\mathscr{M}x, \mathscr{N}y) \leq \sigma \max\{d_\lambda(\mathscr{K}x, \mathscr{L}y), d_{2\lambda}(\mathscr{M}x, \mathscr{K}x), d_{2\lambda}(\mathscr{N}y, \mathscr{L}y)\},$$

where $\sigma \in \left(0, \frac{1}{2}\right)$. Then, there exists only one point (say for instance μ) which verifies $\mathscr{M}\mu = \mathscr{K}\mu = \mu = \mathscr{N}\mu = \mathscr{L}\mu$.

3.4. **Application.** Consider the next equation:

$$\mathscr{M}(x) = \mathscr{P}(x) + \int_0^1 \mathscr{Q}(x,y)\mathscr{R}(y, \mathscr{M}(y))dy \tag{3.1}$$

for $x \in [0,1]$, $\mathscr{P} : [0,1] \to \mathbb{R}$, $\mathscr{Q} : [0,1] \times [0,1] \to \mathbb{R}$, $\mathscr{R} : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous mappings. Let $\mathscr{X} = C([0,1], \mathbb{R})$ be endowed with the modular metric

$$d_\lambda(\mathscr{M}, \mathscr{N}) = \frac{1}{1+\lambda} \sup_{0 \leq x \leq 1} |\mathscr{M}(x) - \mathscr{N}(x)|,$$

for all λ positive and $\mathscr{M}$, $\mathscr{N}$ in $\mathscr{X}$. Define on $\mathscr{X}$ the self-operator $\mathscr{O}$ by

$$\mathscr{O}\mathscr{M}(x) = \mathscr{P}(x) + \int_0^1 \mathscr{Q}(x,y)\mathscr{R}(y, \mathscr{M}(y))dy.$$

Theorem 9. *Suppose that the next requirements hold true:*

(1) $\sup\limits_{0 \leq x \leq 1} \int_0^1 \mathscr{Q}(x,y)dy \leq 1$,

(2) $|\mathscr{R}(y, \mathscr{M}(y)) - \mathscr{R}(y, \mathscr{N}(y))| \leq \sigma|\mathscr{M} - \mathscr{N}|$, *for each* $y \in [0,1]$, $\mathscr{M}$, $\mathscr{N} \in \mathscr{X}$, *where* $\sigma \in \left(0, \frac{1}{2}\right)$.

Then, integral equation (3.1) *has a unique solution.*

Proof. For all $\mathscr{M}$, $\mathscr{N}$ in $\mathscr{X}$, we have

$$\begin{aligned} |\mathscr{O}\mathscr{M}(x)-\mathscr{O}\mathscr{N}(x)| &= \left|\int_0^1 \mathscr{Q}(x,y)[\mathscr{R}(y,\mathscr{M}(y))-\mathscr{R}(y,\mathscr{N}(y))]dy\right| \\ &\leq \int_0^1 |\mathscr{Q}(x,y)||\mathscr{R}(y,\mathscr{M}(y))-\mathscr{R}(y,\mathscr{N}(y))|dy \\ &\leq \sigma\int_0^1 |\mathscr{Q}(x,y)||\mathscr{M}(y)-\mathscr{N}(y)|dy. \end{aligned}$$

Multiplying by $\frac{1}{1+\lambda}$ and taking the supremum on both sides, we get

$$\begin{aligned} d_\lambda(\mathscr{O}\mathscr{M},\mathscr{O}\mathscr{N}) &\leq \sigma d_\lambda(\mathscr{M},\mathscr{N}) \\ &\leq \sigma\max\{d_\lambda(\mathscr{M},\mathscr{N}),d_{2\lambda}(\mathscr{M},\mathscr{O}\mathscr{M}),d_{2\lambda}(\mathscr{O}\mathscr{N},\mathscr{N}),d_\lambda(\mathscr{O}\mathscr{M},\mathscr{N}), \\ &\quad d_\lambda(\mathscr{M},\mathscr{O}\mathscr{N})\}, \end{aligned}$$

so, all the requirements of Theorem 5 with $\mathscr{M}=\mathscr{N}=\mathscr{O}$ and $\mathscr{K}=\mathscr{L}=\mathscr{I}$ (the identity mapping on $\mathscr{X}$) are satisfied, hence, integral equation (3.1) has a unique solution in $\mathscr{X}$. □

4. Conclusion

In this work, we proved the existence and uniqueness of a common fixed point for two pairs of occasionally weakly biased mappings of type $(\mathscr{A})$ in a nonempty closed set endowed with a modular metric. In addition, we derived two other results, present a suitable example which illustrates our main result, furthermore, we provided an application to solve an integral equation. Our results improve some existing ones for instance Theorem 3.2 of [19]. Note that, we investigated our results in the recent weaker framework of occasionally weakly biased mappings of type $(\mathscr{A})$, consequently, some other results in metric as well as modular metric spaces become as results of our findings.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] A.A.N. Abdou, Some Fixed Point Theorems in Modular Metric Spaces, J. Nonlinear Sci. Appl. 09 (2016), 4381–4387. https://doi.org/10.22436/jnsa.009.06.78.

[2] A. Abdou, M.A. Khamsi, On the Fixed Points of Nonexpansive Mappings in Modular Metric Spaces, Fixed Point Theory Appl. 2013 (2013), 229. https://doi.org/10.1186/1687-1812-2013-229.

[3] H. Abobaker, R. Ryan, Modular Metric Spaces, Irish Math. Soc. Bull. 80 (2017), 35–44. https://doi.org/10.33232/bims.0080.35.44.

[4] U. Aksoy, E. Karapinar, I. Erhan, Fixed Point Theorems in Complete Modular Metric Spaces and an Application to Anti-Periodic Boundary Value Problems, Filomat 31 (2017), 5475–5488. https://doi.org/10.2298/FIL1717475A.

[5] H. Bouhadjera, Fixed Points for Occasionally Weakly Biased Mappings of Type $(\mathscr{A})$, Math. Morav. 26 (2022), 113–122. https://doi.org/10.5937/matmor2201113b.

[6] H. Bouhadjera, Fixed Point Results in Incomplete Metric and Dislocated Metric Spaces, Ann. West Univ. Timisoara - Math. Comput. Sci. 59 (2023), 130–150. https://doi.org/10.2478/awutm-2023-0011.

[7] H. Bouhadjera, A General Common Fixed Point Theorem for Occasionally Weakly Compatible Maps, An. Univ. Oradea Fasc. Mat. XVII (2010), 17–22.

[8] H. Bouhadjera, A. Djoudi, General Common Fixed Point Theorems for Weakly Compatible Maps, Gen. Math. 16 (2008), 95–107.

[9] H. Bouhadjera, A. Djoudi, Common Fixed Point Theorems for Pairs of Single and Multivalued D-Maps Satisfying an Integral Type, Ann. Math. Inform. 35 (2008), 43–59.

[10] V.V. Chistyakov, Metric Modulars and Their Application, Dokl. Akad. Nauk 406 (2006), 165–168.

[11] V.V. Chistyakov, Modular Metric Spaces, I: Basic Concepts, Nonlinear Anal. Theory Methods Appl. 72 (2010), 1–14. https://doi.org/10.1016/j.na.2009.04.057.

[12] Z. Djedid, I.M. Batiha, S. Alsharif, S. Abu-Ghurra, M. Aljazzazi, et al., Fixed Point of Hardy-Rogers Contraction Mappings in Non-Solid Cone χ_b-Metric Space with Applications, Adv. Fixed Point Theory 15 (2025), 25. https://doi.org/10.28919/afpt/9304.

[13] M.R. Fréchet, Sur Quelques Points du Calcul Fonctionnel, Rend. Circ. Mat. Palermo 22 (1906), 1–72.

[14] S. Hussain, Non-Unique Fixed Point Theorems in Modular Metric Spaces, Symmetry 11 (2019), 549. https://doi.org/10.3390/sym11040549.

[15] I.H. Jebril, H. Bouhadjera, Some Fixed Point Results in Dislocated Metric Spaces under Minimum Conditions, Adv. Fixed Point Theory 15 (2025), 8. https://doi.org/10.28919/afpt/9069.

[16] I.H. Jebril, H. Bouhadjera, I.M. Batiha, B. Batiha, Common Fixed Point Theorems in Metric Domains, Adv. Fixed Point Theory 15 (2025), 4. https://doi.org/10.28919/afpt/8969.

[17] E. Karapınar, A. Fulga, S.S. Yeşilkaya, Interpolative Meir–Keeler Mappings in Modular Metric Spaces, Mathematics 10 (2022), 2986. https://doi.org/10.3390/math10162986.

[18] A. Mutlu, K. Ozkan, U. Gurdal, A New Fixed Point Theorem in Modular Metric Spaces, Int. J. Anal. Appl. 16 (2018), 472–483.

[19] C. Mongkolkeha, W. Sintunavarat, P. Kumam, Fixed Point Theorems for Contraction Mappings in Modular Metric Spaces, Fixed Point Theory Appl. 2011 (2011), 93. https://doi.org/10.1186/1687-1812-2011-93.

[20] R. Pal, A New Fixed Point Theorem in Modular Metric Spaces, Res. J. Math. Stat. Sci. 8 (2020), 17–20.

[21] Y. Sharma, S. Jain, Coupled Fixed Point Theorems in Modular Metric Spaces Endowed with a Graph, Kyungpook Math. J. 61 (2021), 441–453. https://doi.org/10.5666/KMJ.2021.61.2.441.

[22] A. Younus, M.U. Azam, M. Asif, G. Atta, Fixed Point Theorems for Self and Non-Self Contractions in Modular Metric Spaces Endowed with a Graph, Palestine J. Math. 11 (2022), 385–396.

[23] X. Zhu, Q. Zhao, L. Zhong, Fixed Point Results for F-Contraction Mapping in Modular Metric Spaces, Eur. J. Math. Comput. Sci. 7 (2020), 39–47.